ON THE BETTI NUMBERS OF BIRATIONALLY ISOMORPHIC PROJECTIVE VARIETIES WITH TRIVIAL CANONICAL BUNDLES

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Abstract

Let $X$ and $Y$ be two birationally isomorphic smooth projective $n$-dimensional algebraic varieties $X$ and $Y$ over $\mathbb{C}$ having trivial canonical line bundles. Using methods of the $p$-adic analysis on algebraic varieties over local number fields, we prove that in the above situation the Betti numbers of $X$ and $Y$ must be the same.

1 Introduction

The purpose of this note is to show that the elementary theory of the $p$-adic integrals on algebraic varieties help to prove some cohomological properties of birationally isomorphic algebraic varieties over $\mathbb{C}$. We prove the following theorem which has been used by Beauville in his recent explanation of a Yau-Zaslow formula for the number of rational curves on a $K3$-surface [1] (see also [3, 10]):

**Theorem 1.1** Let $X$ and $Y$ be two irreducible birationally isomorphic smooth $n$-dimensional projective algebraic varieties over $\mathbb{C}$. Assume that the canonical line bundles $\Omega^n_X$ and $\Omega^n_Y$ are trivial. Then $X$ and $Y$ must have the same Betti numbers, i.e.,

$$H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C}) \quad \forall i \geq 0.$$ 

We remark that Theorem 1.1 is obvious for $n = 1$. In the case $n = 2$, Theorem 1.1 follows from the uniqueness of minimal models of surfaces of nonnegative Kodaira dimension, i.e. from the property that any birational isomorphism between two such minimal models extends to a biregular one [4]. The uniqueness of minimal models of $n$-dimensional algebraic varieties of nonnegative Kodaira dimension fails for $n \geq 3$ in general. However, Theorem 1.1 for $n = 3$ can be proved using a result of
Kawamata ([5], §6), who has shown that any two birationally isomorphic minimal models of 3-folds are connected by a sequence of flops (see also [6]). By simple topological arguments, one can prove that if two 3-dimensional projective algebraic varieties over \( \mathbb{C} \) with at worst \( \mathbb{Q} \)-factorial terminal singularities are birationally isomorphic via a flop, then their singular Betti numbers are the same. Since one still knows very little about flops in dimension \( n \geq 4 \), it seems unlikely to expect that a consideration of flops could help to prove 1.1 in arbitrary dimension \( n \geq 4 \). Moreover, for projective algebraic varieties with at worst \( \mathbb{Q} \)-factorial Gorenstein terminal singularities of dimension \( n \geq 4 \) Theorem 1.1 is not true in general. For this reason the condition of smoothness for \( X \) and \( Y \) in 1.1 becomes very important in the case \( n \geq 4 \).

2 Gauge-forms and \( p \)-adic measures

Let \( F \) be a local number field, i.e., a finite extension of \( \mathbb{Q}_p \) for some prime \( p \in \mathbb{Z} \). Let \( R \subset F \) be the maximal compact subring, \( q \subset R \) the maximal ideal, \( F_q = R/q \) the residue field with \( |F_q| = q = p^r \). We denote by \( \| : \| : F \to \mathbb{R}_{\geq 0} \) the multiplicative \( p \)-adic norm:

\[
a \mapsto \|a\| = p^{-\text{Ord}(N_{F/Q_p}(a))},
\]

where

\[
N_{F/Q_p} : F \to \mathbb{Q}_p
\]

is the standard norm mapping.

**Definition 2.1** Let \( \mathfrak{X} \) be an arbitrary reduced algebraic \( S \)-scheme, where \( S = \text{Spec } R \). We denote by \( \mathfrak{X}(R) \) the set of \( S \)-morphisms \( S \to \mathfrak{X} \) (or sections of \( \mathfrak{X} \to S \)). We call \( \mathfrak{X}(R) \) the set of \textbf{R-integral points} in \( \mathfrak{X} \). The set of sections of the morphism \( \mathfrak{X} \times_S \text{Spec } F \to \text{Spec } F \) we denote by \( \mathfrak{X}(F) \) and call the set of \textbf{F-rational points} in \( \mathfrak{X} \).

**Remark 2.2** (i) If \( \mathfrak{X} \) is an affine \( S \)-scheme, then one can identify \( \mathfrak{X}(R) \) with the subset in \( \mathfrak{X}(F) \) consisting of all points \( x \in \mathfrak{X}(F) \) such that \( f(x) \in R \) for all \( f \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \).

(ii) If \( \mathfrak{X} \) is a projective (or proper) \( S \)-scheme, then \( \mathfrak{X}(R) = \mathfrak{X}(F) \).

Now let \( X \) be a smooth \( n \)-dimensional algebraic variety over \( F \). Denote by \( \Omega^n_X \) the canonical line bundle over \( X \). We assume that \( X \) admits an extension \( \mathfrak{X} \) to a regular \( S \)-scheme.

Recall the following definition introduced by A. Weil in [9]:

**Definition 2.3** A global section \( \omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/S}) \) is called a **gauge-form** if the \( n \)-form \( \omega \) has no zeros in \( \mathfrak{X} \). By definition, a gauge-form \( \omega \) defines an isomorphism \( \mathcal{O}_\mathfrak{X} \cong \Omega^n_{\mathfrak{X}/S} \) which sends 1 to \( \omega \), i.e., it exists if and only if \( \Omega^n_{\mathfrak{X}/S} \) is a trivial line bundle.
It was observed by A. Weil that a gauge form $\omega$ determines a canonical $p$-adic measure $d\mu_\omega$ on the locally compact $p$-adic topological space $\mathcal{X}(F)$ of $F$-rational points in $\mathcal{X}$. The $p$-adic measure $d\mu_\omega$ is defined as follows:

Let $x \in \mathcal{X}(F)$ be an $F$-point, $t_1, \ldots, t_n$ local $p$-adic analytic parameters at $x$. Then $t_1, \ldots, t_n$ define a $p$-adic homeomorphism $\theta : U \to \mathbb{A}^n(F)$ of an open subset $U \subset \mathcal{X}(F)$ containing $x$ with an open subset $\theta(U) \subset \mathbb{A}^n(F)$. One should stress that both subsets $U \subset \mathcal{X}(F)$ and $\theta(U) \subset \mathbb{A}^n(F)$ are considered to be open in $p$-adic topology, but not in Zariski topology. We write

$$\omega = \theta^* (g dt_1 \wedge \cdots \wedge dt_n),$$

where $g = g(t)$ is a $p$-adic analytic function on $\theta(U)$ having no zeros. Then a $p$-adic measure $d\mu_\omega$ on $U$ is defined to be the pull-back with respect to $\theta$ of the $p$-adic measure $\|g(t)\| \text{d}t$ on $\theta(U)$, where $\text{d}t$ is a standard $p$-adic Haar measure on $\mathbb{A}^n(F)$ with the normalizing condition

$$\int_{\mathbb{A}^n(R)} \text{d}t = 1.$$ 

It is a standard exercise with the Jacobian to check that two $p$-adic measures $d\mu'_\omega, d\mu''_\omega$ constructed by the above method on any two open subsets $U', U'' \subset \mathcal{X}(F)$ coincide on the intersection $U' \cap U''$.

**Definition 2.4** The measure $d\mu_\omega$ on $\mathcal{X}(F)$ constructed as above we call a $p$-adic measure of Weil associated with a gauge-form $\omega$.

**Theorem 2.5** ([9], Th. 2.2.5) Assume that $\mathcal{X}$ is a regular $S$-scheme as above, $\omega$ is a gauge-form on $\mathcal{X}$, and $d\mu_\omega$ the corresponding $p$-adic measure of Weil on $\mathcal{X}(F)$. Then

$$\int_{\mathcal{X}(R)} d\mu_\omega = \frac{|\mathcal{X}(F_q)|}{q^n},$$

where $\mathcal{X}(F_q)$ is the set of closed points of $\mathcal{X}$ over the finite residue field $F_q$.

**Proof.** Let

$$\phi : \mathcal{X}(R) \to \mathcal{X}(F_q), \ x \mapsto \overline{x} \in \mathcal{X}(F_q)$$

be the natural surjective mapping. The idea of proof of the theorem is based on the fact that if $\overline{x} \in \mathcal{X}(F_q)$ is a closed $F_q$-point of $\mathcal{X}$ and $g_1, \ldots, g_n$ are generators of the maximal ideal of $\overline{x}$ in $\mathcal{O}_{\mathcal{X}, \overline{x}}$ modulo the ideal $q$, then the elements $g_1, \ldots, g_n$ define a $p$-adic analytic homeomorphism

$$\gamma : \phi^{-1}(\overline{x}) \to \mathbb{A}^n(q),$$

where $\phi^{-1}(\overline{x})$ is the fiber of $\phi$ over $\overline{x}$ and $\mathbb{A}^n(q)$ is the set of all $R$-integral points of $\mathbb{A}^n$ whose coordinates belong to the ideal $q \subset R$. Moreover, the $p$-adic norm of the
Jacobian of $\gamma$ is identically equal to 1 on the whole fiber $\phi^{-1}(\mathcal{F})$. The latter follows from the fact that if $n$ formal power series $g_1(t), \ldots, g_n(t) \in R[[t_1, \ldots, t_n]]$ are generators of the prime ideal $(t_1, \ldots, t_n)$, then the series $g_1(t), \ldots, g_n(t)$ converge absolutely in $p$-adic norm on the compact $\mathbb{A}^n(q)$ and the Jacobian of the corresponding mapping

$$\mathbb{A}^n(q) \to \mathbb{A}^n(q), \ (t_1, \ldots, t_n) \mapsto (g_1(t), \ldots, g_n(t))$$

is equal to a nonzero element of $F_q$ modulo $q$ on the whole subset $\mathbb{A}^n(q) \subset \mathbb{A}^n(R)$. So, using the $p$-adic analytic homeomorphism $\gamma$, one obtains

$$\int_{\phi^{-1}(\mathcal{F})} d\mu_\omega = \int_{\mathbb{A}^n(q)} dt = \frac{1}{q^n}$$

for each $\mathcal{F} \in \mathcal{X}(F_q)$.

Now we consider a slightly more general situation. Let us only assume that $\mathcal{X}$ is a regular scheme over $S$, but do not assume the existence of a gauge-form on $\mathcal{X}$ (i.e. of an isomorphism $\mathcal{O}_X \cong \Omega^n_{\mathcal{X}/S}$). Nevertheless under these weaker assumptions we can define a unique natural $p$-adic measure $d\mu$ at least on the compact $\mathcal{X}(R) \subset \mathcal{X}(F)$ (but may be not on the whole $p$-adic topological space $\mathcal{X}(F)$):

Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ be a finite covering of $\mathcal{X}$ by Zariski open $S$-subschemes such that the restriction of $\Omega^n_{\mathcal{X}/S}$ on each $\mathcal{U}_i$ is isomorphic to $\mathcal{O}_{\mathcal{U}_i}$. Then each $\mathcal{U}_i$ admits a gauge-form $\omega_i$ and we define a $p$-adic measure $d\mu_i$ on each compact $\mathcal{U}_i(R)$ as the restriction of the $p$-adic measure of Weil $d\mu_{\omega_i}$ associated with $\omega_i$ on $\mathcal{U}_i(F)$. We note that the gauge-forms $\omega_i$ are defined uniquely up to elements $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}_{\mathcal{X}}^*)$. On the other hand, the $p$-adic norm $\|s_i(x)\|$ equals 1 for any element $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}_{\mathcal{X}}^*)$ and any $R$-rational point $x \in \mathcal{U}_i(R)$. Therefore, the constructed $p$-adic measure on $\mathcal{U}_i(R)$ does not depend on the choice of a gauge-form $\omega_i$. Moreover, the $p$-adic measures $d\mu_i$ on $\mathcal{U}_i(R)$ glue together to a $p$-adic measure $d\mu$ on the whole compact $\mathcal{X}(R)$, since one has

$$\mathcal{U}_i(R) \cap \mathcal{U}_j(R) = (\mathcal{U}_i \cap \mathcal{U}_j)(R) \ \forall i, j \in \{1, \ldots, k\}$$

and

$$\mathcal{U}_1(R) \cup \cdots \cup \mathcal{U}_k(R) = (\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k)(R) = \mathcal{X}(R).$$

**Definition 2.6** The constructed above $p$-adic measure defined on the set $\mathcal{X}(R)$ of $R$-integral points of a $S$-scheme $\mathcal{X}$ will be called the **canonical $p$-adic measure**.

For the canonical $p$-adic measure $d\mu$, we obtain the same property as for the $p$-adic measure of Weil $d\mu_{\omega}$.
Theorem 2.7
\[ \int_{\mathfrak{X}(R)} d\mu = \frac{|\mathfrak{X}(F_q)|}{q^n}. \]

Proof. Using a covering of \( \mathfrak{X} \) by some Zariski open subsets \( U_1, \ldots, U_k \), we obtain
\[ \int_{\mathfrak{X}(R)} d\mu = \sum_{i_1} \int_{U_{i_1}(R)} d\mu - \sum_{i_1 < i_2} \int_{(U_{i_1} \cap U_{i_2})(R)} d\mu + \cdots + (-1)^k \int_{(U_{i_1} \cap \cdots \cap U_{i_k})(R)} d\mu \]
and
\[ |\mathfrak{X}(F_q)| = \sum_{i_1} |U_{i_1}(F_q)| - \sum_{i_1 < i_2} |(U_{i_1} \cap U_{i_2})(F_q)| + \cdots + (-1)^k |(U_{i_1} \cap \cdots \cap U_{i_k})(F_q)|. \]

It remains to apply 2.5 to every intersection \( U_{i_1} \cap \cdots \cap U_{i_s} \).

\[ \square \]

Theorem 2.8 Let \( \mathfrak{X} \) be a regular integral \( S \)-scheme and \( Z \subset \mathfrak{X} \) is a closed reduced subscheme of codimension 1. Then the subset \( Z(R) \subset \mathfrak{X}(R) \) has zero measure with respect to the canonical \( p \)-adic measure \( d\mu \) on \( \mathfrak{X}(R) \).

Proof. Using a covering of \( \mathfrak{X} \) by some Zariski open affine subsets \( U_1, \ldots, U_k \), one can always reduce the situation to the case when \( \mathfrak{X} \) is an affine regular integral \( S \)-scheme and \( Z \subset \mathfrak{X} \) is an irreducible principal divisor defined by an equation \( f = 0 \), where \( f \) is a prime element of \( A = \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \).

Let us consider a special case \( \mathfrak{X} = \mathbb{A}^n_S = \text{Spec } R[X_1, \ldots, X_n] \) and \( Z = \mathbb{A}^{n-1}_S = \text{Spec } R[X_2, \ldots, X_n] \), i.e., \( f = X_1 \). For every positive integer \( m \), we denote by \( Z_m(R) \) the subset in \( \mathbb{A}^n(R) \) consisting of all points \( x = (x_1, \ldots, x_d) \in R^n \) such that the \( x_1 \) belongs to the \( m \)-th power of \( q \). One computes straightforward the \( p \)-adic integral
\[ \int_{Z_m(R)} dx = \int_{\mathbb{A}^1(q^m)} dx_1 \prod_{i=2}^n \left( \int_{\mathbb{A}^1(R)} dx_i \right) = \frac{1}{q^n}. \]

On the other hand, we have
\[ Z(R) = \bigcap_{m=1}^\infty Z_m(R). \]

Hence
\[ \int_{Z(R)} dx = \lim_{m \to \infty} \int_{Z_m(R)} dx = 0, \]
and in this case the statement is proved. Using the Noether normalization theorem, one reduces the more general case to the above special one. \[ \square \]
3 The Betti numbers

Proposition 3.1 Let $X$ and $Y$ be two birationally isomorphic smooth projective $n$-dimensional algebraic varieties over $\mathbb{C}$ having trivial canonical line bundles. Then there exist two Zariski open dense subsets $U \subset X$ and $V \subset Y$ such that $U$ is biregularly isomorphic to $V$ and $\operatorname{codim}_X(X \setminus U) = \operatorname{codim}_Y(Y \setminus V) \geq 2$.

Proof. Consider a birational rational map $\varphi : X \dashrightarrow Y$. Since $X$ is smooth, there exists a Zariski open dense subset $U_0 \subset X$ with $\operatorname{codim}_X(X \setminus U_0) \geq 2$ such that $\varphi$ extends to a regular morphism $\varphi_0 : U_0 \to Y$. Define $Z \subset U$ to be the Zariski closed subset consisting of all points $x \in X$ such that $\varphi_0^{-1}(\varphi_0(x)) \neq x$. Since both line bundles $\Omega^n_X$ and $\Omega^n_Y$ are trivial, $Z$ can not be a divisor in $U$: otherwise $Z$ would be the set of zeros of the $\varphi_0$-pullback of nowhere vanishing holomorphic differential $n$-form $\omega \in H^0(Y, \Omega^n_Y)$. If we set $U_1 = U_0 \setminus Z$, then the restriction of $\varphi_0$ on $U_1$ is a regular injective birational morphism $\varphi_1 : U_1 \to Y$. Again we have $\operatorname{codim}_X(X \setminus U_1) \geq 2$. Let $\psi := \varphi^{-1} : Y \dashrightarrow X$ be the inverse birational rational map. By the same arguments as above, there exists a a Zariski open dense subset $V_1 \subset Y$ with $\operatorname{codim}_Y(Y \setminus V_1) \geq 2$ such that $\psi$ extends to a regular injective birational morphism $\psi_1 : V_1 \to X$. Now we define $U := \varphi_1^{-1}(V_1)$ and $V := \psi_1^{-1}(U)$. By the construction, both $U \subset X$ and $V \subset Y$ are Zariski open subsets whose complements have codimensions at least 2. Moreover, the restriction $\Phi$ of $\varphi_1$ on $U$ induces a biregular isomorphism between $U$ and $V$. \hfill \Box

Proof of Theorem 1.4. Let $X$ and $Y$ be two smooth projective birationally isomorphic varieties of dimension $n$ over $\mathbb{C}$ with the trivial canonical bundles. By 3.1, there exist two Zariski open dense subsets $U \subset X$ and $V \subset Y$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ and $\operatorname{codim}_Y(Y \setminus V) \geq 2$ and a biregular isomorphism $\varphi : U \to V$.

By standard arguments, one can choose a finitely generated $\mathbb{Z}$-subalgebra $R \subset \mathbb{C}$ such that the projective varieties $X$ and $Y$ and the Zariski open subsets $U \subset X$ and $V \subset Y$ can be obtained by the base change $\ast \times_S \operatorname{Spec} \mathbb{C}$ from some regular projective $S$-schemes $\mathcal{X}$ and $\mathcal{Y}$ together with Zariski open $S$-subschemes $U \subset \mathcal{X}$ and $V \subset \mathcal{Y}$, where $S := \operatorname{Spec} R$. Moreover, one can choose $R$ in such a way that both relative canonical line bundles $\Omega^n_{\mathcal{X}/S}$ and $\Omega^n_{\mathcal{Y}/S}$ are trivial, both codimensions $\operatorname{codim}_X(\mathcal{X} \setminus U)$ and $\operatorname{codim}_Y(\mathcal{Y} \setminus V)$ are at least 2, and the biregular isomorphism $\varphi : U \to V$ is obtained by the base change from a biregular $S$-isomorphism $\Phi : U \to V$.

For almost all prime numbers $p \in \mathbb{N}$, there exist a regular $R$-integral point $\pi \in S \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_p$, where $R$ is the maximal compact subring with a maximal ideal $q$ in some local $p$-adic field $F$. By an appropriate choice of $\pi \in S \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_p$, we can get that both $\mathcal{X}$ and $\mathcal{Y}$ have good reduction modulo $q$. Moreover, we can assume that the maximal ideal $I(\pi)$ of the unique closed point in

$$S := \operatorname{Spec} R \xrightarrow{\pi} S \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_p$$

is obtained by the base change from some maximal ideal $J(\pi) \subset R$ over the prime ideal $(p) \subset \mathbb{Z}$.
Let $\omega_X$ and $\omega_Y$ be gauge-forms on $X$ and $Y$ respectively. We denote by $\omega_U$ (resp. by $\omega_V$) the restriction of $\omega_X$ to $U$ (resp. of $\omega_Y$ to $V$). Since $\Phi^*$ is a biregular $S$-morphism, $\Phi^* \omega_Y$ is another gauge-form on $U$. Hence there exists a nowhere vanishing regular function $h \in \Gamma(U, O_X^*)$ such that

$$\Phi^* \omega_V = h \omega_U.$$  

The property $\text{codim}_X(X \setminus U) \geq 2$ implies that $h$ is an element of $\Gamma(X, O_X^*) = \mathcal{R}^*$. Hence, one has $\|h(x)\| = 1$ for all $x \in X(F)$, i.e., the $p$-adic measures of Weil on $U(F)$ associated with $\Phi^* \omega_Y$ and $\omega_U$ are the same. The latter implies the following equality of the $p$-adic integrals

$$\int_{U(F)} d\mu_X = \int_{V(F)} d\mu_Y.$$  

By 2.3 and 2.2(ii), we obtain

$$\int_{U(F)} d\mu_X = \int_{X(F)} d\mu_X = \int_{X(R)} d\mu_X$$

and

$$\int_{V(F)} d\mu_Y = \int_{Y(F)} d\mu_Y = \int_{Y(R)} d\mu_Y.$$  

Now, applying the formula in 2.7, we come to the equality

$$\frac{|X(F_q)|}{q^n} = \frac{|Y(F_q)|}{q^n}.$$  

This shows that the numbers of $F_q$-rational points in $X$ and $Y$ modulo the ideal $J(\pi) \subset \mathcal{R}$ are the same. By the consideration of a cyclotomic extension $\mathcal{R}^{(r)} \subset \mathbb{C}$ containing all complex $(q^r - 1)$-th roots of unity, we can repeat the same arguments and obtain that both projective schemes $X$ and $Y$ have the same number of $F_q^{(r)}$-rational points, where $F_q^{(r)}$ is the degree-$r$ extension of the finite field $F_q$. In particular, we obtain that the zeta-functions of Weil

$$Z(X, p, t) = \exp \left( \sum_{r=1}^{\infty} |X(F_q^{(r)})| \frac{t^r}{r} \right)$$

and

$$Z(Y, p, t) = \exp \left( \sum_{r=1}^{\infty} |Y(F_q^{(r)})| \frac{t^r}{r} \right)$$

are the same. Using the Weil’s conjectures proved by Deligne [8] and the comparison theorem between the étale and singular cohomology, we obtain

$$Z(X, p, t) = \frac{P_1(t)P_2(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$
and
\[ Z(\mathcal{Y}, p, t) = \frac{Q_1(t)Q_3(t) \cdots Q_{2n-1}(t)}{Q_0(t)Q_2(t) \cdots Q_{2n}(t)}, \]
where \(P_i(t)\) and \(Q_i(t)\) are polynomials with integer coefficients having the properties
\[ \deg P_i(t) = \dim H^i(X, \mathbb{C}), \quad \deg Q_i(t) = \dim H^i(Y, \mathbb{C}) \quad \forall i \geq 0. \]

Since the standard archimedean absolute value of each root of polynomials \(P_i(t)\) and \(Q_i(t)\) must be \(q^{-i/2}\) and \(P_i(0) = Q_i(0) = 1 \quad \forall i \geq 0\), the equality \(Z(X, p, t) = Z(\mathcal{Y}, p, t)\) implies \(P_i(t) = Q_i(t) \quad \forall i \geq 0\). Therefore, we have \(\dim H^i(X, \mathbb{C}) = \dim H^i(Y, \mathbb{C}) \quad \forall i \geq 0\).

\[ \square \]

4 Remarks

As we have seen from the proof of Theorem 3.1, the zeta-functions of Weil \(Z(X, p, t)\) and \(Z(\mathcal{Y}, p, t)\) are the same for almost all primes \(p \in \text{Spec} \mathbb{Z}\). This fact being expressed in terms of the associated \(L\)-functions indicates that the established isomorphism \(H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C})\) for all \(i \geq 0\) must have some more deep motivic nature. Recently Kontsevich suggested an idea of a motivic integration \(\int\), which has been developed by Denef and Loeser \(\mathcal{L}\). In particular, this technique allows to prove that not only the Betti numbers, but also the Hodge numbers of \(X\) and \(Y\) in \(\mathbb{L}\) must be the same.

References
