The hyperbolic Pythagorean theorem

The hyperbolic Pythagorean theorem is the following statement.

**Proposition 1** Any right triangle \( \triangle ABC \) with \( \angle C \) being the right angle satisfies \( \cosh(c) = \cosh(a) \cosh(b) \).

**Proof:** See [1, page 181].

To prove the rest of the formulas of hyperbolic trigonometry, we need to show the following.

**Proposition 2** Any right triangle \( \triangle ABC \) with \( \angle C \) being the right angle satisfies \( \cos(A) = \tanh(b) / \tanh(c) \).

**Proof:** It is your homework to fill in the details in the following proof.

Use the Poincaré disc model and assume that the vertex \( A \) is at the center of the disk. (The right angle of \( ABC_\triangle \) is at \( C \).) The lines \( AB \) and \( AC \) are represented by straight lines, the line \( BC \) is represented by an arc of a circle centered at \( O_1 \). Let \( B' \) resp. \( C' \) be the second intersection of \( OB \) resp \( OC \) with this circle and \( B_1 \) be the orthogonal projection of \( O \) to the line \( OB \).

Using that the Euclidean distance \( OB \) equals \( \tanh(c/2) \) and that \( OB \cdot OB' = 1 \) (justify why), prove that the Euclidean distance \( BB' = 2/ \sinh(c) \). Observe that the Euclidean distance \( CC' \) is similarly equal to \( 2/ \sinh(b) \). Due to the Star Trek Lemma, the angle \( \angle BO_1B_1 \) is equal to \( \angle B \). (Why?) Hence

\[
\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.
\]

Finally, using that \( \cos(A) = AB_1/AO_1 \), where \( AB_1 = OB + BB'/2 \) and \( AO_1 = AC + CC'/2 \), prove that

\[
\cos(A) = \frac{\tanh(b)}{\tanh(c)}.
\]

\[\text{\large \Box}\]
Proposition 3 The previous two statements also imply the following equalities:
\[\sin(A) = \frac{\sinh(a)}{\sinh(c)}, \tag{1}\]
\[\frac{\cos(A)}{\sin(B)} = \cosh(a) \quad \text{and} \]
\[\cot(A) \cot(B) = \cosh(a) \cosh(b) \tag{3}\]

Proof: Before proving equation (1), note that this equation was actually shown during the proof of Proposition 2 (for \(B\) whose role is exchangeable with the role of \(A\)). That said, here we show that it follows algebraically from the previous two propositions. By Proposition 2 we have
\[\sin^2(A) = 1 - \cos^2(A) = \frac{\tanh^2(b) - \tanh^2(c)}{\tanh^2(c)}.\]
Using the fact that \(\tanh(x) = \sinh(x)/\cosh(x)\), the above equation may be rewritten as
\[\sin^2(A) = \frac{\sinh^2(c) \cosh^2(b) - \cosh^2(c) \sinh^2(b)}{\sinh^2(c) \cosh^2(b)}.\]
Replacing each \(\sinh^2(x)\) with \(\cosh^2(x) - 1\) in the numerator we get
\[\sin^2(A) = \frac{(\cosh^2(c) - 1) \cosh^2(b) - \cosh^2(c) \sinh^2(b)}{\sinh^2(c) \cosh^2(b)} = \frac{\cosh^2(c) - \cosh^2(b)}{\sinh^2(c) \cosh^2(b)}.\]
By Proposition 1 we may replace \(\cosh^2(c)\) with \(\cosh^2(a) \cosh^2(b)\) and get
\[\sin^2(A) = \frac{\cosh^2(a) \cosh^2(b) - \cosh^2(b)}{\sinh^2(c) \cosh^2(b)} = \frac{\cosh^2(a) - 1}{\sinh^2(c)} = \frac{\sinh^2(a)}{\sinh^2(c)}.\]
Since \(A\) is an acute angle, \(\sin(A)\) is positive and we may take the square root on both sides to obtain equation (1). Combining equation (1) with Proposition 2 yields
\[\frac{\cos(A)}{\sin(B)} = \frac{\tanh(b)}{\tanh(c) \cdot \sinh(b)} = \frac{\sinh(b)}{\cosh(b)}.\]
By Proposition 1 we may replace \(\cosh(b)\) with \(\cosh(a) \cosh(b)\) and get
\[\frac{\cos(A)}{\sin(B)} = \frac{\cosh(a) \cosh(b)}{\cosh(b)}.\]
Equation (2) follows after simplifying by \(\cosh(b)\). Finally, using equation (2) for \(\cos(A)/\sin(B)\) and for \(\cos(B)/\sin(A)\) yields
\[\cot(A) \cot(B) = \frac{\cos(A)}{\sin(B)} \cdot \frac{\cos(B)}{\sin(A)} = \cosh(a) \cosh(b).\]
\[\Box\]

References