# No nonminimally coupled massless scalar hair for spherically symmetric neutral reflecting stars

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It has recently been proved that horizonless compact stars with reflecting boundary conditions *cannot* support spatially regular matter configurations made of minimally coupled scalar fields, vector fields, and tensor fields. In the present paper we extend this intriguing no-hair property to the physically interesting regime of scalar fields with *nonminimal* coupling to gravity. In particular, we prove that static spherically symmetric configurations made of nonminimally coupled massless scalar fields cannot be supported by compact reflecting stars.



## I. INTRODUCTION

The early no-scalar-hair theorems of Chase [1], Bekenstein [2], and Teitelboim [3] have rigorously proved that asymptotically flat black holes with classical absorbing horizons cannot support nontrivial external configurations made of static massless or massive scalar fields *minimally* coupled to gravity [4–6]. Later no-scalar-hair theorems [7–10] have intriguingly ruled out the existence of spherically symmetric static hairy black-hole configurations made of minimally coupled scalar fields with positive semidefinite self-interaction potentials.

For the mathematically more challenging case of scalar fields with *nonminimal* coupling to gravity, one may use the elegant no-hair theorems of Bekenstein and Mayo [9–11] to rigorously exclude, in the physical regimes $\xi < 0$ and $\xi \geq 1/2$ of the dimensionless coupling parameter [12], the existence of spherically symmetric asymptotically flat black holes with classical absorbing horizons that support static spatially regular scalar fields with positive semidefinite self-interaction potentials. Recently, a novel no-scalar-hair theorem has been derived which rigorously rules out the existence of spherically symmetric static hairy black-hole configurations made of massless scalar fields for *generic* values of the physical coupling parameter $\xi$ [13].

The mathematically elegant and physically interesting no-scalar-hair theorems presented in [1–3, 7–10], which rigorously exclude the existence of asymptotically flat static composed black-hole-scalar-field hairy configurations with absorbing event horizons, naturally motivate the following physically intriguing question: can the characteristic no-scalar-hair behavior of static black-hole spacetimes [1–3, 7–10] be extended to the physical regime of *horizonless* curved spacetimes?

In order to address this physically interesting question, we have recently [14] explored the static sector of the non-linearly coupled Einstein-scalar field equations with reflecting (*repulsive*) boundary conditions at the surface of a spherically symmetric *horizonless* compact star. Interestingly, it has been explicitly proved in [14] that horizonless compact objects with reflecting boundary conditions (compact reflecting stars [15]) share the intriguing no-scalar-hair property with the more familiar classical black-hole spacetimes with absorbing (*attractive*) event horizons. In particular, we have revealed the fact [14] that spherically symmetric neutral reflecting stars cannot support non-linear static scalar (spin-0) fields *minimally* coupled ($\xi = 0$) to gravity. The no-hair behavior observed in [14] was later extended in the interesting work of Bhattacharjee and Sarkar [16], who have proved that compact objects (horizonless reflecting stars) cannot support higher-spin [vector (spin-1) and tensor (spin-2)] fields.

The main goal of the present paper is to extend the no-hair behavior recently observed in [14, 16] for horizonless compact objects to the physical regime of non-linear scalar fields with *nonminimal* coupling to gravity. Interestingly, exploring the non-linearly coupled Einstein-scalar field equations, we shall explicitly prove below that spherically symmetric horizonless compact reflecting stars, like the more familiar absorbing black-hole spacetimes, cannot support static nonminimally coupled massless scalar fields with *generic* values of the dimensionless coupling parameter $\xi$.

## II. DESCRIPTION OF THE SYSTEM

We consider a spherically symmetric neutral reflecting star [15] of radius $R_{\rm s}$ which interacts non-linearly with a massless scalar field $\psi$. We shall assume that the scalar field has a nonminimal coupling to the characteristic scalar curvature $R$ of the spacetime [see Eq. (3) below]. The static and spherically symmetric composed star-field configurations can be described by the curved line element [9, 17]

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{1}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. We consider asymptotically flat spacetimes which are characterized by the simple asymptotic functional relations [9]

$$\nu \sim M/r \quad \text{and} \quad \lambda \sim M/r \quad \text{for} \quad r \to \infty \ , \tag{2}$$

where $M$ is the total (asymptotically measured) mass of the static spacetime.

The non-minimally coupled massless scalar field $\psi$ is characterized by the action [9, 18]

$$S = S_{EH} - \frac{1}{2}\int \big(\partial_\alpha\psi\partial^\alpha\psi + \xi R\psi^2\big)\sqrt{-g}d^4x \ , \tag{3}$$

where the dimensionless physical parameter $\xi$ quantifies the strength of the nonminimal coupling of the field to the Ricci scalar curvature $R(r)$ of the spherically symmetric spacetime. The action (3) yields the functional expressions [9, 19]

$$T^t_t = e^{-\lambda}\frac{\xi(4/r-\lambda')\psi\psi' + (2\xi - 1/2)(\psi')^2 + 2\xi\psi\psi''}{1-8\pi\xi\psi^2} \ , \tag{4}$$

$$T^r_r = e^{-\lambda}\frac{(\psi')^2/2 + \xi(4/r+\nu')\psi\psi'}{1-8\pi\xi\psi^2} \ , \tag{5}$$

and

$$T^t_t - T^\phi_\phi = e^{-\lambda}\frac{\xi(2/r-\nu')\psi\psi'}{1-8\pi\xi\psi^2} \tag{6}$$

for the components of the energy-momentum tensor which characterizes the nonminimally coupled massless scalar field. As explicitly proved in [9], the mixed components of the energy-momentum tensor must be finite for generic physically acceptable systems:

$$\{|T^t_t|, |T^r_r|, |T^\theta_\theta|, |T^\phi_\phi|\} < \infty \ . \tag{7}$$

In particular, for asymptotically flat spacetimes with finite total mass (as measured by asymptotic observers), the energy density $\rho \equiv -T^t_t$ of the matter fields is characterized by the asymptotic functional behavior [20]:

$$r^3\rho(r) \to 0 \quad \text{for} \quad r \to \infty \ . \tag{8}$$

In addition, as explicitly proved in [9], causality requirements enforce the simple relations

$$|T^\theta_\theta| = |T^\phi_\phi| \leq |T^t_t| \geq |T^r_r| \tag{9}$$

between the components of the energy-momentum tensors of physically acceptable systems.

The action (3) also yields the compact radial differential equation [9]

$$\partial_\alpha\partial^\alpha\psi - \xi R\psi = 0 \tag{10}$$

for the nonminimally coupled massless scalar field. Substituting into (10) the spherically symmetric curved line element (1), one obtains the characteristic differential equation

$$\psi'' + \frac{1}{2}\big(\frac{4}{r} + \nu' - \lambda'\big)\psi' - \xi R e^\lambda\psi = 0 \ , \tag{11}$$

which determines the spatial behavior of the static nonminimally coupled massless scalar field. Following [14, 16], we shall assume that the non-linear massless scalar fields vanish on the surface $r = R_{\text{s}}$ of the compact reflecting star:

$$\psi(r = R_{\text{s}}) = 0 \ . \tag{12}$$

In addition, as pointed out in [9], for physically acceptable systems the effective gravitational constant, $G_{\text{eff}} = G(1-8\pi G\xi\psi^2)$ [9], must be finite and positive at asymptotic spatial infinity [9]. This physical requirement yields the asymptotic relations

$$-\infty < 8\pi\xi\psi^2 < 1 \quad \text{for} \quad r \to \infty \tag{13}$$

for the radial eigenfunction of the nonminimally coupled scalar field.

## III. THE NON-EXISTENCE THEOREM FOR THE COMPOSED REFLECTING-STAR-NONMINIMALLY-COUPLED-MASSLESS-SCALAR-FIELD CONFIGURATIONS

In the present section we shall explicitly prove that spherically symmetric neutral reflecting stars *cannot* support non-linear matter configurations made of static *nonminimally* coupled massless scalar fields.

Using the Einstein relation $R=-8\pi T$ between the scalar curvature and the trace of the energy-momentum tensor, one obtains from Eqs. (4), (5), and (6) the functional expression

$$R=-\frac{8\pi e^{-\lambda}}{1-8\pi\xi\psi^2}\Big[\xi\big(\frac{12}{r}+3\nu'-3\lambda'\big)\psi\psi'+6\xi\psi\psi''+(6\xi-1)(\psi')^2\Big] \tag{14}$$

for the scalar curvature of the spherically symmetric static spacetime. Substituting (14) into the radial equation (11), one obtains the (rather cumbersome) differential equation

$$\psi''\cdot\Big(1+\frac{48\pi\xi^2\psi^2}{1-8\pi\xi\psi^2}\Big)+\psi'\cdot\Big\{\frac{2}{r}\Big(1+\frac{48\pi\xi^2\psi^2}{1-8\pi\xi\psi^2}\Big)+\frac{1}{2}(\nu'-\lambda')+\frac{8\pi\xi\psi}{1-8\pi\xi\psi^2}\big[3\xi(\nu'-\lambda')\psi+(6\xi-1)\psi'\big]\Big\}=0\ , \tag{15}$$

which determines the spatial behavior of the nonminimally coupled massless scalar field configurations in the curved spacetime.

We shall first prove that the scalar eigenfunction $\psi$ must approach zero at spatial infinity. From Eqs. (2), (6), (8), and (9), one deduces that $\psi\psi'$ must approach zero asymptotically faster than $1/r^2$. That is,

$$r^2\psi\psi'\to 0\quad \text{for}\quad r\to\infty\ . \tag{16}$$

Taking cognizance of the relations (2) and (16), one finds that, in the asymptotic far region $M/r\ll 1$, the scalar radial equation (15) can be approximated by the differential equation [21, 22]

$$\psi''+\frac{2}{r}\psi'=0\quad \text{for}\quad r\gg M\ , \tag{17}$$

whose general mathematical solution is given by

$$\psi(r)=\frac{A}{r}+B\quad \text{for}\quad r\gg M\ , \tag{18}$$

where $\{A,B\}$ are constants.

One immediately realizes that the scalar function (18) with $B\neq 0$ violates the characteristic relation (16) for asymptotically flat spacetimes [20]. We therefore conclude that physically acceptable (finite mass) configurations of the static scalar field, *if* they exist, must be characterized by an asymptotically vanishing radial eigenfunction [see Eq. (18) with $B=0$]

$$\psi(r)=\frac{A}{r}\quad \text{for}\quad r\gg M\ . \tag{19}$$

Taking cognizance of the inner boundary condition (12) of the scalar eigenfunction at the surface of the compact reflecting star, together with the characteristic asymptotic behavior (19) at spatial infinity, one deduces that the characteristic scalar eigenfunction $\psi$ of the nonminimally coupled massless scalar field must have (at least) one extremum point, $r=r_{\text{peak}}$, within the interval $r_{\text{peak}}\in(R_{\text{s}},\infty)$. In particular, the radial scalar eigenfunction is characterized by the simple functional relations

$$\{\psi\neq 0\quad ;\quad \psi'=0\quad ;\quad \psi\cdot\psi''<0\}\quad \text{for}\quad r=r_{\text{peak}} \tag{20}$$

at this extremum point. In addition,

$$\psi\cdot\psi'\geq 0\quad \text{for}\quad r\in[R_{\text{s}},r_{\text{peak}}]\ . \tag{21}$$

### A. The no-scalar-hair theorem for generic inner boundary conditions in the physical regimes $\xi<0$ and $\xi>\frac{1}{6}$

Interestingly, as we shall now show explicitly, one can rule out the existence of asymptotically flat non-linear static configurations ('hair') made of nonminimally coupled massless scalar fields in the physical regimes $\xi<0$ and $\xi>1/6$ *without* using the inner boundary condition (12) at the surface of the central compact object.

Substituting (19) into (4) and (5) and using the asymptotic far region relation (2), one finds the compact expressions

$$T^t_t = (2\xi - 1/2)\frac{\psi^2}{r^2}\cdot[1+O(M/r)] \tag{22}$$

and

$$T^r_r = (1/2 - 4\xi)\frac{\psi^2}{r^2}\cdot[1+O(M/r)] \tag{23}$$

for the components of the energy-momentum tensor which characterizes the nonminimally coupled massless scalar fields. Interestingly, the functional expressions (22) and (23) for the energy-momentum components yield the far-region relation

$$|T^r_r| > |T^t_t| \quad \text{for} \quad \xi < 0 \ \ \text{or} \ \ \xi > 1/6\ . \tag{24}$$

From the inequality (24) one immediately reveals the fact that spatially regular non-linear configurations of the nonminimally coupled massless scalar fields *violate* the characteristic relation (9) imposed by causality on the energy-momentum components of physically acceptable systems. Our analysis therefore rules out the existence of spherically symmetric static configurations made of massless scalar fields nonminimally coupled to gravity in the physical regimes $\xi < 0$ and $\xi > 1/6$.

Before we proceed, it is worth emphasizing again that the no-scalar-hair theorem presented in this subsection is based on the characteristic asymptotic (*far*-region $r \gg M$) behavior of the nonminimally coupled massless scalar fields. Our theorem, which excludes nonminimally coupled massless scalar hair in the physical regimes $\xi < 0$ and $\xi > 1/6$, is therefore valid for both black-hole spacetimes with inner *attractive* boundary conditions (at the absorbing horizon of a classical black-hole spacetime) and for horizonless curved spacetimes with inner *repulsive* boundary conditions (at the surface of a compact reflecting star).

### B. The no-scalar-hair theorem for spherically symmetric neutral reflecting stars with generic values of the nonminimal coupling parameter $\xi$

We start our second no-scalar-hair theorem, which would be valid for spherically symmetric reflecting stars with *generic* values of the physical coupling parameter $\xi$, by showing that the radial function

$$\Omega(r;\xi) \equiv 1 - 8\pi\xi\psi^2 \tag{25}$$

is positive definite in the interval $[R_{\text{s}}, r_{\text{peak}}]$. Obviously, $\Omega > 0$ for $\xi < 0$. As we shall now prove explicitly, $\Omega > 0$ also in the $\xi > 0$ case. We first note that, taking cognizance of the inner boundary condition (12) at the surface of the compact reflecting star, one finds the simple relation

$$\Omega(r = R_{\text{s}}) = 1\ . \tag{26}$$

Now, suppose that $\Omega$ vanishes at some point $r = r_0$ in the interval $[R_{\text{s}}, r_{\text{peak}}]$. Then, in the vicinity of $r_0$, one can expand $\Omega(r)$ in the form

$$\Omega(r) = \alpha(r-r_0)^\beta + O[(r-r_0)^\gamma] \quad ; \quad \gamma > \beta > 0\ , \tag{27}$$

which implies [see Eq. (25)]

$$\psi^2(r) = (8\pi\xi)^{-1}\cdot\{1 - \alpha(r-r_0)^\beta + O[(r-r_0)^\gamma]\}\ . \tag{28}$$

From (28) one finds the leading order behavior

$$\psi\psi'(r) = -\frac{\alpha\beta}{16\pi\xi}\cdot(r-r_0)^{\beta-1} \tag{29}$$

in the vicinity of $r_0$. Note that $\psi\psi' \geq 0$ in the interval $[R_{\text{s}}, r_{\text{peak}}]$ [see Eq. (21)], which implies that $\beta$ is odd [23] and

$$\alpha\cdot\beta < 0\ . \tag{30}$$

It now proves useful to explore the spatial behavior of the linear combination [see Eqs. (4), (5), and (6)]

$$\mathcal{T}\equiv T^t_t+T^r_r-T^\phi_\phi={{e^{-\lambda}}\over{\Omega}}\big[{{6\xi}\over{r}}\psi\psi'+{1\over2}(\psi')^2\big]\ . \tag{31}$$

From Eqs. (27), (28), and (29), one finds the leading order behavior

$$\mathcal{T}={{e^{-\lambda(r_0)}}\over{\alpha(r-r_0)}}\Big[-{{3\alpha\beta}\over{8\pi r_0}}+{{\alpha^2\beta^2}\over{64\pi\xi}}(r-r_0)^{\beta-1}\Big] \tag{32}$$

in the vicinity of $r_0$. Taking cognizance of Eq. (30), one immediately realizes that, for $\xi>0$, the expression inside the square brackets in (32) is positive definite, which implies the singular behavior

$$\mathcal{T}\to\infty\ \ \ \ \text{for}\ \ \ \ r\to r_0 \tag{33}$$

in the vicinity of $r_0$. However, the linear combination $\mathcal{T}\equiv T^t_t+T^r_r-T^\phi_\phi$ of the energy-momentum components must be finite for physically acceptable systems [see Eq. (7)] [9]. One therefore deduces that $\Omega(r)$ *cannot* switch signs in the interval $[R_{\text{s}},r_{\text{peak}}]$. In particular, taking cognizance of Eq. (26), one finds [24]

$$\Omega(r)\equiv1-8\pi\xi\psi^2>0\ \ \ \ \text{for}\ \ \ \ r\in[R_{\text{s}},r_{\text{peak}}]\ . \tag{34}$$

Finally, taking cognizance of the characteristic functional relations (20) and (34), one deduces that, at the extremum point $r=r_{\text{peak}}$ of the scalar field eigenfunction, the first term on the l.h.s of (15) is *non*-zero whereas the second term on the l.h.s of (15) is *zero*. Thus, the equality sign in the radial scalar equation (15) *cannot* be respected at the extremum point $r=r_{\text{peak}}$ of the scalar eigenfunction. We therefore conclude that, for *generic* values of the dimensionless physical parameter $\xi$, the spherically symmetric neutral reflecting stars *cannot* support massless scalar fields nonminimally coupled to gravity.

## IV. SUMMARY

It is by now well established that spherically symmetric asymptotically flat black-hole spacetimes with absorbing event horizons cannot support spatially regular static configurations made of massless scalar fields nonminimally coupled to gravity [9–11, 13].

It has recently been proved that *horizonless* spacetimes describing compact reflecting stars may share the interesting no-hair property with the more familiar classical black-hole spacetimes [14, 16]. In particular, it has been proved that compact objects with reflecting (rather than absorbing) boundary conditions cannot support spatially regular matter configurations made of minimally coupled scalar fields, vector fields, and tensor fields [14, 16].

In the present paper we have explored the possibility of extending the regime of validity of the intriguing no-hair property recently observed [14, 16] for horizonless curved spacetimes. To this end, we have studied analytically the static sector of the non-linearly coupled Einstein-scalar field equations for spatially regular *nonminimally* coupled massless scalar fields.

Using the causality relations (9), which characterize the energy-momentum components of physically acceptable systems [9], we have presented a compact no-scalar-hair theorem which rules out the existence of non-linear static hairy configurations made of massless scalar fields nonminimally coupled to gravity in the physical regimes $\xi<0$ and $\xi>1/6$. It is worth emphasizing again that this theorem (see Sec. IIIa) is valid for *generic* inner boundary conditions. In particular, the theorem excludes the existence of both hairy scalar configurations with attractive inner boundary conditions at the absorbing horizon of a black-hole spacetime and hairy scalar configurations with repulsive inner boundary conditions at the compact surface of a reflecting star.

Finally, we have presented a second no-scalar-hair theorem (see Sec. IIIb) which explicitly proves that spherically symmetric asymptotically flat horizonless neutral stars with compact reflecting surfaces cannot support static massless scalar fields nonminimally coupled to gravity with *generic* values of the dimensionless physical parameter $\xi$.

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[22] Note that the asymptotic property (13) of the physically acceptable [9] nonminimally coupled scalar eigenfunctions implies the asymptotic far-region ($r \gg M$) relation $1 + 48\pi\xi^2\psi^2/(1 - 8\pi\xi\psi^2) \geq 1$.
[23] Note that, for even values of $\beta$, the functional expression (29) for $\psi\psi'$ would switch signs at $r = r_0$ in contradiction with the characteristic relation (21).
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