5.5 Double-Angle and Half-Angle Formulas

In these section we want to find formulas for $\cos 2\theta, \sin 2\theta$, and $\tan 2\theta$ in terms of $\cos\theta, \sin\theta$, and $\tan\theta$ respectively. These are called double angle formulas. Then we will use them to find half-angle formulas for $\cos\frac{\theta}{2}, \sin\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$.

The Cosine of 2θ

We may form an isosceles triangle with an angle of 2θ by flipping a triangle across the horizontal axis on the unit circle. Then the law of cosines would yield the double angle formula for cosine:

$$\cos 2\theta = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

Example 1. *Evaluate* $\cos 2\theta$*, given some trigonometric function of an angle* θ*.*

The Sine of 2θ

We apply the law of sines to the isosceles triangle at the beginning of this section to obtain the law of sines:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

The Tangent of 2θ

Using the previous double angle formulas, we find:

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Example 2. *Given* $\tan\theta$*, find* $\tan 2\theta$*.*

The Cosine and Sine of $\frac{\theta}{2}$

Using the double angle formula for cosine, we obtain

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

and

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}},$$

where the choice of plus or minus depends on the angle θ.

Example 3. *Find an exact expression for sine or cosine of half of a famous angle, in a quadrant with negative value(s).*

The Tangent of $\frac{\theta}{2}$

Using algebra, we may obtain

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Example 4. *Evaluate the tangent of half of a famous angle.*