MT3503 Complex Analysis

MRQ

November 28, 2018

Contents

Introduction

Complex analysis is viewed by many as one of the most spectacular branches of mathematics that we teach to undergraduates. It sits as a piece of interesting mathematics that is used in many other areas, both in pure mathematics and applied mathematics. The starting premise will be readily appreciated by all students who have completed either of the prerequisites for this module. They will have met the definition of the derivative of a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(and this definition appears in both *MT2502* and *MT2503*). One begins complex analysis by basically using the same definition to differentiate a function $f\colon \mathbb{C} \to \mathbb{C}$ of a complex variable. What is surprising is that as one examines such functions is that the behaviour of differentiable functions of a complex variable (that are termed *holomorphic functions*) is somewhat different to that of differentiable functions of a real variable.

Examples of surprising properties of differentiable functions of a complex variable are:

1. If a function f of a complex variable is differentiable on an open set U, then it can be differentiated as many times are you would like (that is, f', f'', f''', $\dots$ all exist).

2. If $f\colon \mathbb{C} \to \mathbb{C}$ is differentiable and *bounded* (that is, $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$) then it is constant.

3. If $f\colon B(c,r) \to \mathbb{C}$ is differentiable at every point of distance at most r from c (that is, z satisfying $|z-a| < r$), then f is given by a *Taylor series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n$$

for all z with $|z-a| < r$.

These facts will all be established *and made precise* during the lecture course. (In particular, we shall specify what the term “open set” means in Chapter 1 and why it is significant for what we do here.) The above facts contrast quite considerably with real-valued functions as the following three examples show. (The details are omitted in these examples.)

Example 0.1 The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geqslant 0 \\ -\frac{1}{2}x^2 & \text{if } x < 0 \end{cases}$$

is differentiable, but f' is not differentiable at $x = 0$ in contrast to Property 1 above. Indeed, one can show that

$$f'(x) = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$$

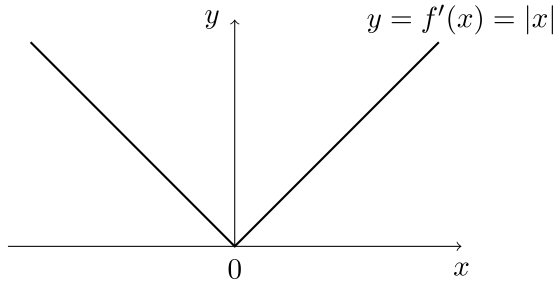


Figure 1: The graph of the derivative of $f(x)$ in Example 0.1.

To verify this, one needs to treat $x = 0$ separately, as those who have covered *MT2502 Analysis* will probably remember. Then $f'(x) = |x|$ is not differentiable at 0 as was covered in the *MT2502* lecture notes and as can be anticipated by looking at the graph of f' (see Figure 1).

Example 0.2 The function $f(x) = \sin x$ is differentiable on $\mathbb{R}$ and satisfies $|\sin x| \leqslant 1$ for all $x \in \mathbb{R}$. This non-constant function stands in contrast to Property 2 above.

Example 0.3 It requires somewhat more work to construct an example illustrating that Property 3 fails with real-valued functions. Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

With considerable care, one can show that f may be differentiated as many times as one wants at all points in $\mathbb{R}$. Indeed, one can use an induction argument to show that there exist polynomials p_n of degree $3n$ such that the nth derivative of f satisfies

$$f^{(n)}(x) = \begin{cases} p_n(1/x)\,\mathrm{e}^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(To verify all this, one needs to understand the behaviour of $p_n(1/x)\,\mathrm{e}^{-1/x^2}$ as $x \to 0$, but I will omit this since it is a considerable detour away from the core of the module.) In particular, the coefficients of the Taylor series of f about 0 are all equal to 0, but

$$f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0$$

for all $x \neq 0$.

Returning to the topic of complex analysis, once we have established many properties of differentiable functions of a complex variable, there are a large suite of applications. The primary applications that we shall cover in the module are:

- evaluation of certain real integrals, e.g., $\int_0^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x$;
- evaluation of certain real series, e.g., $\sum_{n=1}^\infty 1/n^4$.

There are many other examples of applications of complex analysis, for example, in number theory (e.g., the Prime Number Theorem states that the number of primes at most n is asymptotically $n/\log n$ and was proved by employing complex analysis) and in fluid dynamics. These applications are beyond the course, but methods covered within it could be used, for example, to show that the Riemann zeta function is differentiable on a suitable subset of $\mathbb{C}$.

Structure of the lecture course

The following topics will be covered in the lectures:

- **Review of complex numbers:** We begin by reviewing the basic properties of complex numbers extracted from the content of *MT1002 Mathematics*.
- **Holomorphic functions:** We present the basic definitions of limits, continuity and differentiability in the complex setting. In particular, we establish the Cauchy–Riemann Equations. We also discuss (though omit most proofs) how power series define differentiable functions within their radius of convergence.
- **Contour integrals:** We define how to integrate a function of a complex variable along a path in the complex plane. The most significant theorem in complex analysis will be discussed: Cauchy's Theorem says that under sufficient conditions the integral around a closed path of a holomorphic function equals 0.
- **Theoretical consequences of Cauchy's Theorem:** A large number of theorems, including the Properties 1–3 listed above, follow from Cauchy's Theorem.
- **Singularities and poles:** Roughly halfway through the course, we shall discuss the situation of a function that is differentiable in many places but has some points where it cannot be differentiated. These are called *singularities* and we shall discuss them in detail.
- **Laurent's Theorem and Cauchy's Residue Theorem:** Information about the behaviour of functions with isolated singularities and what happens when we integrate such functions around closed paths. The latter theorem is the principal tool for our applications.
- **Applications of contour integration:** We shall give lots of examples showing how the tools developed to calculate real integrals and sum real series.
- **Complex logarithm and multifunctions:** Discussion of the behaviour of certain functions that can take many values at a single point; these arise essentially out of the fact that the argument of a complex number is not uniquely specified.
- **Counting zeros:** We demonstrate how the behaviour of a function around a contour can be used to determine the number of zeros inside it.

Prerequisites

The prerequisite modules to take this lecture course are *MT2502 Analysis* or *MT2503 Multivariate Calculus*. If one thinks about this for a moment, one realises that the only prior mathematics that could be assumed would be something that appears in both or material from courses upon which these both depend (i.e., *MT1002* and school mathematics). In reality, to do complex analysis one does want to pull in material from both modules, but what we shall actually do is state these facts whenever needed, explain how they should be interpreted (particularly in the context we require) but not bother with proofs (specifically, for example, in the case of material about limits, differentiation or convergence from *MT2502*).

Examples of some of the topics that we shall use are:

- basic properties of complex numbers (from *MT1002* or school maths);
- the definition of the derivatives (from both *MT2502* and *MT2503*);
- basic properties of differentiation (from *MT1002*, *MT2502* or school maths);

- partial differentiation (from *MT2503*);
- power series (introduced in both *MT2502* and *MT2503*, though detailed proofs are delayed to *MT3502*).

Recommended texts

The following two textbooks each cover the material in the course and in much the same spirit as these lecture notes. They are precise about the mathematics covered, but not overly technical. There are many other textbooks on complex analysis available and indeed most introductory texts on the subject would be suitable for this module.

- John M. Howie, *Complex Analysis*, Springer Undergraduate Mathematics Series, Springer, 2003.
- H. A. Priestley, *Introduction to Complex Analysis, Second Edition*, OUP, 2003.

Chapter 1

Complex numbers and the topology of the complex plane

We start our journey by reviewing the basic properties of complex numbers. This review material is all found in *MT1002 Mathematics*, though many students will have covered this during their school education (in particular, those who took second-year entry into their programme). The end part of the chapter will discuss the geometry of the complex plane and introduce a topological property that will be required to precisely phrase some of the concepts and results of this module.

Complex numbers

A *complex number* is a number of the form

$$a + bi$$

where a and b are real numbers and the number i satisfies

$$i^2 = -1.$$

The following are consequently examples of complex numbers: any real number (take $b = 0$ in the definition), $3 + 4i$, $\sqrt{2} - (1/\pi)i$, etc. The set of all complex numbers is denoted by $\mathbb{C}$. The *real part* of $z = a + bi$ is the real number a and the *imaginary part* is the real number b. We write $\operatorname{Re} z$ and $\operatorname{Im} z$ for the real part and imaginary part of the complex number z.

Arithmetic in $\mathbb{C}$ is defined as follows. Addition and subtraction is straightforward:

$$(a + bi) + (c + di) = (a + c) + (b + d)i;$$

that is, we simply add (or subtract, when subtracting complex numbers) the real and imaginary parts. Multiplication involves exploiting the fact that $i^2 = -1$:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

To perform division requires the use of the *complex conjugate*. If $z = a + bi$ (with $a, b \in \mathbb{R}$), we write $\bar{z} = a - bi$. Note then that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2.$$

We then calculate

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \left(\frac{ac + bd}{c^2 + d^2} \right) + \left(\frac{bc - ad}{c^2 + d^2} \right) i.$$

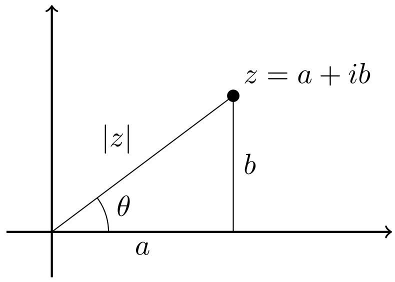


Figure 1.1: A complex number z plotted in the complex plane.

The general formula for the quotient of one complex number by another does not look all that pleasant, but in specific cases it is straightforward to calculate.

The *modulus* of a complex number $z = a + bi$ (with real part a and imaginary part b) is given by

$$|z| = \sqrt{a^2 + b^2}.$$

Thus

$$z\bar{z} = |z|^2 .$$

The formula for division can then be expressed as

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

for $z, w \in \mathbb{C}$.

It will prove to be helpful, indeed essential, to represent complex numbers in a diagram, often called the *Argand diagram* but we shall typically refer to as the *complex plane*. In Figure 1.1, we have a right-angled triangle whose vertices are the complex numbers $z = a + ib$, a and 0. As a consequence, if we write θ for the angle indicated (expressed in radians), then by basic trigonometry

$$a = |z| \cos\theta \qquad \text{and} \qquad b = |z| \sin\theta \tag{1.1}$$

and so

$$z = |z| (\cos\theta + i \sin\theta),$$

which expresses the complex number z in terms of its modulus and this angle θ.

Although we chose the particular θ appearing as the triangle, the choice is only determined up to a particular integer multiple of 2π. We could add or subtract 2π to the value θ and the equations appearing in (1.1) would still be satisfied. We shall refer to any θ that satisfies these equations as the *argument* of z and denote it by $\arg z$. Thus, it is determined from the real and imaginary parts of z by the equation

$$\arg z = \theta = \tan^{-1}(b/a),$$

though as we have noted this is not uniquely determined but could, given one particular value θ, be any member of the set

$$\{\dots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \dots\}.$$

In many situations, we shall restrict the argument to come from a particular range of values to ensure that we have a unique choice of argument within that range. Typical choices include

$$-\pi < \arg z \leqslant \pi \qquad \text{or} \qquad 0 \leqslant \arg z < 2\pi.$$

One often refers to the *principal value* of the argument which is when we take $\arg z$ from the interval $(-\pi, \pi]$ and one sometimes writes $\operatorname{Arg} z$ for the principal value of the argument.

When we write a complex number as $z = |z| (\cos\theta + i \sin\theta)$ in terms of its modulus and its argument, we call this the *modulus-argument form* for z. The formula $\cos\theta + i\sin\theta$ is often abbreviated in various textbooks to $\mathrm{e}^{i\theta}$. Because the notation is familiar to students, we shall it in these notes, but in the next chapter we shall make a definition of the complex exponential function and be able to interpret the notation $\mathrm{e}^{i\theta}$ in terms of that definition.

The following summarises the basic properties of modulus and argument that we shall need. They were either established in the *MT1002* lecture notes or can be deduced quickly from the others. Note that parts of (ii), (iii) and (iv) refer to argument are only true in the sense that we can *choose* a value of the argument satisfying the equation; that is, the result depends upon an appropriate choice of argument from the infinitely many values permitted.

Theorem 1.1 *Let z and w be complex numbers.*

(i) $|\operatorname{Re} z| \leqslant |z|$ *and* $|\operatorname{Im} z| \leqslant |z|$.

(ii) $|zw| = |z|\,|w|$ *and* $\arg(zw) = \arg z + \arg w$ *(up to some integer multiple of 2π).*

(iii) $|\bar{z}| = |z|$ *and* $\arg\bar{z} = -\arg z$ *(up to some integer multiple of 2π).*

(iv) $|1/z| = 1/|z|$ *and* $\arg(1/z) = -\arg z$ *(up to some integer multiple of 2π).*

(v) $\overline{zw} = \bar{z}\bar{w}$.

(vi) ***De Moivre's Theorem:*** *If $|z| = r$ and $\arg z = \theta$, then*

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

for all integers n.

The proof will be omitted in the lectures, but the details are included in these notes for completeness.

Proof: In parts (i)–(iv), write $r = |z|$, $s = |w|$, $\theta = \arg z$ and $\phi = \arg w$. Then $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$.

(i) Under the above assumption, $\operatorname{Re} z = r\cos\theta$ and $\operatorname{Im} z = r\sin\theta$. Therefore

$$|\operatorname{Re} z| = r\,|\cos\theta| \leqslant r = |z|$$

and

$$|\operatorname{Im} z| = r\,|\sin\theta| \leqslant r = |z|,$$

using the facts that $|\cos\theta| \leqslant 1$ and $|\sin\theta| \leqslant 1$.

(ii) Now observe

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big) \\ &= rs\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big), \end{aligned}$$

using the trigonometric addition formulae. Hence

$$|zw| = rs = |z|\,|w|$$

and

$$\arg(zw) = \theta + \phi = \arg z + \arg w.$$

(iii) The conjugate is given by

$$\bar{z} = r(\cos\theta - i\sin\theta) = r\big(\cos(-\theta) + i\sin(-\theta)\big),$$

since sine is an odd function and cosine is an even function. Hence

$$|\bar{z}| = r = |z| \qquad \text{and} \qquad \arg\bar{z} = -\theta = -\arg z.$$

(iv) This is most easily deduced from part (ii). Recall that the formula for division says $1/z = \bar{z}/|z|^2$. Hence

$$|1/z| = |\bar{z}|/|z|^2 = 1/|z|$$

and

$$\arg(1/z) = \arg(\bar{z}/|z|^2) = \arg\bar{z} = -\arg z$$

using parts (i) and (ii).

(v) For this part, write $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (ad + bc)i$ and

$$\bar{z}\bar{w} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i = \overline{zw}.$$

(vi) We first deal with the case when n is a positive integer. We then proceed by induction on n. The base case $n = 1$ states that $z = r(\cos\theta + i\sin\theta)$ which is simply the definition of r as the modulus of z and θ as the argument of z. Assume as inductive hypothesis that $z^n = r^n(\cos n\theta + i\sin n\theta)$; that is, $|z^n| = r^n$ and $\arg(z^n) = n\theta$. Then, by (i),

$$\left|z^{n+1}\right| = |z^n|\,|z| = r^n\,r = r^{n+1}$$

and

$$\arg(z^{n+1}) = \arg(z^n) + \arg z = n\theta + \theta = (n+1)\theta.$$

Thus $z^{n+1} = r^{n+1}\big(\cos(n+1)\theta + i\sin(n+1)\theta\big)$, as required. This completes the induction and establishes the equation when n is positive.

When $n = 0$, $z^0 = 1$, $r^0 = 1$ and $\cos n\theta = \cos 0 = 1$ and $\sin n\theta = \sin 0 = 0$. Thus the formula holds.

Finally, when n is negative, say $n = -m$ for some $m > 0$, then

$$|z^n| = \left|z^{-m}\right| = |1/z^m| = 1/|z^m| = 1/r^m = r^{-m} = r^n$$

and

$$\arg(z^n) = \arg(1/z^m) = -\arg(z^m) = -m\theta = n\theta$$

using the formula already established for z^m. This then establishes the required formula for n negative. □

We use the modulus of a complex number to define distance in the complex plane. Note that $|z|$ is the distance of the complex number z from the origin. Thus, if z and w are complex numbers, we think of $|z - w|$ as the distance from w to z. (See Figure 1.2.)

What justifies the use of $|z - w|$ as the distance between two complex numbers is that the modulus satisfies the triangle inequality. Indeed, having something like the triangle inequality is essential for performing analysis and so establishing this inequality is the first step in being able to study complex analysis.

Theorem 1.2 (Triangle Inequality) *Let $z, w \in \mathbb{C}$. Then*

(i) $|z + w| \leqslant |z| + |w|$;

(ii) $|z - w| \geqslant \big||z| - |w|\big|$.

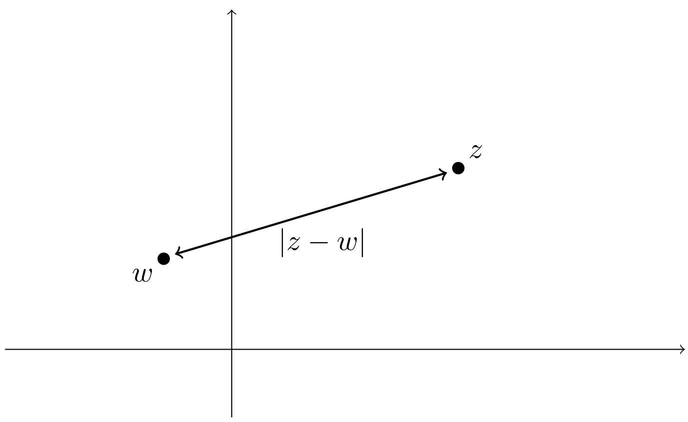


Figure 1.2: Use of modulus for the distance between two complex numbers.

PROOF: (i)

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\ &= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\ &\leqslant |z|^2 + 2\,|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2\,|z|\,|w| + |w|^2 \\ &= \big(|z|+|w|\big)^2. \end{aligned}$$

Therefore, upon taking square roots of both sides,

$$|z+w| \leqslant |z| + |w|\,.$$

(ii) Note that $|z| - |w|$ is a *real* number, so the right-hand side denotes the magnitude of this real number. To verify the inequality, we use part (i) in the following:

$$|z| = |z - w + w| \leqslant |z-w| + |w|\,,$$

so

$$|z-w| \geqslant |z| - |w|\,. \tag{1.2}$$

Then interchanging the roles of z and w in this inequality we obtain

$$|z-w| = |w-z| \geqslant |w| - |z|\,. \tag{1.3}$$

Putting Equations (1.2) and (1.3) together gives

$$|z-w| \geqslant \big||z| - |w|\big|,$$

since the absolute value of $|z| - |w|$ is the right-hand side of either (1.2) or (1.3). □

Remark/Warning: Note that when we have used an inequality above, it *always* involved *real* numbers. The modulus $|z|$ of a complex number z is real and so it makes sense to write assertions involving inequalities and moduli of complex numbers. However, there is no inequality defined on the complex numbers that interacts with its addition and multiplication in a helpful way. As a consequence, in this course one should never need to write "$z_1 \leqslant z_2$" for two (non-real) complex numbers z_1 and z_2 and, if one were to do so, it is unlikely to have meaning.

In short, *don't write "$z_1 \leqslant z_2$" for $z_1, z_2 \in \mathbb{C}$!*

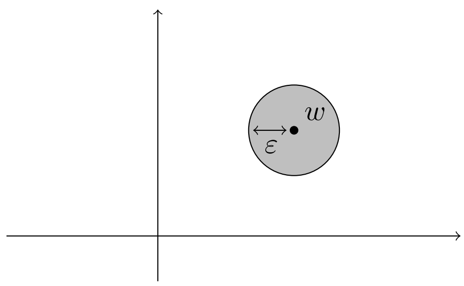


Figure 1.3: The open disc of radius ε about the complex number w.

Open sets

It will be unavoidable to make reference to a few concepts from topology during the module. However, since this is not a course about topology, these lecture notes will avoid dwelling on them as much as possible. Furthermore, the lecturer has taken the decision to also avoid "front-loading" the preliminaries with such concepts. Instead, they will be introduced in as brief a manner as possible when needed. There is therefore only one concept from topology that we shall introduce at this point, namely what it means for a subset of $\mathbb{C}$ to be 'open.' This concept will appear throughout right from the start of our work, which is why we introduce it now. It will enable us to talk about limits of functions and what it means for a function to be differentiable.

Definition 1.3 If $w \in \mathbb{C}$ and $r > 0$ is a real number, the *open disc* (or *open ball*) of radius r about w is

$$B(w, r) = \{ z \in \mathbb{C} \mid |z - w| < r \}.$$

(See Figure 1.3 for an illustration of the open disc about a complex number w.)

Open discs are used in the following definition.

Definition 1.4 A subset U of $\mathbb{C}$ is called *open* if for every $w \in U$ there exists some $\varepsilon > 0$ such that

$$B(w, \varepsilon) \subseteq U.$$

This looks perhaps like a technical definition and a bit mysterious at first glance. It is worth our spending a little time to unpack the main idea and to explain how it will fit in our context (so as hopefully to make it less mysterious!).

The basic idea is as follows: Suppose that U is an open subset of $\mathbb{C}$ and consider some complex number w selected from the set U. The definition (Definition 1.4) and that the open disc $B(w, \varepsilon)$ is contained inside U (as illustrated in Figure 1.4). That is, all the complex numbers in the disc $B(w, \varepsilon)$ are all in U and so the complex number w is surrounded by points that lie in U. Thus we can approach w from *any* directon without leaving the set U.

The context where we shall use this concept is that we shall consider complex-valued functions $f \colon U \to \mathbb{C}$ defined upon an open subset U of $\mathbb{C}$. If $w \in U$, we can approach w from all directions while staying in the domain of f and consequently we can analyze the behaviour of the values $f(z)$ as z approaches w. In particular, this will allow us to discuss the existence and value of the limit of $f(z)$ as z approaches w.

For this reason, many of the theorems stated in the lecture course will be phrased in terms of open sets U. However, since this is not a course on topology, it will generally be easy to verify that any particular set we are interested in as being open. We shall not be considering esoteric examples at all. As an example:

Lemma 1.5 *If $w \in \mathbb{C}$ and $r > 0$ is any real number, the open disc $B(w, r)$ is open.*

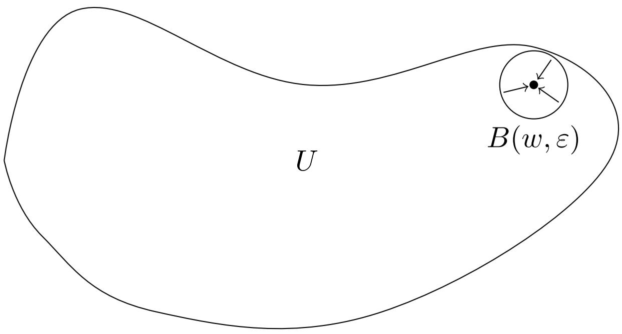


Figure 1.4: An open subset U in $\mathbb{C}$.

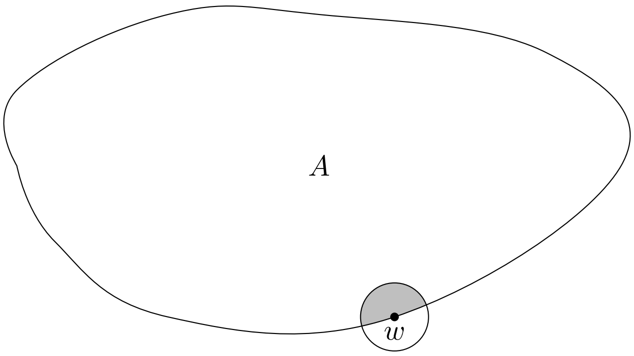
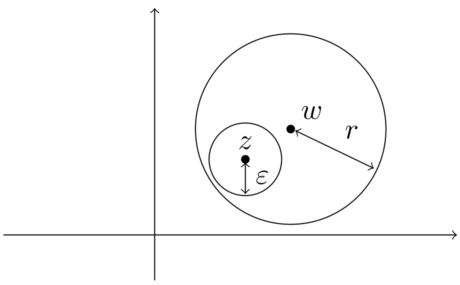


Figure 1.5: Sets with points on the boundary are not open.

This is reasonably important as we should not call an object an "open" disc if it were not itself open in the sense of the terminology we have introduced. The proof of this lemma will be omitted during the lectures (as it belongs most naturally in a module on topology) and will instead be illustrated by a picture.

Proof: This can be established quite easily once we draw a reasonable diagram of what is going on. The diagram below illustrates what we are trying to achieve, but it would perhaps become more transparent to understand our choice of ε if the arrows were drawn radially outwards from w so that they pass through u.

Let $U = B(w,r)$. Let $z \in U$. We must find some radius ε such that the ball about z of this radius is kept within U.



Since $z \in U$, the modulus $|z-w| < r$. Take $\varepsilon = r - |z-w|$. Note then that $\varepsilon > 0$. We shall show that

$$B(z,\varepsilon) \subseteq U.$$

If $v \in B(z,\varepsilon)$, then $|v-z| < \varepsilon$. So

$$|v-w| = |(v-z) + (z-w)|$$

$$
\begin{aligned}
&\leqslant |v-z| + |z-w| \\
&< \varepsilon + |z-w| = r.
\end{aligned}
$$

So $v \in B(w,r) = U$. This shows that

$$B(z,\varepsilon) \subseteq U,$$

as required. We conclude that $U = B(w,r)$ is indeed open. $\square$

Almost any precise verification of openness of a set will follow roughly the same type of argument. We shall often find open sets are those that can be defined by strict inequalities involving the modulus of complex numbers (such as $|z-w| < r$ as in $B(w,r)$).

Chapter 2

Holomorphic Functions

The primary goal of this course is to understand the behaviour of functions of a complex-variable that are differentiable. As a consequence, the main topic of this section will be differentiation. We shall begin with a brief general discussion.

Given a function $f\colon D \to \mathbb{C}$ of a complex variable, defined on some subset $D \subseteq \mathbb{C}$, one can associate two *real-valued* functions, namely

$$\operatorname{Re} f\colon D \to \mathbb{R} \qquad \text{and} \qquad \operatorname{Im} f\colon D \to \mathbb{R}.$$

Here, of course, $\operatorname{Re} f(z)$ is the real part and $\operatorname{Im} f(z)$ is the imaginary part of the value $f(z)$.

If we write a complex number $z \in D$ in terms of its real and imaginary parts as $z = x+iy$, then we can view $f\colon D \to \mathbb{C}$ as a function of two real variables. This is quite common and the usual notation is to write $u(x,y)$ and $v(x,y)$ for the real and imaginary parts of the function $f(x+iy)$. Thus

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

for $x+iy \in D$ and, in writing this, we have associated a pair of real-valued functions of two real variables

$$u\colon \tilde{D} \to \mathbb{R} \qquad \text{and} \qquad v\colon \tilde{D} \to \mathbb{R},$$

for a suitable subset $\tilde{D}$ of $\mathbb{R}^2$, to the complex function $f\colon D \to \mathbb{C}$. To be precise, the set $\tilde{D}$ is given by $\tilde{D} = \{\, (x,y) \in \mathbb{R}^2 \mid x+iy \in D \,\}$, the subset of $\mathbb{R}^2$ that corresponds to the subset D of $\mathbb{C}$.

Example 2.1 *Calculate the real and imaginary parts of the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ expressed as functions of two real variables.*

Solution: Write $z = x+iy$. Then

$$f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy,$$

so the real and imaginary parts are

$$u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v(x,y) = 2xy,$$

respectively. □

Differentiability

The definition of what it means for a complex-valued function to be differentiable is essentially the same definition as for a real-valued function (as was given in *MT2502* and *MT2503*).

Definition 2.2 Let U be an open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ be a function of a complex variable defined on U. We say that f is *differentiable* at a point $c \in U$ if the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. When this limit does exist, we call

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

the *derivative* of f at c. We also write $\frac{\mathrm{d}f}{\mathrm{d}z}$ for the function $f'(z)$ (whenever this function is defined).

Note: The definition of the derivative is also expressed, by writing w for $z + h$ (for $h \neq 0$) as

$$\frac{\mathrm{d}f}{\mathrm{d}z} = f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}.$$

Many textbooks will use this equivalent formulation.

When we speak of the limit in the above definition, and the equivalent formulation, what we mean is that the value of

$$\frac{f(w) - f(c)}{w - c}$$

approaches a single value, that we denote by $f'(c)$, as w approaches c no matter which route one uses. The requirement that U is an open set containing the point c, at which we differentiate f, is to allow us to approach c from any direction. Furthermore, our definition means that the value of the above formula can be made to arbitrarily close to $f'(c)$ provided that we are sufficiently close to c. In view of this intuition, the following formal definition is also used. (This formal version is precisely as found in *MT2502*.) We shall, however, have little cause to manipulate limits as expressed in the following formulation. This ε–δ definition is really only used in a few of our proofs.

Definition 2.3 (Formal Definition) Let U be an open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ be a function of a complex variable defined on U. We say that f is *differentiable* at a point $c \in U$ if for all $\varepsilon > 0$, there exists some $\delta > 0$ (depending upon ε) such that $0 < |h| < \delta$ implies

$$\left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < \varepsilon.$$

Since the definition of differentiability is expressed as a limit in exactly the same way as was used for real-valued functions, one can deduce the same basic facts about differentiability as for real-valued functions using precisely the same methods. We list these now. (Some proofs appear in the lecture notes, but they will not be proved during the lectures.)

Theorem 2.4 *Let U be an open subset of $\mathbb{C}$. If the function $f\colon U \to \mathbb{C}$ is differentiable at $c \in U$, then it is continuous at c (that is, $\lim_{z \to c} f(z) = f(c)$).*

PROOF: [Proof omitted during lectures and is non-examinable.]

Observe that if f is differentiable at c then

$$\begin{aligned} \lim_{z \to c} \bigl(f(z) - f(c) \bigr) &= \lim_{z \to c} \left(\left(\frac{f(z) - f(c)}{z - c} \right) (z - c) \right) \\ &= f'(c) \cdot 0 = 0. \end{aligned}$$

Hence $\lim_{z \to c} f(z) = f(c)$; that is, f is continuous at c. □

Our second list is the standard list of rules for differentiability:

Theorem 2.5 *Let f and g be functions of a complex variable. Then*

(i) ***Sum Rule:*** *If f and g are both differentiable at a point c, then $f+g$ is differentiable at c and*

$$(f+g)'(c) = f'(c) + g'(c).$$

(ii) *If f is differentiable at a point c and $\alpha \in \mathbb{C}$, then αf is differentiable at c and*

$$(\alpha f)'(c) = \alpha f'(c).$$

(iii) ***Product Rule:*** *If f and g are both differentiable at a point c, then $f \cdot g$ is differentiable at c and*

$$(f \cdot g)'(c) = f'(c)\, g(c) + f(c)\, g'(c);$$

that is, we have the usual general formula for differentiation of products:

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(f(z)\, g(z)\big) = f'(z)\, g(z) + f(z)\, g'(z).$$

(iv) ***Chain Rule:*** *If g is differentiable at a point c and f is differentiable at $g(c)$, then the composite $f \circ g$ is differentiable at c and*

$$(f \circ g)'(c) = f'(g(c))\, g'(c).$$

(v) ***Quotient Rule:*** *If f and g are both differentiable at a point c and $g(c) \neq 0$, then f/g is differentiable at c and*

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)\, f'(c) - f(c)\, g'(c)}{g(c)^2}.$$

Proof: [Omitted in lectures and currently also in the lecture notes. In a future version of the notes, the proofs may be added for the sake of completeness. Proofs will not be examinable.] □

We shall be interested in functions that are not just differentiable at individual points in $\mathbb{C}$ or individual points of some open subset, but rather are differentiable on the *whole* of some open subset of $\mathbb{C}$. Accordingly, we make the following definition:

Definition 2.6 Let U be an open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ be a function of a complex variable defined on U. We say that f is *holomorphic* on U if it is differentiable at every point of U.

A function $f\colon \mathbb{C} \to \mathbb{C}$ that is holomorphic on the whole complex plane $\mathbb{C}$ (that is, differentiable at every point of $\mathbb{C}$) is sometimes called an *entire* function.

Note, for example, that the Sum Rule (Theorem 2.5(i)) then says that if f and g are holomorphic on some open subset U then the sum $f+g$ is also holomorphic on U and

$$(f+g)'(z) = f'(z) + g'(z) \qquad \text{for all } z \in U.$$

Similar observations apply to scalar multiples, products, composites and quotients of holomorphic functions.

Example 2.7 (i) Since sums and products of holomorphic functions are holomorphic on the complex plane $\mathbb{C}$, it follows straightaway that polynomials are holomorphic on $\mathbb{C}$. So, for example, if $f(z) = z^n$, then f is holomorphic on $\mathbb{C}$ and

$$f'(z) = nz^{n-1}.$$

More generally, if $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, a polynomial with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$, then g is holomorphic on $\mathbb{C}$ and

$$g'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots n a_n z^{n-1}.$$

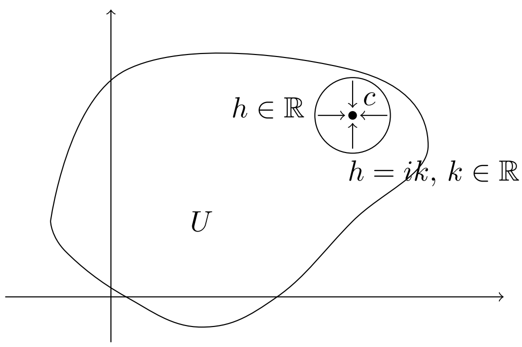


Figure 2.1: The Cauchy–Riemann Equations: Limits as $h \to 0$ through real and imaginary values.

(ii) If we use the quotient rule, then it follows that for polynomials $f(z)$ and $g(z)$, the quotient $f(z)/g(z)$ is holomorphic on any open set U such that U contains none of the roots of g.

In a course on calculus, the next typical examples of differentiable real-valued functions after polynomials are usually exponential functions such as e^x and trigonometric functions such as $\sin x$ and $\cos x$. We wish to do something similar for functions of a complex variable. It is, however, not entirely obvious exactly how to define such functions. One possibility is to exploit the properties that we expect the functions to have and so, for example, if $z = x + iy$ is a complex number with real and imaginary parts x and y we might define

$$\mathrm{e}^z = \mathrm{e}^x\,\mathrm{e}^{iy} = \mathrm{e}^x(\cos y + i \sin y).$$

The problem with this definition, however, is that it is more complicated to show e^z given by this formula is indeed differentiable.

The most straightforward solution is to define the functions we seek as power series. This will also have the advantage of building some general theory that will both provide us with a good source of holomorphic functions and fit within the direction of the mathematics we develop. Accordingly the last part of this chapter will be devoted to a discussion of power series.

The Cauchy–Riemann Equations

Let $f\colon U \to \mathbb{C}$ be a function of a complex variable defined upon some open subset $U \subseteq \mathbb{C}$. If f is differentiable at some point $c \in U$, this means

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists. We interpret this as saying

$$\frac{f(c+h) - f(c)}{h}$$

approaches the same value (in $\mathbb{C}$) no matter how h approaches 0. In particular, if we let $h \to 0$ through real numbers or through purely imaginary numbers (i.e., $h = ik$ where $k \in \mathbb{R}$), then we should obtain the same limit (see Figure 2.1). Exploiting this is what leads to what are known as the Cauchy–Riemann equations.

Write u and v for the real and imaginary parts of f and view them as functions of two real variables x and y by also writing $z = x + iy$ for $z \in U$. Thus

$$f(x + iy) = u(x, y) + i\,v(x, y)$$

whenever $x + iy \in U$. In particular, we write our fixed value as $c = a + ib$ for $a, b \in \mathbb{R}$. Now if h is a sufficiently small real number such that $a + ib + h \in U$, then

$$\frac{f(a + ib + h) - f(a + ib)}{h} = \frac{u(a + h, b) - u(a, b)}{h} + i\,\frac{v(a + h, b) - v(a, b)}{h}. \tag{2.1}$$

Now if we view $u(x, y)$ as a function of x alone then as $h \to 0$ the quotient

$$\frac{u(a + h, b) - u(a, b)}{h}$$

approaches the derivative of u with respect to x evaluated at $x = a$ (treating y as the constant b). Those who have completed *MT2503* know what we get. If we differentiate $u(x, y)$ with respect to x, treating y as constant, then the result is the partial derivative, denoted

$$\frac{\partial u}{\partial x} \quad \text{and} \quad u_x.$$

Those who have come in via *MT2502* route need not worry. The partial derivative is exactly what has just been described: view $u(x, y)$ as a function of x only, treat y as constant, and differentiate with respect to x to obtain $\frac{\partial u}{\partial x}$. Similarly, the second term in Equation (2.1) has limit equal to the partial derivative of v with respect to x as $h \to 0$, so we conclude

$$\lim_{h \to 0} \frac{f(a + ib + h) - f(a + ib)}{h} = \frac{\partial u}{\partial x}(a, b) + i\,\frac{\partial v}{\partial x}(a, b), \tag{2.2}$$

evaluating both partial derivatives at (a, b).

On the other hand, we could consider a purely imaginary h, $h = ik$ with $k \in \mathbb{R}$ sufficiently small such that $a + ib + ik \in U$. Then

$$\begin{aligned}
\frac{f(a + ib + ik) - f(a + ib)}{ik} &= \frac{u(a, b + k) - u(a, b)}{ik} + i\,\frac{v(a, b + k) - v(a, b)}{ik} \\
&= \frac{v(a, b + k) - v(a, b)}{k} - i\,\frac{u(a, b + k) - u(a, b)}{k}.
\end{aligned}$$

We now let $k \to 0$. In this case, we are, in the first quotient, viewing $v(x, y)$ as a function of y, treating x as the constant a, and as $k \to 0$ we obtain the partial derivative of v with respect to y evaluated at (a, b). The second term above has limit equal to the partial derivative of u with respect to y and so we conclude

$$\lim_{k \to 0} \frac{f(a + ib + ik) - f(a + ib)}{ik} = \frac{\partial v}{\partial y}(a, b) - i\,\frac{\partial u}{\partial y}(a, b). \tag{2.3}$$

Now recall that $f'(c)$ is the limit of

$$\frac{f(c + h) - f(c)}{h}$$

as $h \to 0$ *no matter* how h approaches 0. Consequently, Equations (2.2) and (2.3) (which are the limits as we approach along a horizontal or vertical line in the complex plane) must be equal:

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

(when all partial derivatives are evaluated at (a, b)). Now u and v are real-valued functions, so if we take the real and imaginary parts of this last equation, we have established the Cauchy–Riemann Equations as stated in the following theorem:

Theorem 2.8 (Cauchy–Riemann Equations) *Let $f\colon U \to \mathbb{C}$ be a function of a complex variable, with U an open subset of $\mathbb{C}$, and suppose that f is differentiable at $c = a + ib \in U$ (where $a, b \in \mathbb{R}$). Write*

$$f(x + iy) = u(x, y) + i\,v(x, y)$$

where $u\colon \tilde{U} \to \mathbb{R}$ and $v\colon \tilde{U} \to \mathbb{R}$ (and where $\tilde{U} = \{\, (x, y) \in \mathbb{R}^2 \mid x + iy \in U \,\}$). Then the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

of u and v exist at (a, b) and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

□

In particular, if a function $f\colon U \to \mathbb{C}$ is holomorphic on the open subset U, then the Cauchy–Riemann Equations are satisfied at every point of U. We mention one more observation that appears above: when $f(x + iy) = u(x, y) + i\,v(x, y)$, the derivative of f when it exists can be expressed in terms of the partial derivatives as

$$f'(x + iy) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

for each $z = x + iy \in U$.

Example 2.9 Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z^2.$$

Writing $z = x + iy$, we have already observed the real and imaginary parts of f are given by

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy$$

(see Example 2.1). Let us calculate the partial derivatives:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x, & \frac{\partial u}{\partial y} &= -2y, \\ \frac{\partial v}{\partial x} &= 2y, & \frac{\partial v}{\partial y} &= 2x. \end{aligned}$$

Observe that the Cauchy–Riemann Equations do indeed hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

(This should not be surprising: we have shown that the Cauchy–Riemann Equations hold for *every* holomorphic function. In particular, they hold for the function $f(z) = z^2$.)

Note that Theorem 2.8 says that *if* f is holomorphic on a set U, *then* it satisfies the Cauchy–Riemann Equations at every point of U. This means that one way that the Cauchy–Riemann Equations can be used to show that a particular function *is not* holomorphic by showing that these equations do not hold. However, there are examples of functions that satisfy the Cauchy–Riemann Equations but are still not holomorphic (see Example 2.11 below). Thus, the Cauchy–Riemann Equations are *necessary* for holomorphic functions, but they are not *sufficient* to establish differentiability.

The following examples illustrate the use of the Cauchy–Riemann Equations to show a function is not differentiable.

Example 2.10 *Show that the functions* (i) $f(z) = \bar{z}$ *and* (ii) $g(z) = |z|$ *are not differentiable at any point of* $\mathbb{C}$.

SOLUTION: (i) Writing u and v for the real and imaginary parts of

$$f(x + iy) = x - iy,$$

we observe

$$u(x, y) = x \qquad \text{and} \qquad v(x, y) = -y.$$

Thus

$$\frac{\partial u}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial v}{\partial y} = -1.$$

In particular, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, so the Cauchy–Riemann Equations are never satisfied. Consequently, f cannot be differentiated at any point of $\mathbb{C}$.

(ii) When

$$g(x + iy) = (x^2 + y^2)^{1/2},$$

the real and imaginary parts of g are

$$u(x, y) = (x^2 + y^2)^{1/2} \qquad \text{and} \qquad v(x, y) = 0.$$

To calculate the partial derivatives, we need to treat the point $(x, y) = (0, 0)$ separately. For $(x, y) \neq (0, 0)$, we calculate

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2 + y^2)^{1/2}}, \qquad \frac{\partial u}{\partial y} = \frac{y}{(x^2 + y^2)^{1/2}}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

In particular,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \qquad \text{if } x \neq 0$$

and

$$\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y} \qquad \text{if } y \neq 0.$$

In the case of the partial derivatives at $(0, 0)$, observe

$$\frac{u(h, 0) - u(0, 0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{for real } h > 0 \\ -1 & \text{for real } h < 0. \end{cases}$$

This has no limit as $h \to 0$ and hence $\frac{\partial u}{\partial x}$ does not exist at the origin. Thus the Cauchy–Riemann Equations are not satisfied at $z = 0$ either since not all the partial derivatives involved exist.

In conclusion, there are no points in the complex plane where $g(z) = |z|$ satisfies the Cauchy–Riemann Equations and hence g is not differentiable on $\mathbb{C}$. □

Comments: One way to interpret the previous example and the fact that the use of complex conjugate and the modulus causes the Cauchy–Riemann Equations to fail is that one could think intuitively of holomorphic functions as those that are genuinely functions of z alone and do not involve complex conjugates $\bar{z}$. This will not be made precise at all. Indeed it happens to be the case that if $f(z)$ is holomorphic on an open set, then $\overline{f(\bar{z})}$ is holomorphic on another open set (but, of course, here one has used complex conjugation *twice*). Nevertheless, the Cauchy–Riemann Equations give some intuition into which sort of functions are holomorphic.

Example 2.11 *Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ given by*

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Show that f satisfies the Cauchy–Riemann Equations at $z = 0$ but is not differentiable there.

Solution: We need to calculate the partial derivatives of the real and imaginary parts at $(0, 0)$. Consider a non-zero real number x. Then, according to our above formula for f,

$$f(x) = x$$

so the real and imaginary parts of f along the real axis are

$$u(x, 0) = x \qquad \text{and} \qquad v(x, 0) = 0.$$

This enables us to calculate the partial derivatives of u and v with respect to x at $(0, 0)$:

$$\frac{\partial u}{\partial x}(0, 0) = 1 \qquad \text{and} \qquad \frac{\partial v}{\partial x}(0, 0) = 0.$$

Similarly, for a non-zero real numbers y,

$$f(iy) = \frac{i^5 y^5}{y^4} = iy,$$

so

$$u(0, y) = 0 \qquad \text{and} \qquad v(0, y) = y,$$

from which we calculate

$$\frac{\partial u}{\partial y}(0, 0) = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial y}(0, 0) = 1.$$

Hence the Cauchy–Riemann Equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, are satisfied at $z = 0$.

Now consider $h = \frac{(1+i)}{\sqrt{2}} k$ for some real number k. Note

$$f(h) = \left(\frac{1 + i}{\sqrt{2}} \right)^5 k,$$

so that

$$\frac{f(h) - f(0)}{h} = \left(\frac{1 + i}{\sqrt{2}} \right)^4 = (\cos \tfrac{\pi}{4} + i \sin \tfrac{\pi}{4})^4 = -1.$$

Hence the limit of $(f(h) - f(0))/h$ as $h \to 0$ through complex numbers of the form $h = (1+i)k/\sqrt{2}$ is -1, whereas (from the partial derivatives we have calculated earlier) the limit through real numbers h is $\frac{\partial u}{\partial x}(0, 0) + i \frac{\partial v}{\partial x}(0, 0) = 1$. Since these values are different, we conclude that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

does not exist, so f is not differentiable at $z = 0$. □

This last example confirms that although the Cauchy–Riemann Equations are necessary for a function to be holomorphic, they are not sufficient to show this. All they really tell us is about the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

as h approaches 0 along the real axis and along the imaginary axis. The Equations do not give any information about what the limit might be if h approaches 0 along a different path. It is this missing information that explains why the Cauchy–Riemann Equations alone are not sufficient to establish a function is holomorphic. What is surprising is that they are *almost* enough to establish that a function is holomorphic. It turns out if one also assumes that the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ are continuous, then one can establish that the function is holomorphic. In this lecture course, we merely state the following (partial) converse of the Cauchy–Riemann Equations.

Theorem 2.12 *Let $f \colon U \to \mathbb{C}$ be a function of a complex variable defined upon an open subset U of $\mathbb{C}$. Write*

$$f(x+iy) = u(x,y) + i\,v(x,y),$$

for each $x + iy \in U$, and suppose that the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y}$$

of the functions u and v exist and are continuous on U. Suppose in addition that the Cauchy–Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

hold at every point of U. Then f is holomorphic on U.

There is just one additional comment that we make about the hypotheses of this theorem. The assumption that the partial derivatives are continuous is not as strong or unexpected as one might think. If f is a holomorphic function, then we shall observe later (amongst other things) that the derivative f' is itself also holomorphic and so, in particular, continuous. Consequently, the real and imaginary parts $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ of f' are necessarily continuous. In view of this, one should not be so surprised that continuity of the partial derivatives appear amongst the hypotheses in this theorem.

Proof: Currently omitted. See Howie [1, Theorem 4.3]. □

Before turning to power series, we mention one result about holomorphic functions, analogous to the result for real-valued functions that is important. The proof actually depends upon the analogous result for real-valued functions.

Theorem 2.13 *Let $w \in \mathbb{C}$ and $r > 0$. Suppose that $f \colon B(w,r) \to \mathbb{C}$ is holomorphic on the open disc $B(w,r)$. If $f'(z) = 0$ for all $z \in B(w,r)$, then f is constant on $B(w,r)$.*

Proof: First write

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

for $x + iy \in B(w,r)$. Note that

$$f'(x+iy) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

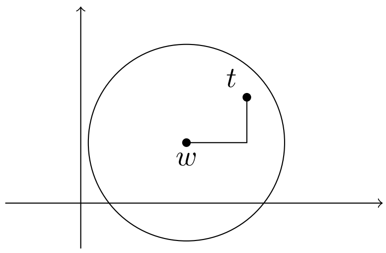


Figure 2.2: Joining $w = a + ib$ to $t = c + id$ via $c + ib$.

for all points $x + iy \in B(w, r)$ (with these equations having been established in our proof of the Cauchy–Riemann Equations). Since $f'(z) = 0$ for all $z \in B(w, r)$, we conclude by taking real and imaginary parts that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

on the disc $B(w, r)$.

Write $w = a+ib$ (for $a, b \in \mathbb{R}$). Consider some $t = c+id \in B(w, r)$ (where also $c, d \in \mathbb{R}$). We can join t to w using horizontal and vertical lines in the complex plane (as shown in Figure 2.2). For the sake of being concrete, let us consider the horizontal line from $w = a + ib$ to $c + ib$. Viewed as a function of x alone, the function $u(x, b)$ has zero derivative: $\partial u/\partial x = 0$ on the line from w to $c + ib$. From results about functions of a real-variable, we conclude that $u(x, b)$ is constant on this line. Thus $u(a, b) = u(c, b)$. By the same argument, using the partial derivative $\partial v/\partial x$, we conclude $v(a, b) = v(c, b)$. Thus

$$f(w) = u(a, b) + i\,v(a, b) = u(c, b) + i\,v(c, b) = f(c + ib).$$

Similarly, using the vertical line from $c + ib$ to t and exploiting the partial derivatives $\partial u/\partial y$ and $\partial v/\partial y$, we establish $f(c + ib) = f(t)$.

Putting this together, we conclude

$$f(t) = f(w)$$

for all $t \in B(w, r)$, so f is constant on this disc. □

By exploiting the fact that the whole complex plane is the union of open discs with ever growing radii, one deduces the following.

Corollary 2.14 *If f is holomorphic on $\mathbb{C}$ and $f'(z) = 0$ for all $z \in \mathbb{C}$, then f is constant.*

Proof: By the theorem, f is constant on every open disc $B(0, r)$ for all radii $r > 0$. If $z \in \mathbb{C}$, there exists some radius r with $r > |z|$ and we conclude $f(z) = f(0)$. Hence f is constant on $\mathbb{C}$. □

Power series

The purpose of this section is to consider functions of the form

$$\sum_{n=0}^{\infty} c_n z^n,$$

where $c_0, c_1, c_2, \ldots \in \mathbb{C}$, and more generally of the form

$$\sum_{n=0}^{\infty} c_n (z - a)^n$$

where also $a \in \mathbb{C}$. For a particular complex number value of z, we can evaluate the sum

$$s_N = \sum_{n=0}^{N} c_n (z-a)^n$$

for each natural number N. In this way, we define a sequence

$$s_0, s_1, s_2, \dots$$

of partial sums. This sequence (s_N) may or may not converge to a complex number. Consequently, we can define a complex-valued function $f\colon D \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where D is some set of complex numbers for which this series converges. It would be possible to make the concept of convergence precise with an ε–δ definition as was done in *MT2502*. We shall, however, just state a theorem, Theorem 2.16 below, that tells that such a power series defines a complex-valued function on a certain open disc.

Definition 2.15 A *power series* is a function of the form

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

(where a and c_0, c_1, $\dots$ are complex numbers) whenever this converges (i.e., for whichever values of $z \in \mathbb{C}$ the series of complex numbers converges).

Power series were discussed in both *MT2502* and *MT2503* with various facts presented. We summarise the basic facts that we require in the following theorem. The primary place where power series are considered is within the module *MT3502 Real Analysis*, albeit for power series of real numbers. It is for this reason that the proofs are omitted. These proofs can be found (in the context of power series involving real numbers) in *MT3502*.

Theorem 2.16 *Let a and c_0, c_1, $\dots$ be complex numbers and consider the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$.*

(i) *There exists* a radius of convergence R, *either a real number in $[0, \infty)$ or $R = \infty$, such that the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges (absolutely) for $|z-a| < R$ and does not converge for $|z-a| > R$.*

(ii) *The function $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is differentiable at every z satisfying $|z-a| < R$ with derivative*

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}.$$

Moreover, the latter power series has the same radius of convergence R as the original series.

The term "absolute convergence" appearing in the theorem is actually stronger than convergence. To say $\sum_{n=0}^{\infty} c_n (z-a)^n$ is absolutely convergent means that the series converges but also the series of real numbers formed by taking the modulus of each term

$$\sum_{n=0}^{\infty} |c_n (z-a)^n|$$

is convergent. In fact, convergence of the latter series of real numbers obtained by taking the modulus always implies convergence of the former series of complex numbers.

Note that when $R = 0$, we are saying that the power series only converges when $z = a$. (Our power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ necessarily converges at $z = a$ since all but the first term equal 0 at this value of z.) When $R = \infty$, we are saying the power series converges for all $z \in \mathbb{C}$. The second part of the theorem tells us that a power series can be differentiated term-by-term inside the radius of convergence.

The following gives one way of calculating the radius of convergence of a power series. Note, however, that there can be choices of coefficients c_n such that the limit appearing in the proposition does not exist. In those cases, more care would be needed to verify what the radius of convergence actually is. Nevertheless, the proposition can be used in many interesting cases.

Proposition 2.17 *Let a and c_0, c_1, ... be complex numbers. Suppose that*

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

exists. Then this limit R is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$.

The proof of this result depends upon the *Ratio Test*: a basic fact about convergence of *real numbers* that is covered in *MT1002* and proved in *MT2502*. For the sake of all students, especially those who may have managed to avoid it during their route to this module, we recall the result:

> **Ratio Test:** Let $\{b_0, b_1, b_2, \dots\}$ be a collection of *positive* real numbers. Suppose that the limit $\ell = \lim_{n\to\infty} b_{n+1}/b_n$ exists. Then
>
> (i) if $\ell < 1$, the series $\sum_{n=0}^{\infty} b_n$ converges;
>
> (ii) if $\ell > 1$, the series $\sum_{n=0}^{\infty} b_n$ does not converge.

Proof of Proposition 2.17: As noted above, the power series certainly converges for $z = a$. Assume that $0 < |z-a| < R$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{\left|c_{n+1}(z-a)^{n+1}\right|}{|c_n(z-a)^n|} &= \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |z-a| \\ &= |z-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \\ &= \frac{|z-a|}{R} < 1. \end{aligned}$$

Hence, by the Ratio Test, $\sum_{n=0}^{\infty} |c_n(z-a)^n|$ converges. Thus the power series converges absolutely for $|z-a| < R$. This establishes that the radius of convergence is at least R.

Conversely, suppose $|z-a| > R = \lim_{n\to\infty} |c_n/c_{n+1}|$. Then

$$|z-a| > |c_n/c_{n+1}| \qquad \text{for sufficiently large } n,$$

so

$$\left|c_{n+1}(z-a)^{n+1}\right| > |c_n(z-a)^n| \qquad \text{for sufficiently large } n.$$

This means that $c_n(z-a)^n \not\to 0$ as $n \to \infty$. Hence $\sum_{n=0}^{\infty} c_n(z-a)^n$ does not converge for $|z-a| > R$. Hence the radius of convergence is precisely R. □

Example 2.18 (Complex exponential function) Define

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

to be the complex *exponential function.* (Note that we are not making an assertion about what "powers" involving complex numbers in the exponent at this stage. Instead, we are merely defining the symbol "e^z" to mean the power series on the right-hand side, whenever it converges.)

The coefficients in this power series are $c_n = 1/n!$. So

$$\left|\frac{c_n}{c_{n+1}}\right| = \frac{(n+1)!}{n!} = n+1 \to \infty$$

as $n \to \infty$. Proposition 2.17 tells us that the radius of convergence of the power series is $R = \infty$. Hence the power series defines a function that is holomorphic on the whole complex plane $\mathbb{C}$ with derivative obtained by term-by-term differentiation of the power series:

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^z) = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = \mathrm{e}^z.$$

Thus, the complex exponential function already shares one property with the real version of the function.

Fix some complex number ζ and consider some other complex variable z. (So we are permitting z to vary, but we are treating ζ as a constant.) Define

$$f(z) = \mathrm{e}^z\, \mathrm{e}^{\zeta - z}$$

The product and chain sum rules (Theorem 2.5) tells us that $f(z)$ is holomorphic on the complex plane $\mathbb{C}$ and that the derivative of f is

$$f'(z) = \mathrm{e}^z\, \mathrm{e}^{\zeta-z} - \mathrm{e}^z\, \mathrm{e}^{\zeta-z} = 0.$$

It follows (by Corollary 2.14) that f is constant on $\mathbb{C}$. Note $\mathrm{e}^0 = 1$ by definition, so from $f(z) = f(0)$ for all $z \in \mathbb{C}$ we deduce

$$\mathrm{e}^z\, \mathrm{e}^{\zeta-z} = \mathrm{e}^0 \mathrm{e}^{\zeta} = \mathrm{e}^{\zeta}.$$

This formula now holds for all ζ and z in $\mathbb{C}$, so substituting $\zeta = z+w$ for some complex numbers z and w yields

$$\mathrm{e}^z\, \mathrm{e}^w = \mathrm{e}^{z+w}$$

for all complex-numbers w and z.

Example 2.19 (Complex trigonometric functions) Having defined e^z, one way we *could* define $\sin z$ and $\cos z$ is in terms of the exponential function. We shall use an alternative method, namely via power series, as follows:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$
$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots.$$

To determine the radius of convergence of these series takes a little more care than applying the formula of Proposition 2.17 without thought. The problem with that formula is that in the above power series a lot of the coefficients c_n equal 0, so the limit in Proposition 2.17 does not exist. The solution is to adapt the *method*, namely the use of the Ratio Test, employed to prove that proposition.

If $z \in \mathbb{C}$, put $a_n = \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ for each integer $n \geqslant 0$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|z|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|z|^{2n+1}}$$

$$= \frac{|z|^2}{(2n+3)(2n+2)} \to 0 \qquad \text{as } n \to \infty.$$

Hence $\sum_{n=0}^{\infty} |a_n|$ converges by the Ratio Test, so the power series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ is absolutely convergent for all $z \in \mathbb{C}$. The radius of convergence is therefore $R = \infty$.

A similar argument can be applied to the series for $\cos z$.

In conclusion,

- the power series for $\sin z$ and $\cos z$ converge everywhere in the complex plane (i.e., the radius of convergence $R = \infty$);
- $\sin z$ and $\cos z$ are holomorphic on $\mathbb{C}$ and have derivatives

$$\frac{\mathrm{d}}{\mathrm{d}z}(\sin z) = \cos z \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\cos z) = -\sin z.$$

  (These are obtained by differentiating each power series term-by-term and observing that the result is the other power series.)

Furthermore, since a power series converges absolutely inside its radius of convergence, we are free to manipulate the power series as follows: For any $z \in \mathbb{C}$,

$$\begin{aligned} \cos z + i \sin z &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \frac{z^6}{6!} - \dots \\ &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \dots \\ &= \mathrm{e}^{iz}. \end{aligned}$$

So

$$\mathrm{e}^{iz} = \cos z + i \sin z \tag{2.4}$$

for all complex numbers z.

We can make further observations by substituting into the series above. If one substitutes $-z$ for z in the power series for $\sin z$ and $\cos z$, then we obtain immediately

$$\sin(-z) = -\sin z \qquad \text{and} \qquad \cos(-z) = \cos z.$$

Therefore

$$\mathrm{e}^{-iz} = \cos z - i \sin z$$

from Equation (2.4) and when we add or subtract these two equations, we deduce

$$2\cos z = \mathrm{e}^{iz} + \mathrm{e}^{-iz} \qquad \text{and} \qquad 2i \sin z = \mathrm{e}^{iz} - \mathrm{e}^{-iz}.$$

Hence

$$\cos z = \tfrac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$

and

$$\sin z = \tfrac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz}) = -\tfrac{i}{2}(\mathrm{e}^{iz} - \mathrm{e}^{-iz}).$$

(If we wanted to define $\cos z$ and $\sin z$ in terms of the exponential function, it is these two formulae that we would have used. In the above, we have verified that the functions defined by the power series satisfy the formulae that we wanted.)

Note that we can now also justify formulae referred to in Chapter 1 when we presented the modulus-argument form of a complex number. Indeed, if θ is a real number, Equation 2.4 says

$$\mathrm{e}^{i\theta} = \cos\theta + i \sin\theta$$

and so if θ is a suitable argument for the complex number z, we are permitted to write

$$z = |z| (\cos\theta + i \sin\theta) = |z| \, \mathrm{e}^{i\theta}$$

in terms of this complex number's modulus and argument. We also calculate that

$$\mathrm{e}^{x+iy} = \mathrm{e}^{x} \, \mathrm{e}^{iy} = \mathrm{e}^{x}(\cos y + i \sin y)$$

for any $z = x + iy \in \mathbb{C}$.

Further complex trigonometric functions can be defined in terms of the two we currently have, e.g.,

$$\tan z = \frac{\sin z}{\cos z}$$

when $\cos z \neq 0$. We can also define complex hyperbolic functions in terms of the exponential function by

$$\sinh z = \tfrac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}) \qquad \text{and} \qquad \cosh z = \tfrac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}).$$

Finally, we observe that when we substitute a real number into the power series we have used, we get the original functions e^{x}, $\sin x$ and $\cos x$ with which we are familiar. Indeed, the most natural way to view what we have done is the following:

(i) Start with a differentiable function of a real variable (e.g., e^{x}).

(ii) Find the power series expansion of this function: the *Taylor series* (see, for example, *MT2503*; for e^{x} this is $\sum_{n=0}^{\infty} x^{n}/n!$).

(iii) Use this power series to extend the function to the complex numbers.

Essentially, the above procedure can be viewed as exactly what we have done in the last part of this chapter.

Chapter 3

Contour Integration and Cauchy's Theorem

In this chapter, we introduce the main tool that is used to study complex analysis, namely the integral of a (holomorphic) function around a suitable curve in the complex plane. The last half of the chapter is devoted to a discussion of the most important theorem of complex analysis, Cauchy's Theorem, which states that the integral of a holomorphic function around a suitably well-behaved closed curve is zero. This will turn out to be a very powerful tool that will be used throughout the rest of the lecture course.

Curves

Before we define the integral of a function of a complex variable around along a curve, we must first define the technical terms stating what we mean by a curve and which properties we use when defining the integral.

Definition 3.1 (i) A *curve* (also known as a *path*) is a continuous function

$$\gamma\colon [a,b] \to \mathbb{C}$$

defined on some closed interval $[a,b]$ in $\mathbb{R}$. We write

$$\gamma^* = \{\, \gamma(t) \mid t \in [a,b] \,\}$$

for the *image* of the curve (that is, the actual subset of $\mathbb{C}$ that is traced as one follows the curve).

(ii) A curve $\gamma\colon [a,b] \to \mathbb{C}$ is *smooth* if γ is differentiable (with one-sided derivatives at the end-points a and b) and the derivative γ' is continuous.

In line with the position that we have taken in these notes, we shall not give a formal ε–δ definition, of the flavour found in *MT2502*, of the concepts of continuity and differentiability as they appear in Definition 3.1. For us, to see that $\gamma\colon [a,b] \to \mathbb{C}$ is a curve is to require that

$$\gamma(c) = \lim_{t \to c} \gamma(t)$$

at every parameter $c \in [a,b]$, with one-sided limits at $c = a$ and b. Similarly to say γ is smooth is to require that

$$\gamma'(c) = \lim_{t \to c} \frac{\gamma(t) - \gamma(c)}{t - c}$$

at every point $c \in [a,b]$, again with one-sided limits at $c = a$ and b. The reality, however, is that every example we consider will be built from functions that we already understand quite well.

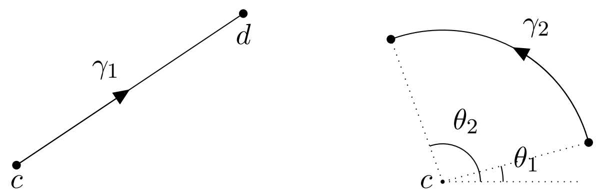


Figure 3.1: (i) A line segment, and (ii) a circular arc.

(See Example 3.2, where we merely use linear polynomials and exponential functions of t.) We will be able to recognize smooth curves γ when we draw the image γ^* on the complex plane. Curves will be drawn without any breaks and smooth curves will, indeed, be smooth: they have no sharp angles or changes in direction.

Observation: Furthermore note that, by taking real and imaginary parts, we can write any curve γ in the form

$$\gamma(t) = x(t) + i\,y(t)$$

where $x, y\colon [a,b] \to \mathbb{R}$ are continuous real-valued functions. Moreover, γ is smooth if and only if x and y are differentiable with continuous derivatives.

Example 3.2 The two most common examples of smooth curves that we shall use are lines and circular arcs.

(i) If $c, d \in \mathbb{C}$, the *line segment* from c to d is

$$\gamma_1\colon [0,1] \to \mathbb{C}$$

given by

$$\gamma_1(t) = c + (d-c)t \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Observe that $\gamma_1(0) = c$ and $\gamma_1(1) = d$.

(ii) If $c \in \mathbb{C}$, $r > 0$ and θ_1, θ_2 are angles chosen in some appropriate range (with $\theta_1 < \theta_2$), the (anti-clockwise) *circular arc*

$$\gamma_2\colon [\theta_1, \theta_2] \to \mathbb{C}$$

is given by

$$\gamma_2(t) = c + r\mathrm{e}^{it}.$$

The derivatives of these curves are calculated using the usual rules of differentiation:

$$\gamma_1'(t) = d - c$$
$$\gamma_2'(t) = ri\mathrm{e}^{it}.$$

Both are continuous functions of t (indeed γ_1' is constant), so line segments and circular arcs are examples of smooth curves. These examples of curves are illustrated in Figure 3.1.

In fact, most of the examples of curves that we shall use in our applications will be constructed by putting together various line segments and circular arcs.

The following concepts further expand upon the definition of curve as given in Definition 3.1.

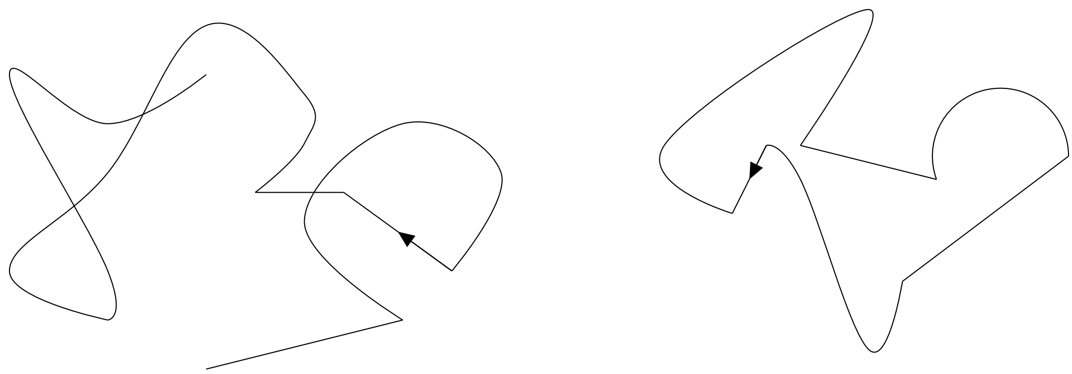


Figure 3.2: (i) A piecewise smooth curve that is not simple, and (ii) a contour.

Definition 3.3 (i) A *piecewise smooth* curve is a curve $\gamma\colon [a,b] \to \mathbb{C}$ where there are real numbers

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b$$

such that $\gamma\colon [a_i, a_{i+1}] \to \mathbb{C}$ is a smooth curve for $i = 0, 1, \ldots, n-1$.

(We often write $\gamma|_{[a_i,a_{i+1}]}$ for the restriction of γ to the domain $[a_i, a_{i+1}]$; that is, viewing γ as function on the subset of $[a_i, a_{i+1}]$ only. A piecewise smooth curve is one where each restriction $\gamma|_{[a_i,a_{i+1}]}$ is smooth.)

(ii) A curve $\gamma\colon [a,b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.

(iii) A curve $\gamma\colon [a,b] \to \mathbb{C}$ is *simple* if whenever $a \leqslant t_1 < t_2 \leqslant b$ (except possibly for $t_1 = a$ and $t_2 = b$) necessarily $\gamma(t_1) \neq \gamma(t_2)$.

So:

- a piecewise smooth curve is one obtained by gluing together finitely many smooth curves;
- a closed curve is one which finishes where it starts;
- a simple curve is one which has no crossings, except possibly the start and the end coincide (that is, simple curve are permitted to be closed).

Note that if γ is piecewise smooth as in (i) in the definition, then at $a_1, a_2, \ldots, a_{n-1}$, the left-hand and right-hand derivatives

$$\lim_{h\to 0^+} \frac{f(a_i - h) - f(a_i)}{-h} \qquad \text{and} \qquad \lim_{h\to 0^+} \frac{f(a_i + h) - f(a_i)}{h}$$

(for $1 \leqslant i \leqslant n-1$) exist, since $\gamma\colon [a_{i-1}, a_i] \to \mathbb{C}$ and $\gamma\colon [a_i, a_{i+1}] \to \mathbb{C}$ are smooth, but *these are not necessarily equal.*

Finally, putting these concepts together:

Definition 3.4 A *contour* is a piecewise smooth, simple, closed curve.

Example 3.5 One example (that is fairly typical of our future applications) of a contour is the following. Let ε and R be positive real numbers satisfying $0 < \varepsilon < R$. Then define

$$\gamma(t) = \begin{cases} \varepsilon + t(R - \varepsilon) & \text{if } 0 \leqslant t \leqslant 1 \\ R\mathrm{e}^{i\pi(t-1)} & \text{if } 1 < t \leqslant 2 \\ -R - (t-2)(\varepsilon - R) & \text{if } 2 < t \leqslant 3 \\ \varepsilon\mathrm{e}^{i\pi(4-t)} & \text{if } 3 < t \leqslant 4. \end{cases}$$

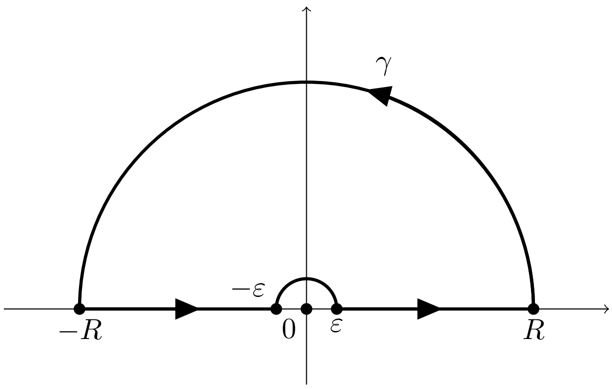


Figure 3.3: The contour γ in Example 3.5.

Such a formula looks complicated, but one should interpret this curve simply as gluing together the following smooth curves:

(i) the line segment from ε to R, followed by

(ii) a semi-circular arc centred on 0 from R to $-R$ anticlockwise, followed by

(iii) the line segment from $-R$ to $-\varepsilon$, finally followed by

(iv) a semi-circular arc centred on 0 from $-\varepsilon$ to ε clockwise.

See Figure 3.3 for a graphical illustration of this contour.

We usually will not need to worry about writing down precise formulae for any contour that we work with. We will be able to just parametrise each smooth piece of the contour separately and work with them piece-by-piece. The reason why we can safely do so is basically found in Proposition 3.11 below.

Definition 3.6 Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth curve. The *length* of γ is

$$L(\gamma) = \int_a^b \left|\gamma'(t)\right| \, \mathrm{d}t.$$

If γ is a piecewise smooth curve, say

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that $\gamma|_{[a_i,a_{i+1}]}$ is smooth for each i, then the *length* of γ is

$$L(\gamma) = \sum_{i=0}^{n-1} L(\gamma|_{[a_i,a_{i+1}]}),$$

the sum of the lengths of each smooth piece $\gamma|_{[a_i,a_{i+1}]}$.

Thus, with the above notation, the length of the piecewise smooth curve γ is

$$L(\gamma) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} \left|\gamma'(t)\right| \, \mathrm{d}t,$$

though one should note that $\gamma'(t)$ is possibly undefined at the finite collection of points a_1, a_2, $\dots$, a_{n-1}.

Example 3.7 *Calculate the lengths of (i) the line segment γ_1 from c to d, and (ii) the circular arc γ_2 of radius r subtended by angles θ_1 and θ_2, as given in Example 3.2.*

Solution: (i) The line segment is parametrised as

$$\gamma_1(t) = c + (d-c)t \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Then

$$\gamma_1'(t) = d - c,$$

so the length of γ_1 is

$$L(\gamma_1) = \int_0^1 |d-c| \, \mathrm{d}t = |d-c|,$$

which is indeed the expected length of the line from c to d.

(ii) The circular arc is parametrised as

$$\gamma_2(t) = c + r\mathrm{e}^{it} \qquad \text{for } \theta_1 \leqslant t \leqslant \theta_2.$$

Then

$$\gamma_2'(t) = ri\mathrm{e}^{it},$$

so the length of γ_2 is

$$L(\gamma_2) = \int_{\theta_1}^{\theta_2} \left| ri\mathrm{e}^{it} \right| \mathrm{d}t = r \int_{\theta_1}^{\theta_2} \mathrm{d}t = r(\theta_2 - \theta_1),$$

which is indeed the length of a circular arc of radius r subtended by angles θ_1 to θ_2. □

We can use the above calculation to determine the length of the contour γ given in Example 3.5. For a piecewise smooth curve, one simply adds up the length of each smooth piece. Thus, for this curve γ, one determines

$$\begin{aligned} L(\gamma) &= (R-\varepsilon) + \pi R + (R-\varepsilon) + \pi\varepsilon \\ &= 2(R-\varepsilon) + \pi(R+\varepsilon). \end{aligned}$$

Integration along a curve

We can now define what we mean by the integral of a function f of a complex variable evaluated along some curve γ in the complex plane.

Definition 3.8 Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve and f be a function of a complex variable whose domain contains γ^* such that f is continuous on γ^*. The *integral* of f along γ is defined to be

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t)) \, \gamma'(t) \, \mathrm{d}t.$$

Notational Warning: Some previous iterations of this lecture course use the notation

$$\oint_\gamma f(z) \, \mathrm{d}z$$

for the integral of a function f around a piecewise smooth curve in the specific case that γ is *closed.* In this lecture course, we shall not use that notation since we are sticking quite close to that used in Howie [1] and Priestley [2].

We shall now explain how to interpret and understand the integral that we have just defined, and also how to practically calculate this integral. In terms of interpretation, we have assumed that γ is piecewise smooth so there is a partition of $[a, b]$, say

$$a = a_0 < a_1 < \cdots < a_n = b,$$

such that γ is smooth on each $[a_i, a_{i+1}]$ (for $0 \leqslant i \leqslant n-1$); that is, γ' exists on $[a_i, a_{i+1}]$ and is continuous on this subinterval. Now f is continuous on γ^*, so when we compose our functions, we conclude

$$t \mapsto f(\gamma(t))\, \gamma'(t)$$

is a continuous function on $[a_i, a_{i+1}]$; that is,

$$f(\gamma(t))\, \gamma'(t) = u(t) = i\, v(t)$$

where u and v are continuous real-valued functions on $[a_i, a_{i+1}]$. We shall now use the fact that a continuous (real-valued) function is integrable on a closed and bounded interval to conclude

$$\int_{a_i}^{a_{i+1}} f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t = \int_{a_i}^{a_{i+1}} u(t)\, \mathrm{d}t + i \int_{a_i}^{a_{i+1}} v(t)\, \mathrm{d}t$$

makes sense. (In this module, we treat this "fact" as a "black box" that we shall just quote and use. It is a theorem that is proved in *MT3502*.) Consequently, the integral

$$\int_a^b f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t$$

has some value and this is what we mean by the integral

$$\int_\gamma f(z)\, \mathrm{d}z$$

of f along the curve γ.

However, just knowing that $\int_\gamma f(z)\, \mathrm{d}z$ does have a value does not actually tell us what this value is. Answering that question is the main thrust of what we do in this module. The most elementary method is to rely upon the method already learnt for calculating integrals: the Fundamental Theorem of Calculus (i.e., we recognize the integrand as the derivative of some function and perform integration as "reverse differentiation"). Here we expand upon that idea and also demonstrate how the breaking of $f(\gamma(t))\, \gamma'(t)$ into real and imaginary parts is unnecessary. This will simplify the whole process.

Indeed, suppose that we can recognize the integrand

$$f(\gamma(t))\, \gamma'(t) = u(t) + i\, v(t)$$

as the derivative of some complex-valued function $F \colon [a, b] \to \mathbb{C}$. Then

$$u(t) = (\operatorname{Re} F)'(t) \qquad \text{and} \qquad v(t) = (\operatorname{Im} F)'(t),$$

the derivatives of the real and imaginary parts of F. Hence, by the Fundamental Theorem of Calculus for real-valued functions:

$$\begin{aligned}
\int_\gamma f(z)\, \mathrm{d}z &= \int_a^b f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t \\
&= \int_a^b u(t)\, \mathrm{d}t + i \int_a^b v(t)\, \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b (\operatorname{Re} F)'(t)\,\mathrm{d}t + i \int_a^b (\operatorname{Im} F)'(t)\,\mathrm{d}t \\
&= \operatorname{Re} F(t)\Big|_{t=a}^{b} + i \operatorname{Im} F(t)\Big|_{t=a}^{b} \\
&= F(t)\Big|_{t=a}^{b} = F(b) - F(a).
\end{aligned}
$$

In conclusion, we did not actually need to break $f(\gamma(t))\,\gamma'(t)$ into real and imaginary parts. Once we recognize it is as the derivative of some function F, we can simply perform "reverse differentiation" and then evaluate $F(t)$ between the limits (these being the end-points of the parametrisation interval). The following two examples implement exactly this method. We shall also extend this idea in a version of the Fundamental Theorem of Calculus suitable for integrals along a curve (see Theorem 3.12 below).

Example 3.9 *Calculate*

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z$$

where γ is a circular contour of radius 1 about the origin.

Solution: We parametrise γ as

$$\gamma(t) = \mathrm{e}^{it} \qquad \text{for } 0 \leqslant t \leqslant 2\pi.$$

Then $\gamma'(t) = i\mathrm{e}^{it}$ and so

$$
\begin{aligned}
\int_\gamma \frac{1}{z}\,\mathrm{d}z &= \int_0^{2\pi} \frac{1}{\mathrm{e}^{it}} \cdot i\mathrm{e}^{it}\,\mathrm{d}t \\
&= i \int_0^{2\pi} \mathrm{d}t = 2\pi i.
\end{aligned}
$$

□

Example 3.10 *Calculate*

$$\int_\gamma \mathrm{e}^z\,\mathrm{d}z$$

where γ is a square contour with corners 0, 1, $1+i$ and i.

Solution: We parametrise the four parts of the square contour as

$$
\begin{aligned}
\gamma_1(t) &= t && \text{for } 0 \leqslant t \leqslant 1, \\
\gamma_2(t) &= 1 + it && \text{for } 0 \leqslant t \leqslant 1, \\
\gamma_3(t) &= 1 + i - t && \text{for } 0 \leqslant t \leqslant 1, \\
\gamma_4(t) &= (1-t)i && \text{for } 0 \leqslant t \leqslant 1
\end{aligned}
$$

(see Figure 3.4). Then

$$
\begin{aligned}
\int_\gamma \mathrm{e}^z\,\mathrm{d}z &= \int_{\gamma_1} \mathrm{e}^z\,\mathrm{d}z + \int_{\gamma_2} \mathrm{e}^z\,\mathrm{d}z + \int_{\gamma_3} \mathrm{e}^z\,\mathrm{d}z + \int_{\gamma_4} \mathrm{e}^z\,\mathrm{d}z \\
&= \int_0^1 \mathrm{e}^t\,\mathrm{d}t + \int_0^1 \mathrm{e}^{1+it} \cdot i\,\mathrm{d}t + \int_0^1 \mathrm{e}^{1+i-t} \cdot (-1)\,\mathrm{d}t + \int_0^1 \mathrm{e}^{(1-t)i} \cdot (-i)\,\mathrm{d}t \\
&= \mathrm{e}^t\Big|_{t=0}^{1} + \mathrm{e}^{1+it}\Big|_{t=0}^{1} + \mathrm{e}^{1+i-t}\Big|_{t=0}^{1} + \mathrm{e}^{(1-t)i}\Big|_{t=0}^{1}
\end{aligned}
$$

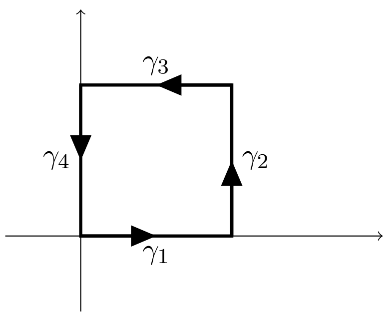


Figure 3.4: The square contour γ in Example 3.10.

$$\begin{aligned} &= (\mathrm{e} - 1) + (\mathrm{e}^{1+i} - \mathrm{e}) + (\mathrm{e}^{i} - \mathrm{e}^{1+i}) + (1 - \mathrm{e}^{i}) \\ &= 0. \end{aligned}$$

□

One thing that is apparent in the second calculation (and to a lesser extent in the first) is that we chose a parametrisation of our contour that was convenient to us. We should verify that contour integration does indeed behave as well as we want.

Proposition 3.11 *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve and $f\colon \gamma^* \to \mathbb{C}$ be continuous. Then*

(i)
$$\int_{\overleftarrow{\gamma}} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z,$$
where $\overleftarrow{\gamma}$ denotes the curve γ traced backwards:
$$\overleftarrow{\gamma}(t) = \gamma(a+b-t) \qquad \text{for } a \leqslant t \leqslant b.$$

(ii) *If $a < c < b$, $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$, then*
$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

(iii) ***Reparametrisation:*** *Let $\psi\colon [c,d] \to [a,b]$ be a differentiable real-valued functions with* positive *continuous derivative. Define $\tilde{\gamma} = \gamma \circ \psi\colon [c,d] \to \mathbb{C}$ (the composite of γ and ψ). Then*
$$\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = \int_{\gamma} f(z)\,\mathrm{d}z.$$

The reason for requiring $\psi'(t) > 0$ always in part (iii) is to ensure that we do not backtrack as we trace $\tilde{\gamma}$.

Proof: (i) By the Chain Rule,
$$\overleftarrow{\gamma}'(t) = -\gamma'(a+b-t).$$
Therefore
$$\begin{aligned} \int_{\overleftarrow{\gamma}} f(z)\,\mathrm{d}z &= \int_a^b f(\overleftarrow{\gamma}(t))\,\overleftarrow{\gamma}'(t)\,\mathrm{d}t \\ &= -\int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)\,\mathrm{d}t. \end{aligned}$$

Substitute $s = a + b - t$, noting that $\mathrm{d}t/\mathrm{d}s = -1$, to conclude

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z &= -\int_b^a f(\gamma(s))\,\gamma'(s)\,\frac{\mathrm{d}t}{\mathrm{d}s}\,\mathrm{d}s \\ &= \int_b^a f(\gamma(s))\,\gamma'(s)\,\mathrm{d}s \\ &= -\int_a^b f(\gamma(s))\,\gamma'(s)\,\mathrm{d}s \\ &= -\int_\gamma f(z)\,\mathrm{d}z.\end{aligned}$$

(ii) This part is the most straightforward. It follows by applying the general fact that

$$\int_a^b g(t)\,\mathrm{d}t = \int_a^c g(t)\,\mathrm{d}t + \int_c^b g(t)\,\mathrm{d}t$$

for any continuous function $g\colon [a,b] \to \mathbb{R}$ to the real and imaginary parts of the integrand appearing in the definition of $\int_\gamma f(z)\,\mathrm{d}z$.

(iii) By definition

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z &= \int_c^d f(\tilde{\gamma}(t))\,\tilde{\gamma}'(t)\,\mathrm{d}t \\ &= \int_c^d f(\gamma(\psi(t)))\,\gamma'(\psi(t))\,\psi'(t)\,\mathrm{d}t.\end{aligned}$$

Substitute $s = \psi(t)$, noting that $\mathrm{d}s/\mathrm{d}t = \psi'(t)$, so

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(s))\,\gamma'(s)\,\mathrm{d}s \\ &= \int_\gamma f(z)\,\mathrm{d}z,\end{aligned}$$

as required. (Note that the assumption $\psi'(t) > 0$ is actually necessary here for the application of Integration by Substitution. Precise statements of the hypotheses needed for Integration by Substitution make this explicit.) $\square$

Let us now establish how our basic method for calculating integrals can be used to establish a version of the Fundamental Theorem of Calculus for integrals along a curve.

Theorem 3.12 (Fundamental Theorem of Calculus for Integrals along a Curve) *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $F\colon U \to \mathbb{C}$ be a function of a complex variable defined on an open subset U containing γ^*. Assume that F is holomorphic on U with derivative $f = F'$ that is continuous on γ^*. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Thus, under these strong conditions, we can calculate the integral of f along the curve γ from the values of F at the end-points of the curve.

PROOF: Assume that γ has the property that γ is smooth on each $[a_i, a_{i+1}]$ where

$$a = a_0 < a_1 < \cdots < a_n = b.$$

Define $g = F \circ \gamma \colon [a,b] \to \mathbb{R}$. By (a suitable extension of) the Chain Rule, g is differentiable on $[a_i, a_{i+1}]$ and

$$g'(t) = F'(\gamma(t))\,\gamma'(t) = f(\gamma(t))\,\gamma'(t).$$

Hence

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t \\
&= \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} g'(t)\,\mathrm{d}t \\
&= \sum_{i=0}^{n-1} \bigl(g(a_{i+1}) - g(a_i)\bigr) && \text{(by our basic method)} \\
&= g(a_n) - g(a_0).
\end{aligned}$$

Thus

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

$\square$

Corollary 3.13 (Easy Version of Cauchy's Theorem) *Let $\gamma \colon [a,b] \to \mathbb{C}$ be a contour and let $F \colon U \to \mathbb{C}$ be a function of a complex variable defined on an open subset U containing γ^*. Assume that F is holomorphic on U with derivative $f = F'$ that is continuous on γ^*. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof: By the Fundamental Theorem of Calculus for Path Integrals,

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = 0$$

since γ is closed (that is, $\gamma(b) = \gamma(a)$). $\square$

The "Easy Version" of Cauchy's Theorem basically applies to any function $f(z)$ that we can recognise as the derivative of a function. For example, it tells us that if $f(z)$ is, for example, any one of e^z or z^n for some $n = 0, 1, \dots$ (all of which arise as the derivative of another function) then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for any contour γ. The "Easy Version" does not apply to functions that we are not able to recognise as the (continuous) derivative of a holomorphic function. For such functions (which are the ones we are now consequently most interested in) we shall need a more powerful version of Cauchy's Theorem.

Theorem 3.14 (Crude Estimation Theorem) *Let $\gamma \colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $f \colon \gamma^* \to \mathbb{C}$ be continuous. Suppose that $|f(z)| \leqslant M$ for all $z \in \gamma^*$. Then*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leqslant M \cdot L(\gamma),$$

where $L(\gamma)$ denotes the length of γ.

PROOF: Let
$$c = \int_\gamma f(z)\,\mathrm{d}z.$$
Then
$$|c| = c\,\mathrm{e}^{i\theta}$$
for some $\theta \in [0, 2\pi]$. Hence
$$\begin{aligned}
|c| = \operatorname{Re}|c| &= \operatorname{Re}\left(\mathrm{e}^{i\theta}\int_\gamma f(z)\,\mathrm{d}z\right) \\
&= \operatorname{Re}\left(\mathrm{e}^{i\theta}\int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t\right) \\
&= \operatorname{Re}\left(\int_a^b \mathrm{e}^{i\theta} f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t\right) \\
&= \int_a^b \operatorname{Re}\left(\mathrm{e}^{i\theta} f(\gamma(t))\,\gamma'(t)\right)\,\mathrm{d}t \\
&\leqslant \int_a^b \left|\mathrm{e}^{i\theta} f(\gamma(t))\,\gamma'(t)\right|\,\mathrm{d}t \\
&= \int_a^b |f(\gamma(t))| \cdot \left|\gamma'(t)\right|\,\mathrm{d}t \\
&\leqslant \int_a^b M\left|\gamma'(t)\right|\,\mathrm{d}t \\
&= M\int_a^b \left|\gamma'(t)\right|\,\mathrm{d}t \\
&= M \cdot L(\gamma),
\end{aligned}$$
as claimed. □

Cauchy's Theorem

We have already used the Fundamental Theorem of Calculus for Path Integrals to establish the "Easy Version" of Cauchy's Theorem; that is,
$$\int_\gamma f(z)\,\mathrm{d}z = 0$$
for any function f that occurs as the (continuous) derivative of a holomorphic function defined on an open set U containing γ^*. The purpose of this section is to gain a (partial) understanding of why a more general version of this result is true.

In order to state the main version of Cauchy's Theorem that we shall use, we need the following fact that is intuitively clear, but rather challenging (i.e., well beyond this lecture course) to prove in full generality.

Theorem 3.15 (Jordan Curve Theorem) *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a contour. Then the complex plane can be expressed as the union of three disjoint subsets:*

(i) *γ^*, the image of γ;*

(ii) *$I(\gamma)$, the* interior *of γ, which is open, bounded and connected;*

(iii) *$E(\gamma)$, the* exterior *of γ, which is open, unbounded and connected.*

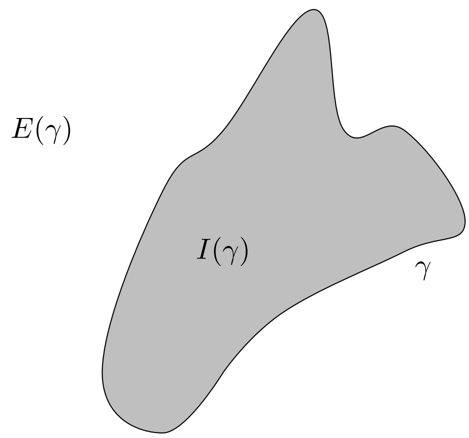


Figure 3.5: The Jordan Curve Theorem: A contour γ has an interior and an exterior.

We shall not spend any particular time on the technical terms appearing in the statement of the Jordan Curve Theorem. When saying the interior and exterior are open, we are merely using the term introduced in Definition 1.4 as elsewhere in the lecture course. The interior being bounded means that there is some bound on the values of the modulus: there exists M such that

$$|z| \leqslant M \qquad \text{for all } z \in I(\gamma),$$

while $|z|$ can be arbitrarily large for $z \in E(\gamma)$. Equivalently, it means that the interior $I(\gamma)$ is contained in some disc $B(0, M)$.

The term *connected* is a topological concept that we shall not state precisely in this module. For subsets of the complex plane it is equivalent to the following:

> A subset A of $\mathbb{C}$ is (*path*) *connected* if every pair $a, b \in A$ of points in A can be joined by a path in A.

See Figure 3.5 for an illustration of the Jordan Curve Theorem.

Using the language arising in the Jordan Curve Theorem, we can state our most general form of Cauchy's Theorem that we shall use.

Theorem 3.16 (Cauchy's Theorem) *Let γ be a contour and let f be a holomorphic function on some open subset U such that $\gamma^* \cup I(\gamma) \subseteq U$ (that is, U contains both the contour γ and its interior). Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

We shall present two proofs of special cases of Cauchy's Theorem. The first is essentially the one appearing in some previous versions of this lecture course. The proof depends upon an important theorem that appears in *MT2506 Vector Calculus* and this version of Cauchy's Theorem assumes stronger conditions, namely the continuity of the derivative f', than the general version. Since we wish to use Cauchy's Theorem to show that for *any* holomorphic function, the derivative f' is differentiable we shall still want the more general version where we do not assume the additional property.

The advantage of presenting this proof though is that we are able to link the concept of contour integral to the concept of line integral as introduced in *MT2506* and provide some context for those who have covered that module. The following proof is not examinable and those who have not studied *MT2506* will not miss anything significant as a consequence of the missing background. A proof of an alternative version of Cauchy's Theorem will be presented afterwards.

Theorem 3.17 *Let γ be a contour and let f be a holomorphic function on an open subset U with $\gamma^* \cup I(\gamma) \subseteq U$. Suppose in addition that f' is continuous on U. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

PROOF: The first step is to replace γ, if necessary, by its reverse $\overleftarrow{\gamma}$ so that (with use of Proposition 3.11(i)) we can assume that γ is positively oriented (i.e., is traced anti-clockwise).

Write $f(x+iy) = u(x,y)+i\,v(x,y)$ in terms of its real and imaginary parts to define functions $u, v \colon \tilde{U} \to \mathbb{R}$ where

$$\tilde{U} = \{\, (x,y) \in \mathbb{R}^2 \mid x + iy \in U \,\}.$$

Similarly, write $\gamma(t) = x(t) + i\,y(t)$ to define two real-valued functions $x, y \colon [a,b] \to \mathbb{R}$. Then by definition

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big)\,\big(x'(t) + iy'(t)\big)\,\mathrm{d}t \\
&= \int_a^b \big(u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\big)\,\mathrm{d}t \\
&\qquad + i\int_a^b \big(v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t)\big)\,\mathrm{d}t.
\end{aligned}$$

Consider the first term: It is

$$\int_a^b \left(u(x(t),y(t))\,\frac{\mathrm{d}x}{\mathrm{d}t} - v(x(t),y(t))\,\frac{\mathrm{d}y}{\mathrm{d}t} \right)\mathrm{d}t = \oint_{\tilde{\gamma}} \big(u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y\big)$$

in the notation of *MT2506* (and where $\tilde{\gamma}(t) = (x(t),y(t))$ is the curve in $\tilde{U}$ corresponding to γ). We now use Green's Theorem:

> **Green's Theorem:** Under the hypotheses that $\tilde{\gamma}$ is a piecewise smooth, positively-oriented, simple curve bounding an area A and such that the functions P and Q have continuous partial derivatives in some domain containing R, the following equation holds:
>
> $$\oint_{\tilde{\gamma}} \big(P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y\big) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Hence, in our context,

$$\oint_{\tilde{\gamma}} \big(u(x,y)\,\mathrm{d}x - v(x,y)\,dy\big) = -\iint_{I(\tilde{\gamma})} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Now the Cauchy–Riemann Equations (Theorem 2.8) tell us that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

always, so the above integral on the right-hand size is zero. Hence

$$\int_a^b \big(u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\big)\,\mathrm{d}t = 0.$$

The same argument, relying upon the other Cauchy–Riemann Equation tells us that

$$\int_a^b \big(v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t)\big)\,\mathrm{d}t = 0.$$

Hence

$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

as claimed. $\square$

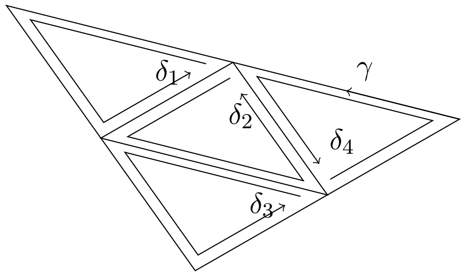


Figure 3.6: Subdividing the triangular contour γ into four smaller triangles.

Theorem 3.18 (Cauchy's Theorem for a Triangle) *Let γ be a triangular contour and let f be holomorphic on an open set containing γ and its interior. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

PROOF: The first step is to subdivide γ, by dividing each edge of γ in half, into four smaller triangular contours which, temporarily, we label δ_1, δ_2, δ_3 and δ_4 (see Figure 3.6). Then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\delta_i} f(z)\,\mathrm{d}z$$

since the integrals along each interior edge cancel as they are traversed in opposite directions (see Proposition 3.11(i)). Therefore

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leqslant \sum_{i=1}^{4} \left| \int_{\delta_i} f(z)\,\mathrm{d}z \right|.$$

It follows that at least one of the four smaller contours δ_i satisfies

$$\left| \int_{\delta_i} f(z)\,\mathrm{d}z \right| \geqslant \frac{1}{4} \left| \int_\gamma f(z)\,\mathrm{d}z \right|.$$

We define γ_1 to be one of the new triangular contours δ_i that satisfies this inequality. Note that each edge of γ_1 has half the length of the corresponding edge of γ, so $L(\gamma_1) = \frac{1}{2}L(\gamma)$.

We now repeat the process with the triangular contour γ_1. We divide it into four subtriangles and, by the same argument, find one of them, called γ_2, to be one of these new triangles satisfying

$$\left| \int_{\gamma_2} f(z)\,\mathrm{d}z \right| \geqslant \frac{1}{4} \left| \int_{\gamma_1} f(z)\,\mathrm{d}z \right| \geqslant \frac{1}{16} \left| \int_\gamma f(z)\,\mathrm{d}z \right|.$$

Continuing to repeat the process, we construct a sequence of triangular contours γ_1, γ_2, γ_3, ... with the following properties:

$$\left| \int_{\gamma_n} f(z)\,\mathrm{d}z \right| \geqslant \frac{1}{4^n} \left| \int_\gamma f(z)\,\mathrm{d}z \right| \tag{3.1}$$

$$L(\gamma_n) = \frac{1}{2^n} L(\gamma). \tag{3.2}$$

Now, for each $n \in \mathbb{N}$, pick a point c_n in the interior of the triangular contour γ_n. At each stage we choose γ_{n+1} via subdivision of the interior of γ_n, so $I(\gamma_{n+1}) \subseteq I(\gamma_n)$. Hence if $m \geqslant n$, then both c_m and c_n lie inside the triangular contour γ_n. Hence

$$|c_m - c_n| \leqslant L(\gamma_n) = \frac{1}{2^n} L(\gamma)$$

whenever $m \geqslant n$. Hence
$$|c_m - c_n| \to 0 \qquad \text{as } m, n \to \infty.$$
This means that (c_n) is a *Cauchy sequence*, so necessarily converges to some point $c \in \mathbb{C}$. (The fact that Cauchy sequences in $\mathbb{C}$ converge can be deduced very quickly, using the real and imaginary parts, from the fact that the same is true for Cauchy sequences of real numbers, as is shown in *MT2502*.) Moreover, for every choice of n, this limit c lies in the union $\gamma_n^* \cup I(\gamma_n)$ (that is, c lies either on the contour γ_n or inside it). The reason for this is that if it were the case that $c \in E(\gamma_n)$, the exterior of γ, then there is an $\varepsilon > 0$ with $B(c, \varepsilon) \subseteq E(\gamma)$ because the exterior is open. However, then $B(c, \varepsilon)$ contains none of the points c_m with $m \geqslant n$, contrary to the fact that c is the limit of the sequence (c_n).

Take $\varepsilon > 0$. As f is differentiable at c, there exists $\delta > 0$ such that
$$\left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < \varepsilon \tag{3.3}$$
when $0 < |h| < \delta$. Consider the open disc $B(c, \delta)$ of radius δ about c. Since c lies inside or on every γ_n and $L(\gamma_n) = \frac{1}{2^n} L(\gamma) \to 0$ as $n \to \infty$, there exists n such that
$$\gamma_n^* \subseteq B(c, \delta)$$
(that is, the triangular contour lies inside the open disc of radius δ about c; see Figure 3.7). Therefore if z lies on γ_n^*, then $z = c + h$ for some h with $|h| < \delta$, so
$$\big| f(z) - f(c) - f'(c)\,(z - c) \big| \leqslant \varepsilon\,|z - c|$$
by Equation (3.3). Note that we already know
$$\int_{\gamma_n} 1 \,\mathrm{d}z = \int_{\gamma_n} z \,\mathrm{d}z = 0$$
as observed once we had established our "Easy Version" of Cauchy's Theorem (Corollary 3.13). Hence
$$\begin{aligned} \int_{\gamma_n} f(z)\,\mathrm{d}z &= \int_{\gamma_n} \big(f(z) - f(c) - f'(c)\,(z-c)\big)\,\mathrm{d}z + \big(f(c) - f'(c)c\big) \int_{\gamma_n} 1\,\mathrm{d}z + f'(c) \int_{\gamma_n} z\,\mathrm{d}z \\ &= \int_{\gamma_n} \big(f(z) - f(c) - f'(c)\,(z-c)\big)\,\mathrm{d}z. \end{aligned}$$
Since $|f(z) - f(c) - f'(c)\,(z-c)| < \varepsilon\,|z-c| \leqslant \varepsilon\,L(\gamma_n)$ for z on γ_n^*, we deduce
$$\begin{aligned} \left| \int_{\gamma_n} f(z)\,\mathrm{d}z \right| &= \left| \int_{\gamma_n} \big(f(z) - f(c) - f'(c)\,(z-c)\big)\,\mathrm{d}z \right| \\ &\leqslant \varepsilon\,L(\gamma_n) \cdot L(\gamma_n) \\ &= \frac{\varepsilon}{4^n} L(\gamma)^2 \end{aligned}$$
using the Crude Estimation Theorem (3.14). Hence, using Equation (3.1),
$$\left| \int_{\gamma} f(z)\,\mathrm{d}z \right| \leqslant \varepsilon \cdot L(\gamma)^2.$$
Since $\varepsilon > 0$ is arbitrary, we conclude
$$\int_{\gamma} f(z)\,\mathrm{d}z = 0.$$

□

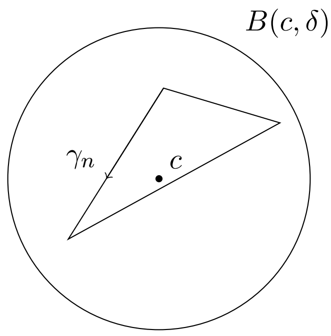


Figure 3.7: The triangular contour γ_n contained inside the open disc of radius δ about c.

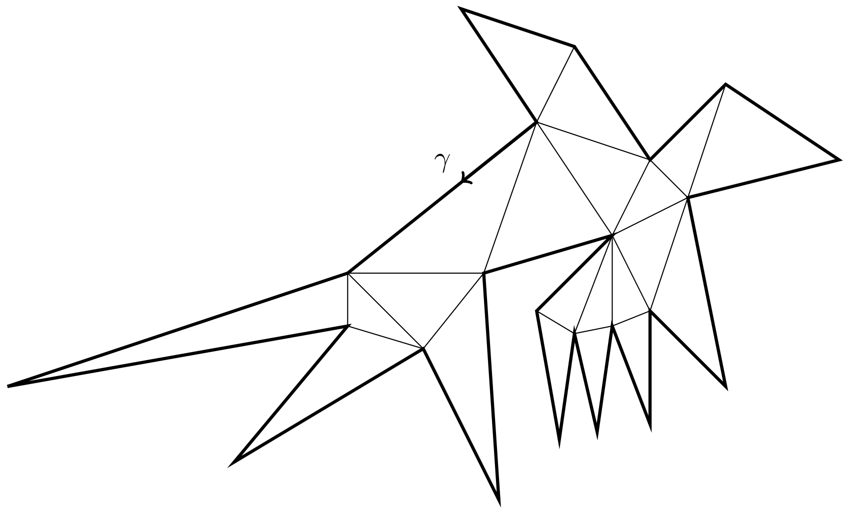


Figure 3.8: Triangulation of the polygonal contour γ.

Corollary 3.19 (Cauchy's Theorem for Polygonal Contours) *Let γ be a polygonal contour (that is, it is built from a finite collection of line segments joined together) and let f be holomorphic on an open set containing γ and its interior. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

PROOF: The first stage is to triangulate γ; that is, subdivide the interior of γ into triangles (see Figure 3.8). In this way, we construct a collection γ_1, γ_2, $\dots$, γ_k of triangular contours such that

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{k} \int_{\gamma_i} f(z)\,\mathrm{d}z$$

(since the integrals along the interior edges cancel in the sum using Proposition 3.11(i)). Hence, by Cauchy's Theorem for a Triangle (Theorem 3.18),

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

□

A full proof of Cauchy's Theorem, as stated in Theorem 3.16, is beyond this lecture course. In fact, the work done so far takes us quite a long way towards a full proof. A strategy (which can actually be fully implemented) is the following: If f is holomorphic on an open set containing an arbitrary contour γ and its interior, approximate γ by a *polygonal* contour $\tilde{\gamma}$ in such a way that the integrals

$$\int_{\gamma} f(z)\,\mathrm{d}z \qquad \text{and} \qquad \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z$$

are close (i.e., within some given $\varepsilon > 0$). Then $\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = 0$ by the case already established. From this, one deduces the general version of Cauchy's Theorem. The main challenge remaining in this approach is obtaining the polygonal approximation $\tilde{\gamma}$ such that the integrals are within ε if each other. This requires much care and that is the reason we omit this aspect of the proof of the general result.

Chapter 4

Consequences of Cauchy's Theorem

Now that we know what Cauchy's Theorem says, we can obtain a variety of consequences. We shall be using our general form of Cauchy's Theorem (Theorem 3.16) thoughout this section.

The first observation will be a useful tool throughout our work.

Theorem 4.1 (Deformation Theorem) *Let U be an open set, γ be a positively oriented contour such that γ and its interior are contained in U, a be a point in the interior of γ, and γ_1 be a positively oriented circular contour, centred at a, contained together with its interior inside γ. Suppose f is holomorphic on $U \setminus \{a\}$. Then*

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

Note: The term "positively oriented" appeared in the previous chapter. For completeness, we recall that a contour is *positively oriented* if it is traced anti-clockwise.

The upshot of the Deformation Theorem is that one can replace γ by a nice (i.e., circular, for example) and small contour without changing the integral. See Figure 4.1 for an illustration of how one would apply the Deformation Theorem.

Proof: Pick two points w and z on the contour γ and two points u and v on the contour γ_1 in such a way that we can join w to u by a curve γ_2 and z to v by a curve γ_3 that do not cross. Write γ_4 and γ_5 for the two pieces that w and z subdivide γ into and write γ_6 and γ_7 for those that u and v divide γ_1 into. See Figure 4.2.

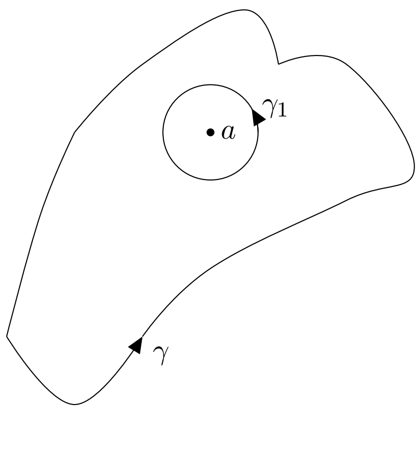


Figure 4.1: Application of the Deformation Theorem

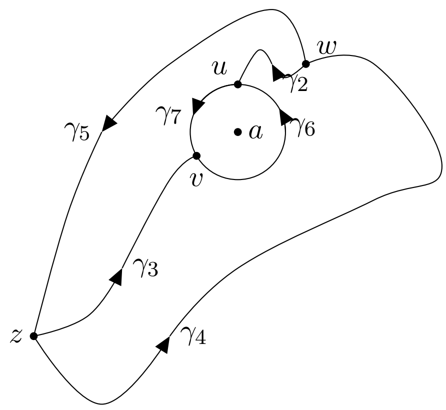


Figure 4.2: Proof of the Deformation Theorem: Subdivision of the contours

Let us write δ_1 for the contour obtained by following γ_3, $\overleftarrow{\gamma_7}$, $\overleftarrow{\gamma_2}$ and γ_5 and δ_2 for that obtained by following γ_2, $\overleftarrow{\gamma_6}$, $\overleftarrow{\gamma_3}$ and γ_4. Note that, by our construction, a does *not* lie in the interior of δ_1 or δ_2. Since f is holomorphic on $U \setminus \{a\}$, we can now apply Cauchy's Theorem (Theorem 3.16) to conclude

$$\int_{\delta_1} f(z)\,\mathrm{d}z = \int_{\delta_2} f(z)\,\mathrm{d}z = 0.$$

Adding $\int_{\delta_1} f(z)\,\mathrm{d}z$ to $\int_{\delta_2} f(z)\,\mathrm{d}z$, and then expanding into each contributing curve, we obtain

$$\int_{\gamma_3} f(z)\,\mathrm{d}z - \int_{\gamma_7} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z + \int_{\gamma_5} f(z)\,\mathrm{d}z \\ + \int_{\gamma_2} f(z)\,\mathrm{d}z - \int_{\gamma_6} f(z)\,\mathrm{d}z - \int_{\gamma_3} f(z)\,\mathrm{d}z + \int_{\gamma_4} f(z)\,\mathrm{d}z = 0.$$

Thus

$$\int_{\gamma_4} f(z)\,\mathrm{d}z + \int_{\gamma_5} f(z)\,\mathrm{d}z = \int_{\gamma_6} f(z)\,\mathrm{d}z + \int_{\gamma_7} f(z)\,\mathrm{d}z;$$

that is,

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z,$$

as claimed. □

Cauchy's Integral Formula and its consequences

We shall use the Deformation Theorem throughout our work in this section, including in the proof of the following result.

Theorem 4.2 (Cauchy's Integral Formula) *Let f be a holomorphic function on an open set U, let γ be a positively oriented contour which together with its interior are contained inside U, and let a be a point in the interior of γ. Then*

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

Proof: First, since f is differentiable at a, there exists $\delta > 0$ such that $0 < |z-a| < \delta$ implies

$$\left| \frac{f(z) - f(a)}{z-a} - f'(a) \right| < 1.$$

Now take any $\varepsilon > 0$ satisfying $0 < \varepsilon < \delta$ and such that the positively oriented circular contour γ_ε of radius ε about a is, together with its interior, contained inside γ. Now $f(z)/(z-a)$ is holomorphic on $U \setminus \{a\}$, so by the Deformation Theorem (Theorem 4.1)

$$\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{\gamma_\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

Thus, in effect, our first step is to replace γ by the circular contour γ_ε. We parametrise γ_ε as $\gamma_\varepsilon(t) = a + \varepsilon\,\mathrm{e}^{it}$ for $0 \leqslant t \leqslant 2\pi$, so that

$$\int_{\gamma_\varepsilon} \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\varepsilon\,\mathrm{e}^{it}} \cdot \varepsilon i\,\mathrm{e}^{it}\,\mathrm{d}t = 2\pi i$$

(as in Example 3.9). Thus

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z - f(a) &= \frac{1}{2\pi i}\left(\int_{\gamma_\varepsilon} \frac{f(z)}{z-a}\,\mathrm{d}z - f(a)\int_{\gamma_\varepsilon} \frac{1}{z-a}\,\mathrm{d}z\right) \\ &= \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z. \end{aligned}$$

Now if z lies on the contour γ_ε, then $|z-a| = \varepsilon < \delta$, so

$$\begin{aligned} \left|\frac{f(z)-f(a)}{z-a}\right| &= \left|\frac{f(z)-f(a)}{z-a} - f'(a) + f'(a)\right| \\ &\leqslant \left|\frac{f(z)-f(a)}{z-a} - f'(a)\right| + |f'(a)| \\ &< 1 + |f'(a)|. \end{aligned}$$

Hence, using the Crude Estimation Theorem (Theorem 3.14),

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z - f(a)\right| &= \frac{1}{2\pi}\left|\int_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z\right| \\ &\leqslant \frac{1}{2\pi}\cdot(1+|f'(a)|)\cdot 2\pi\varepsilon \\ &= \varepsilon(1+|f'(a)|). \end{aligned}$$

This is true for any ε satisfying $0 < \varepsilon < \delta$, while the left-hand side is independent of ε. Hence, we may let $\varepsilon \to 0$ and conclude that the left-hand side is zero. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z = f(a),$$

as claimed. □

Example 4.3 *Evaluate*

$$\int_\gamma \frac{z}{z-3}\,\mathrm{d}z$$

where (i) γ is a positively oriented circle of radius 2 about 2, and (ii) γ is a positively oriented circle of radius 2 about 0.

SOLUTION: (i) Take $f(z) = z$. This function is holomorphic on $\mathbb{C}$ and the complex number 3 lies inside γ (see Figure 4.3(i)), so Cauchy's Integral Formula (Theorem 4.2) says

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-3}\,\mathrm{d}z = f(3).$$

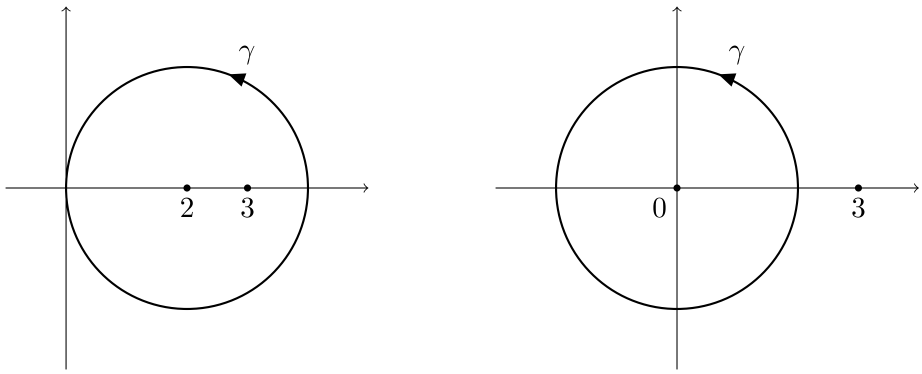


Figure 4.3: The contours γ in Example 4.3, parts (i) and (ii)

Hence

$$\int_\gamma \frac{z}{z-3}\,\mathrm{d}z = 2\pi i\, f(3) = 6\pi i.$$

(ii) Take $g(z) = z/(z-3)$. This function is holomorphic on $\mathbb{C} \setminus \{3\}$, which contains γ and its interior, so

$$\int_\gamma \frac{z}{z-3}\,\mathrm{d}z = 0$$

by Cauchy's Theorem (Theorem 3.16). □

Example 4.4 *Evaluate*

$$\int_\gamma \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z$$

where γ is the positively oriented square contour with corners $-2-3i$, $4-3i$, $4+3i$ and $-2+3i$.

Solution: Note that

$$z^3 - 9z = z(z-3)(z+3)$$

and two of the roots, 0 and 3, lie inside γ, but -3 is in the exterior of γ. Divide the contour γ into two using a vertical line joining $1-3i$ to $1+3i$, so that 0 and 3 lie in different parts of the subdivided contour. Let γ_1 and γ_2 denote the contours as shown in Figure 4.4. The integrals along the inner vertical line cancel, so we conclude

$$\int_\gamma \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z = \int_{\gamma_1} \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z + \int_{\gamma_2} \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z.$$

Now $f(z) = \mathrm{e}^z/(z^2-9)$ is holomorphic on and inside γ_1 (as this contour contains 0 but not 3), so

$$\int_{\gamma_1} \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z = \int_{\gamma_1} \frac{f(z)}{z}\,\mathrm{d}z = 2\pi i\, f(0),$$

by Cauchy's Integral Formula (Theorem 4.2). Similarly, $g(z) = \mathrm{e}^z/z(z+3)$ is holomorphic on and inside γ_2, so

$$\int_{\gamma_2} \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z = \int_{\gamma_2} \frac{g(z)}{z-3} = 2\pi i\, g(3).$$

Hence

$$\int_\gamma \frac{\mathrm{e}^z}{z^3-9z}\,\mathrm{d}z = 2\pi i \left(\frac{\mathrm{e}^0}{-9} + \frac{\mathrm{e}^3}{3\times 6} \right)$$

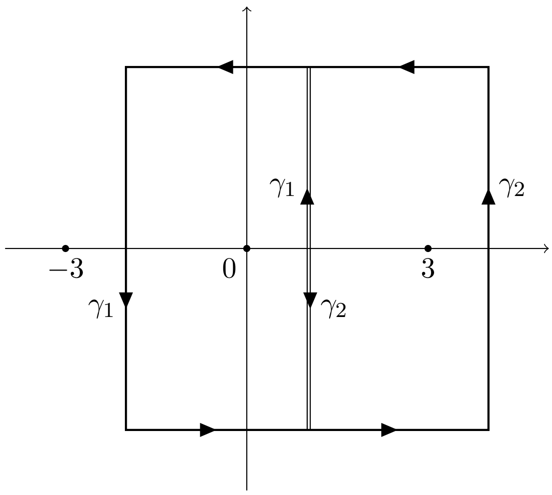


Figure 4.4: Example 4.4: Subdivision of γ into γ_1 and γ_2.

$$\begin{aligned} &= \frac{2\pi i}{18}(\mathrm{e}^3 - 2) \\ &= \frac{\pi i}{9}(\mathrm{e}^3 - 2). \end{aligned}$$

□

Having established Cauchy's Integral Formula (Theorem 4.2), we can now deduce a variety of further consequences including the properties of holomorphic listed in the introduction.

Theorem 4.5 (Liouville's Theorem) *Let f be a bounded holomorphic function on $\mathbb{C}$. Then f is constant.*

Proof: Suppose $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. Fix two points $a, b \in \mathbb{C}$. Now consider any radius R such that $R \geqslant 2\max\{|a|, |b|\}$. Let γ be the positively oriented circular contour of radius R about 0 (see Figure 4.5). Since a and b both lie inside γ, Cauchy's Integral Formula (Theorem 4.2) tells us

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z \qquad \text{and} \qquad f(b) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-b}\,\mathrm{d}z.$$

Hence

$$\begin{aligned} f(a) - f(b) &= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b}\right)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,(a-b)}{(z-a)(z-b)}\,\mathrm{d}z. \end{aligned}$$

Now if z lies on the contour γ, then

$$|z-a| \geqslant |z| - |a| = R - |a| \geqslant \tfrac{1}{2}R$$

(since $|a| \leqslant \frac{1}{2}R$) and

$$|z-b| \geqslant \tfrac{1}{2}R$$

similarly. Therefore

$$|f(a) - f(b)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(z)\,(a-b)}{(z-a)(z-b)}\,\mathrm{d}z\right|$$

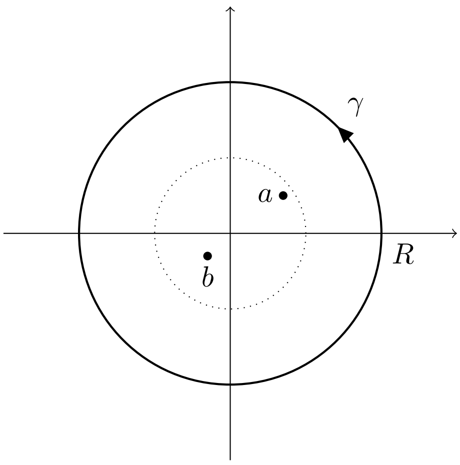


Figure 4.5: The proof of Liouville's Theorem

$$\begin{aligned}
&\leqslant \frac{1}{2\pi} \cdot \frac{M\,|a-b|}{\left(\frac{1}{2}R\right)^2} \cdot L(\gamma) \\
&= \frac{1}{2\pi} \cdot \frac{M\,|a-b|}{\frac{1}{4}R^2} \cdot 2\pi R \\
&= \frac{4M\,|a-b|}{R},
\end{aligned}$$

by the Crude Estimation Theorem (Theorem 3.14). The left-hand side is independent of R and we are permitted to take R as large as we want. Therefore, if we let $R \to \infty$, we conclude

$$|f(a) - f(b)| = 0;$$

that is,

$$f(a) = f(b).$$

We conclude that f is indeed constant. □

Liouville's Theorem means that, for example, although the function $\sin x$ is bounded when viewed as a function of a real variable, the function $\sin z$, of a complex variable, is unbounded. Indeed, if $z = iy$ (for y real), then

$$|\sin(iy)| = |\sinh y| \to \infty \qquad \text{as } y \to \infty.$$

(See Problem Sheet II, Question 10, for some background.)

We can also deduce the following important observation as a consequence of Liouville's Theorem. It is a fact about algebra, namely the roots of polynomial equations, but is most easily proved using complex analysis.

Theorem 4.6 (Fundamental Theorem of Algebra) *Let $p(z)$ be a non-constant polynomial with complex coefficients. Then there exists some $\zeta \in \mathbb{C}$ such that $p(\zeta) = 0$.*

To prove the Fundamental Theorem of Algebra, we shall make use of an important fact about continuous functions requiring a brief piece of terminology. This fact is usually established within a course on topology (for example, *MT4526 Topology*).

A subset K of $\mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus K$ is open. (Note then that the terms "closed" and "not open" have different meanings and should not be confused.) It can be shown that a subset K of $\mathbb{C}$ that is both closed and bounded is what is known as *compact* and then *a continous function $f\colon K \to \mathbb{C}$ defined on a closed and bounded subset K is bounded: there is some $M > 0$ such that*

$$|f(z)| \leqslant M \qquad \textit{for all } z \in K.$$

Examples of closed and bounded subsets of $\mathbb{C}$ include:

(i) the image γ^* of any contour (since the complement equals the union $I(\gamma) \cup E(\gamma)$ of the interior and the exterior, both of which are open);

(ii) the image of any contour together with its interior (as $\mathbb{C} \setminus (\gamma^* \cup I(\gamma)) = E(\gamma)$ is open).

The special case of (ii) that we need in the proof below is the "closed" disc $\{\, z \in \mathbb{C} \mid |z| \leqslant R \,\}$ about 0 since this is the circular contour of radius R together with its interior.

Proof of the Fundamental Theorem of Algebra: Let

$$p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

where $c_0, c_1, \dots, c_n \in \mathbb{C}$, $c_n \neq 0$ and $n \geqslant 1$. Suppose, seeking to obtain a contradiction, that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $1/p(z)$ is holomorphic on $\mathbb{C}$.

Now observe, for non-zero z, that

$$\begin{aligned} |p(z)| &= |z|^n \left| c_n + \frac{c_{n-1}}{z} + \cdots + \frac{c_1}{z^{n-1}} + \frac{c_0}{z^n} \right| \\ &\geqslant |z|^n \left(|c_n| - \frac{|c_{n-1}|}{|z|} - \cdots - \frac{|c_1|}{|z|^{n-1}} - \frac{|c_0|}{|z|^n} \right) \\ &\to \infty \qquad \text{as } |z| \to \infty, \end{aligned}$$

using repeated use of the Triangle Inequality (Theorem 1.2). Therefore there exists some $R > 0$ such that

$$\left| \frac{1}{p(z)} \right| \leqslant 1 \qquad \text{if } |z| \geqslant R.$$

Now since $1/p(z)$ is continuous, it is bounded on the set $K = \{\, z \in \mathbb{C} \mid |z| \leqslant R \,\}$. Hence

$$\left| \frac{1}{p(z)} \right| \leqslant M \qquad \text{if } |z| \leqslant R.$$

Putting this together, we conclude that $1/p(z)$ is bounded on $\mathbb{C}$:

$$\left| \frac{1}{p(z)} \right| \leqslant \max\{1, M\} \qquad \text{for all } z \in \mathbb{C}.$$

Hence, Liouville's Theorem (Theorem 4.5) tells us that $1/p(z)$ is constant, which contradicts the assumption that $p(z)$ is a non-constant polynomial.

We conclude that there must exist some $\zeta \in \mathbb{C}$ such that $p(\zeta) = 0$. $\square$

Cauchy's Formula for Derivatives and applications

Theorem 4.7 (Cauchy's Formula for Derivatives) *Let f be a holomorphic function on the open set U. Then, for any natural number n, f has an nth derivative $f^{(n)}$ on U given by*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

for any positively oriented contour γ that, together with its interior, is contained in U and such that a lies in the interior of γ.

Once we have established this result, we have fulfilled the promise that we would show that a holomorphic function can be differentiated as many times as wanted.

PROOF: We proceed by induction on n. The case $n = 0$ is Cauchy's Integral Formula (Theorem 4.2). Let us then assume $n \geqslant 1$, that $f^{(n-1)}$ exists on U and is given by the formula in the statement (of course, replacing n by $n-1$ in that formula). Let γ be any contour that, together with its interior, is contained in U and let a be a point in the interior of γ. Now if h is small enough that $a+h$ also lies in the interior of γ, put

$$\begin{aligned} E(h) &= \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} - \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z \\ &= \frac{(n-1)!}{2\pi i h}\int_\gamma f(z)\left(\frac{1}{(z-a-h)^n} - \frac{1}{(z-a)^n} - \frac{nh}{(z-a)^{n+1}}\right)\mathrm{d}z \\ &= \frac{(n-1)!}{2\pi i h}\int_\gamma f(z)\, q(z,h)\,\mathrm{d}z, \end{aligned}$$

where

$$q(z,h) = \frac{1}{(z-a-h)^n} - \frac{1}{(z-a)^n} - \frac{nh}{(z-a)^{n+1}}.$$

Choose r such that the positively oriented circular contour γ_r of radius $2r$ about a is contained in the interior of γ. Assume $|h| < r$. Then, by the Deformation Theorem (Theorem 4.1),

$$E(h) = \frac{(n-1)!}{2\pi i h}\int_{\gamma_r} f(z)\left(\frac{1}{(z-a-h)^n} - \frac{1}{(z-a)^n} - \frac{nh}{(z-a)^{n+1}}\right)\mathrm{d}z.$$

Consider the line segment from a to $a+h$, which (borrowing notation from that used for certain subsets of the real line) we shall denote here by $[a, a+h]$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{(z-w)^n}\right) = \frac{n}{(z-w)^{n+1}},$$

so by the Fundamental Theorem of Calculus for integrals along curves (Theorem 3.12),

$$\int_{[a,a+h]} \frac{n}{(z-w)^{(n+1)}}\,\mathrm{d}w = \frac{1}{(z-a-h)^n} - \frac{1}{(z-a)^n}$$

for any z not on the line segment $[a, a+h]$ (including, for example, any $z \in \gamma_r^*$). Thus

$$\begin{aligned} q(z,h) &= \int_{[a,a+h]} \left(\frac{n}{(z-w)^{n+1}} - \frac{n}{(z-a)^{n+1}}\right)\mathrm{d}w \\ &= n\int_{[a,a+h]} \left(\frac{1}{(z-w)^{n+1}} - \frac{1}{(z-a)^{n+1}}\right)\mathrm{d}w. \end{aligned}$$

(Note that the second term in the integrand is constant (independent of w) so when we integrate it along $[a, a+h]$ the effect is just to multiply by $(a+h) - a = h$.)

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{(z-\zeta)^{n+1}}\right) = \frac{n+1}{(z-\zeta)^{n+2}},$$

so

$$\int_{[a,w]} \frac{n+1}{(z-\zeta)^{n+2}}\,\mathrm{d}\zeta = \frac{1}{(z-w)^{n+1}} - \frac{1}{(z-a)^{n+1}}$$

for z not on the line segment $[a, w]$. Hence

$$q(z,h) = n(n+1)\int_{[a,a+h]}\int_{[a,w]} \frac{1}{(z-\zeta)^{n+2}}\,\mathrm{d}\zeta\,\mathrm{d}w$$

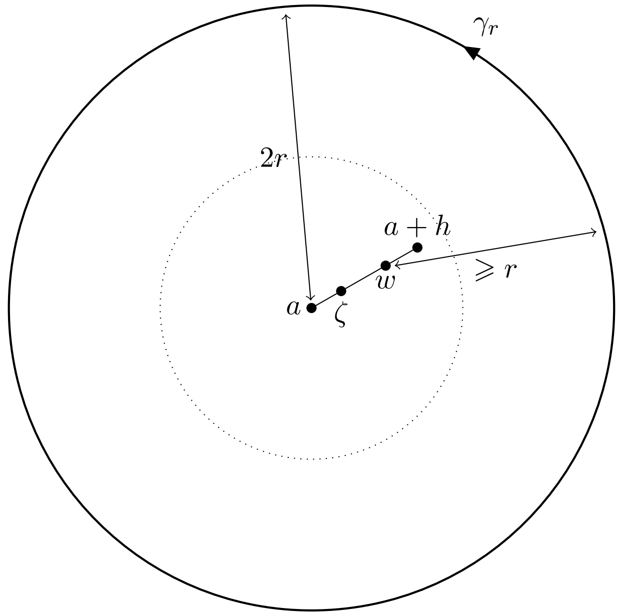


Figure 4.6: Arrangement of the variables in the proof of Cauchy's Formula for Derivatives.

and so

$$E(h) = \frac{(n+1)!}{2\pi i h} \int_{\gamma_r} \int_{[a,a+h]} \int_{[a,w]} \frac{f(z)}{(z-\zeta)^{n+2}} \, \mathrm{d}\zeta \, \mathrm{d}w \, \mathrm{d}z.$$

(See Figure 4.6 for an illustration of the arrangement of $a+h$, w on the line segment $[a, a+h]$ between a and $a+h$, and ζ on the line segment $[a, w]$ between a and w.)

We now apply the Crude Estimation Theorem (Theorem 3.14) to bound the function $E(h)$. Note that if z lies on γ_r^*, w lies on the line segment $[a, a+h]$ and ζ lies on the line segment $[a, w]$, then

$$|z-\zeta| \geqslant r, \qquad L([a,w]) = |w-a| \leqslant |h|, \qquad \text{and} \qquad L([a,a+h]) = |h|,$$

and f is bounded on γ_r^* (as this image γ_r^* is a closed and bounded subset of $\mathbb{C}$), say

$$|f(z)| \leqslant M \qquad \text{for } z \text{ on } \gamma_r^*.$$

Hence, by the Crude Estimation Theorem,

$$\begin{aligned} |E(h)| &\leqslant \frac{(n+1)!}{2\pi |h|} \cdot 4\pi r \cdot |h|^2 \cdot \frac{M}{r^{n+2}} \\ &= \frac{2\,(n+1)!\,M}{r^{n+1}} \, |h| . \end{aligned}$$

Therefore

$$E(h) \to 0 \qquad \text{as } h \to 0;$$

that is,

$$\lim_{h \to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z,$$

so f is n-times differentiable at a and $f^{(n)}(a)$ is given by the claimed formula. □

Example 4.8 *Evaluate the integral*

$$\int_\gamma \frac{\mathrm{e}^{z^2}}{(z-1)^2} \, \mathrm{d}z$$

where γ is a positively oriented circular contour of radius 2 about 0.

Solution: Write $f(z) = \mathrm{e}^{z^2}$. Note that f is holomorphic on $\mathbb{C}$ and that 1 lies inside the contour γ. Hence, by Cauchy's Formula for Derivatives,

$$f'(1) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-1)^2} \, \mathrm{d}z.$$

By the Chain Rule, $f'(z) = 2z\,\mathrm{e}^{z^2}$, so

$$\int_\gamma \frac{f(z)}{(z-1)^2} \, \mathrm{d}z = 2\pi i \cdot 2\mathrm{e} = 4\pi \mathrm{e} i.$$

$\square$

Theorem 4.9 (Taylor's Theorem) *Suppose f is a holomorphic function on an open disc $B(a,r)$ for some $a \in \mathbb{C}$ and some $r > 0$. Then*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for all $z \in B(a,r)$, where each c_n is given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z = \frac{f^{(n)}(a)}{n!}$$

for any contour γ contained inside $B(a,r)$ with a in its interior.

We know already (see Theorem 2.16) that any power series is holomorphic inside its radius of convergence. Taylor's Theorem provides a converse to that theorem: it says that every function f that is holomorphic on some open set U can be expressed as a power series on any open disc inside U. In this context, we mention the following terminology:

Definition 4.10 A function f is said to be *analytic* on an open set U if f is given by a power series in every open disc inside U.

Putting together Theorems 2.16 and 4.9, we observe:

Corollary 4.11 *A function $f \colon U \to \mathbb{C}$ is holomorphic on an open set U if and only if it is analytic on U.*

This tells us that the terms "holomorphic" and "analytic" are essentially equivalent for functions of a complex variable. This explains why many sources use the terms interchangeable (including the Course Catalogue description for this module!).

Proof of Taylor's Theorem: Note first that the fact the formula for c_n given by the above integral equals $f^{(n)}(a)/n!$ follows by Cauchy's Formula for Derivatives (Theorem 4.7). To tidy up and simplify the remained of the proof, apply the translation mapping a to 0 and replace $f(z)$ by $f(z+a)$. Thus, we can assume that $a = 0$, so that f is holomorphic on $B(0,r)$. Fix some $z \in B(0,r)$ (so $|z| < r$) and we claim that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

for $z \in B(0,r)$, where each coefficient is given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{n+1}} \, \mathrm{d}w$$

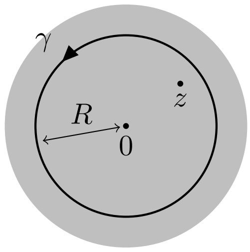


Figure 4.7: The contour used to prove of Taylor's Theorem.

and, by application(s) of the Deformation Theorem (Theorem 4.1), we assume that γ is a positively oriented circular contour about 0 of radius R with $|z| < R < r$. (See Figure 4.7.) We shall then use Cauchy's Integral Formula (Theorem 4.2) to note

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, \mathrm{d}w.$$

We use the formula for the sum of a geometric progression to observe

$$1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots + \left(\frac{z}{w}\right)^N = \frac{1-(z/w)^{N+1}}{1-(z/w)},$$

so

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-(z/w)} = \frac{1}{w}\left(\sum_{n=0}^{N} \left(\frac{z}{w}\right)^n + \frac{(z/w)^{N+1}}{1-(z/w)}\right).$$

Substituting this into our expression from Cauchy's Integral Formula, we obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{N} \left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\,\mathrm{d}w\right) z^n + R_N(z) \\ &= \sum_{n=0}^{N} c_n z^n + R_N(z), \end{aligned} \tag{4.1}$$

where the remainder term is given by

$$R_N(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,(z/w)^{N+1}}{w-z}\,\mathrm{d}w.$$

Now f is bounded on the contour γ (as γ^* is closed and bounded), say

$$|f(w)| \leqslant M \qquad \text{for all } w \in \gamma^*,$$

while $|w| = R$ and

$$|w-z| \geqslant |w| - |z| = R - |z|$$

for $w \in \gamma^*$. Hence, by the Crude Estimation Theorem (Theorem 3.14),

$$\begin{aligned} |R_N(z)| &\leqslant \frac{1}{2\pi} \cdot \frac{M}{R-|z|} \left(\frac{|z|}{R}\right)^{N+1} \cdot 2\pi R \\ &= \frac{RM}{R-|z|}\left(\frac{|z|}{R}\right)^{N+1} \\ &\to 0 \qquad \text{as } N \to \infty \end{aligned}$$

since $|z|/R < 1$. Hence, letting $N \to \infty$ in Equation (4.1) gives

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

as required. $\square$

Chapter 5

Interlude: Harmonic functions

In this section, we present an application of the theory developed, or perhaps more accurately how the theory fits within a particular branch of applied mathematics. What we present links most closely to the Cauchy–Riemann Equations (Theorem 2.8), but some of the observations made in the previous two chapters are also important here.

Consider a holomorphic function $f\colon U \to \mathbb{C}$ defined upon some subset U of $\mathbb{C}$. As previously, we take real and imaginary parts, so write

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

to define two real-valued functions $u, v\colon \tilde{U} \to \mathbb{R}$ defined upon the subset

$$\tilde{U} = \{\, (x,y) \in \mathbb{R}^2 \mid x+iy \in U \,\}$$

of $\mathbb{R}^2$. We know that u and v satisfy the Cauchy–Riemann Equations at every point of $\tilde{U}$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We have also observed that f possesses nth derivatives for all $n \geqslant 1$ (see Cauchy's Formula for Derivatives, Theorem 4.7), so f' can be differentiated. We can therefore differentiate the above equations further:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x\,\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y\,\partial x}.$$

Moreover, f'' can be differentiated, so is certainly continuous. Therefore

$$\frac{\partial^2 v}{\partial x\,\partial y} \qquad \text{and} \qquad \frac{\partial^2 v}{\partial y\,\partial x}$$

both exist and are continuous. This last fact is a sufficient condition for the two mixed second-order derivatives are equal:

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$$

Putting this together, we conclude

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

that is, u satisfies *Laplace's equation.* The same argument applies to the imaginary part v of the holomorphic function f.

In this context, we make the following definition.

Definition 5.1 Let V be an open subset of $\mathbb{R}^2$. A function $u\colon V \to \mathbb{R}$, defined upon V, is said to be *harmonic* on V if

(i) u has continuous second-order partial derivatives on V, and

(ii) u satisfies *Laplace's equation*
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The operator
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
is often by denoted by ∇^2 or by Δ (the choice between the two appears to be individual whim), so Laplace's equation can also be written as
$$\nabla^2 u = 0 \qquad \text{or} \qquad \Delta u = 0.$$

Harmonic functions appear in many areas of applied mathematics, such as electromagnetism, fluid dynamics, etc. In this chapter, however, we concentrate on the link between harmonic functions and holomorphic functions. One direction in the link has already been observed:

Theorem 5.2 *Let $f\colon U \to \mathbb{C}$ be a holomorphic function defined upon some open subset U of $\mathbb{C}$. Write*
$$f(x+iy) = u(x,y) + i\,v(x,y)$$
to define functions $u, v\colon \tilde{U} \to R$ where the domain $\tilde{U} = \{\,(x,y) \in \mathbb{R}^2 \mid x+iy \in U\,\}$ is an open subset of $\mathbb{R}^2$. Then u and v are harmonic functions on $\tilde{U}$. □

We establish a weaker version than a full converse to the above theorem. There is a more general converse, for a harmonic function defined on an arbitrary open subset of $\mathbb{R}^2$, but the proof is a bit more challenging (as we need to consider integrals along more complicated curves) but the same ideas work.

Theorem 5.3 *Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, $r > 0$ and suppose that $u\colon B(\mathbf{a}, r) \to \mathbb{R}$ is a harmonic function on the open disc $B(\mathbf{a}, r) = \{\,\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{a}| < r\,\}$. Then there exists a holomorphic function $f\colon B(c, r) \to \mathbb{C}$ (where $c = a_1 + ia_2$) such that u is the real part of f:*
$$u(x,y) = \operatorname{Re} f(x+iy) \qquad \text{for } (x,y) \in B(\mathbf{a}, r).$$

Proof: Write $D = B(c, r)$ for the open disc in $\mathbb{C}$ that corresponds to the original open disc $B(\mathbf{a}, r)$. Define $g\colon D \to \mathbb{C}$ by
$$g(x+iy) = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}.$$

(The idea here is that g is a function that would equal the derivative of the function f that we seek.) The real and imaginary parts of g are given by
$$\tilde{u} = \frac{\partial u}{\partial x} \qquad \text{and} \qquad \tilde{v} = -\frac{\partial u}{\partial y}.$$

Let us calculate the partial derivates of $\tilde{u}$ and $\tilde{v}$:

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial \tilde{u}}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \\
\frac{\partial \tilde{v}}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial x\,\partial y} \\
\frac{\partial \tilde{v}}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Now, by hypothesis, u is a harmonic function, so its second-order partial derivatives are continuous and it satisfies Laplace's equation. From this we conclude

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}.$$

In conclusion, $\tilde{u}$ and $\tilde{v}$ satisfy the Cauchy–Riemann Equations,

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{v}}{\partial y} \qquad \text{and} \qquad \frac{\partial \tilde{v}}{\partial x} = -\frac{\partial \tilde{u}}{\partial y},$$

while these partial derivatives are continuous since the second-order partial derivatives of u are, by assumption, continuous. Our partial converse to the Cauchy–Riemann Equations (Theorem 2.12) tells us that g is holomorphic on D. However, we would like to show that g is the derivative of a holomorphic function. To do this, define a new function $F\colon D \to \mathbb{C}$ by

$$F(z) = \int_{[a,z]} g(w)\,\mathrm{d}w$$

where $[a,z]$ denotes the line segment from a to z (which is contained within the disc D). We shall show that F is holomorphic.

Let $z \in D$ and let $\varepsilon > 0$. Since g is, in particular, continuous there exists some $\delta > 0$ such that $B(z,\delta) \subseteq D$ and such that $|w-z| < \delta$ implies

$$|g(w) - g(z)| < \tfrac{1}{2}\varepsilon.$$

Now by Cauchy's Theorem applied to the holomorphic function g (in fact, the version for a triangle, Theorem 3.18, is sufficient)

$$\int_{[a,z]} g(w)\,\mathrm{d}w + \int_{[z,z+h]} g(w)\,\mathrm{d}w - \int_{[a,z+h]} g(w)\,\mathrm{d}w = 0$$

(see Figure 5.1). Hence

$$\begin{aligned}
\frac{F(z+h) - F(z)}{h} - g(z) &= \frac{1}{h}\int_{[z,z+h]} g(w)\,\mathrm{d}w - g(z) \\
&= \frac{1}{h}\int_{[z,z+h]} \big(g(w) - g(z)\big)\,\mathrm{d}w
\end{aligned}$$

(since the value $g(z)$ is constant as w varies along the line segment $[z,z+h]$). By the Crude Estimation Theorem,

$$\left|\frac{F(z+h) - F(z)}{h} - g(z)\right| \leqslant \frac{1}{|h|} \cdot \tfrac{1}{2}\varepsilon \cdot |h| < \varepsilon$$

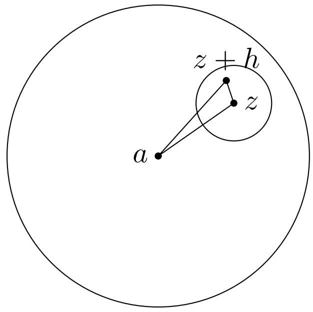


Figure 5.1: Proof of Theorem 5.3: Integrating around a triangle

when $|h| < \delta$ (since then $|w - z| < \delta$ for all w on the line segment $[z, z+h]$). It follows that

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = g(z).$$

In conclusion, F is a holomorphic function on D with derivative equal to g. The derivative of F therefore equals

$$F'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

Write $\hat{u}$ for the real part of F. Since F satisfies the Cauchy–Riemann Equations, we conclude

$$\frac{\partial \hat{u}}{\partial x} = \operatorname{Re} F'(x+iy) = \frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial \hat{u}}{\partial y} = -\operatorname{Im} F'(x+iy) = \frac{\partial u}{\partial y}.$$

Hence $\hat{u} - u$ has partial derivatives that are zero on D, so we conclude that $\hat{u} - u$ is constant, say $\hat{u}(x,y) = u(x,y) + k$ for some constant k. Then $f(z) = F(z) - k$ is the required holomorphic function on D such that the real part of $f(x+iy)$ equals $u(x,y)$. $\square$

Definition 5.4 If $u(x,y)$ is a harmonic function, then a function $v(x,y)$ such that

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

is holomorphic, with real part u, is called a *harmonic conjugate* of u.

The previous theorem tells us that, at least on an open disc, a harmonic function always possesses a harmonic conjugate. As noted before the theorem, the result exists in greater generality so harmonic functions do indeed always possess harmonic conjugates. The proof given does not give us a pleasant way to find the harmonic conjugate of a harmonic funciton, since it tells us to integrate along a curve to find the required holomorphic function. Instead, the best method to employ is to solve the Cauchy–Riemann Equations.

Example 5.5 *Consider the function* $u \colon \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$u(x,y) = x^3 - 3xy^2 - 2y.$$

(i) *Show that* u *is harmonic on* $\mathbb{R}^2$.

(ii) *Find a harmonic conjugate of* u.

(iii) *Find a holomorphic function* $f(z)$*, expressed as a function of a single complex variable* z*, such that* $u(x,y)$ *is the real part of* $f(x+iy)$.

SOLUTION: (i) We calculate the following partial derivatives of u:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial u}{\partial y} = -6xy - 2$$

and

$$\frac{\partial^2 u}{\partial x^2} = 6x, \qquad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = -6y, \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

Observe that the second-order partial derivatives are continuous and satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic on $\mathbb{R}$.

(ii) If u and v are the real and imaginary parts of a holomorphic function f, then they satisfy the Cauchy–Riemann Equations. Thus

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2,$$

so, upon integrating with respect to y,

$$v(x, y) = 3x^2y - y^3 + g(x)$$

for some function $g(x)$ of x alone (i.e., independent of y). Now substitute into $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$:

$$6xy + g'(x) = 6xy + 2$$

Hence $g'(x) = 2$ and we conclude $g(x) = 2x + c$ for some constant c. Any constant c will work here, since we are just adding a constant to the holomorphic function f, so we shall take $c = 0$. Thus

$$v(x, y) = 2x + 3x^2y - y^3$$

is a harmonic conjugate of $u(x, y)$.

(iii) For $v(x, y)$ as above,

$$\begin{aligned} f(x + iy) &= u(x, y) + i\,v(x, y) \\ &= x^3 - 3xy^2 - 2y + i(2x + 3x^2y - y^3) \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 + 2i(x + iy) \\ &= (x + iy)^3 + 2i(x + iy); \end{aligned}$$

that is,

$$f(z) = z^3 + 2iz.$$

□

Chapter 6

Singularities, Poles and Residues

We have spent considerable time considering functions $f: U \to \mathbb{C}$ that are holomorphic on an open set U. We now turn to understanding in greater depth the behaviour of functions f that are holomorphic on essentially a large proportion of an open set U but where the functions fail to be differentiable for some reason at some of the points in U. The term *singularity* is used for the points at which a function $f: U \to \mathbb{C}$ is not holomorphic, but there is some inconsistency in sources relating to this terminology. Some use singularity to mean simply a point at which f is not holomorphic, while others require a singularity to be a limit of a sequence of points at which f *is* holomorphic. In view of this, we shall generally avoid using the term "singularity" but make good use of the following precise term, which all sources seem to agree upon.

Definition 6.1 Let $f: D \to \mathbb{C}$ be a function of a complex variable defined upon some subset D of $\mathbb{C}$. A point $a \in \mathbb{C}$ is called an *isolated singularity* of f if there exists some $r > 0$ such that f is defined and holomorphic on the punctured disc

$$B'(a,r) = \{ z \in \mathbb{C} \mid 0 < |z-a| < r \},$$

but f is not differentiable at a.

So a function f has an isolated singularity at a if it is holomorphic on an open disc about a *except* either it fails to be differentiable at a or is simply not defined at a.

Laurent's Theorem

The purpose of this chapter is to understand the behaviour of functions around isolated singularities and, in particular, what happens when we integrate such functions around contours. The first observation is the following:

Theorem 6.2 (Laurent's Theorem) *Let* $0 \leqslant R < S \leqslant \infty$ *and*

$$A = \{ z \in \mathbb{C} \mid R < |z-a| < S \},$$

an open annulus centred on $a \in \mathbb{C}$*. Assume* f *is holomorphic on* A*. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

for all $z \in A$*, where the coefficients are given by*

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w$$

for all integers n*, where* γ *is a positively oriented circular contour of some radius about* a *whose image is contained inside* A*.*

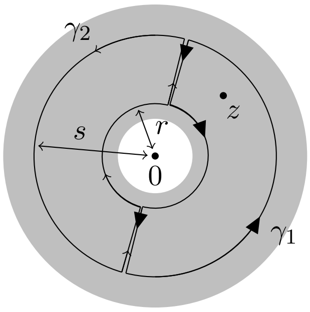


Figure 6.1: The contours in the proof of Laurent's Theorem

When we take $R = 0$ in Laurent's Theorem, the result then applies to the case that f has an isolated singularity at a. Accordingly we make the following definition:

Definition 6.3 Let f be a function of a complex variable with an isolated singularity at a point $a \in \mathbb{C}$. The *Laurent series* (or *Laurent expansion*) of f at a is a (doubly infinite) series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

valid for all z in some punctured disc $B'(a,r) = B(a,r) \setminus \{a\}$ about a.

Laurent's Theorem tells us that such a Laurent series always exists. To interpret what the doubly infinite series means, one views

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{m=1}^{\infty} c_{-m} (z-a)^{-m}$$

as the sum of two series, one that involves powers of $(z-a)$ and the other involving powers of $(z-a)^{-1}$. Equivalently, the Laurent series equals

$$\lim_{M,N\to\infty} \sum_{n=-M}^{N} c_n (z-a)^n$$

where, in the limit, $M, N \to \infty$ independently of each other.

Proof of Laurent's Theorem (Sketch): As in the proof of Taylor's Theorem (Theorem 4.9), first translate, purely for notational convenience, so that we can assume that f is holomorphic on the annulus

$$A = \{ z \in \mathbb{C} \mid R < |z| < S \}$$

centred on 0; that is, we assume $a = 0$. Fix $z \in A$ and choose radii r and s with $R < r < |z| < s < S$. Consider the contours γ_1 and γ_2 as shown in Figure 6.1.

Since f is holomorphic on A, the interior of γ_1 and γ_2 both lie inside A, and z lies in the interior of γ_1 but not γ_2, Cauchy's Integral Formula (Theorem 4.2) says

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} \, \mathrm{d}w$$

while

$$0 = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z} \, \mathrm{d}w$$

by Cauchy's Theorem (Theorem 3.16). If we add the two together and note that the integrals along the line segments cancel, we conclude

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w-z} \, \mathrm{d}w - \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z} \, \mathrm{d}w$$

where γ_r and γ_s are *positively oriented* circular contours of radii r and s, respectively, about 0. We use the formulae

$$\frac{1}{w-z} = \frac{1}{w} \left(\frac{1}{1-(z/w)} \right) = \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w} \right)^n \qquad \text{when } |z| < |w|$$

and similarly

$$\frac{1}{w-z} = -\frac{1}{z} \sum_{m=0}^{\infty} \left(\frac{w}{z} \right)^m \qquad \text{when } |z| > |w|$$

to conclude

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w} \right)^n \mathrm{d}w + \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{z} \sum_{m=0}^{\infty} \left(\frac{w}{z} \right)^m \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\gamma_s} \left(\sum_{n=0}^{\infty} \frac{f(w)}{w^{n+1}} z^n \right) \mathrm{d}w + \frac{1}{2\pi i} \int_{\gamma_r} \left(\sum_{m=0}^{\infty} f(w)\, w^m\, z^{-m-1} \right) \mathrm{d}w. \end{aligned}$$

(Note that $|z| < |w| = s$ when $w \in \gamma_s^*$ and $|z| > |w| = r$ when $w \in \gamma_r^*$.) Now interchange the integral and the sums:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w^{n+1}} \, \mathrm{d}w \right) z^n + \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} f(w)\, w^m \, \mathrm{d}w \right) z^{-m-1} \\ &= \sum_{n=-\infty}^{\infty} c_n z^n, \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w^{n+1}} \, \mathrm{d}w \qquad \text{for } n \geqslant 0$$

and

$$c_n = \frac{1}{2\pi i} \int_{\gamma_s} f(w)\, w^{-(n+1)} \, \mathrm{d}w \qquad \text{for } n \leqslant -1.$$

Finally use of the Deformation Theorem (Theorem 4.1) shows that

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{n+1}} \, \mathrm{d}w$$

for both formulae, where γ is a positively oriented circular contour of any radius about 0 within A, as in the statement of the theorem. □

Why is this a sketch proof? The answer is that the interchange of the integral and the summation requires some justification. To fully justify it one typically observes the two series converge *uniformly* on the images γ_r^* and γ_s^* of the contours, respectively. The topic of uniform convergence is covered elsewhere (namely *MT3502 Real Analysis*) so we omit the details.

Theorem 6.4 *If f is holomorphic on an open annulus $A = \{ z \in \mathbb{C} \mid R < |z-a| < S \}$, the coefficients in any Laurent series for f at a are uniquely determined by f.*

What this theorem means is that if we manage to find a valid Laurent series for f that holds in the annulus A, then it is the unique one we were looking for. The proof basically involves taking possible Laurent series for f, integrating $f(z)/(z-a)^{n+1}$ around the contour γ as in Laurent's Theorem and observing that out drops (a suitable multiple of) a specific coefficient in the series we started with. We need to interchange integration and summation, and again this depends upon uniform convergence. In view of this, we omit the proof of Theorem 6.4 since it depends entirely on material outside the module.

Classifying isolated singularities

The use of the Laurent series enables us to make a set of definitions to classify the singularities of a function f.

Definition 6.5 Let f be a function of a complex variable that is holomorphic on a punctured disc $B'(a,r)$ with an isolated singularity at a. Let

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be the Laurent series of f at a. Then

(i) f has a *removable singularity* at a if $c_n = 0$ for all negative n;

(ii) f has a *pole of order m* at a (for some positive integer m) if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$;

(iii) f has an *isolated essential singularity* at a if there is no m such that $c_n = 0$ for all $n < -m$.

We also use the terms

- *simple pole* for a pole of order 1,
- *double pole* for a pole of order 2,
- *triple pole* for a pole of order 3, etc.

A function f is said to be *meromorphic* on an open set U if it is holomorphic on U except for a collection of poles.

Example 6.6 (i) Consider the function

$$f(z) = \frac{1}{(z-1)^3}.$$

This is holomorphic on $\mathbb{C} \setminus \{1\}$. It has an isolated singularity at 1. Its Laurent series expresses f as a series in powers of $(z-1)$. However, it is already given as a series in powers of $(z-1)$, so the Laurent series is

$$f(z) = \cdots + 0 \cdot (z-1)^{-4} + 1 \cdot (z-1)^{-3} + 0 \cdot (z-1)^{-2} + 0 \cdot (z-1)^{-1} \\ + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \dots .$$

Hence f has a triple pole at $z = 1$.

(ii) Consider the function

$$g(z) = \frac{\sin z}{z}.$$

This is holomorphic on $\mathbb{C} \setminus \{0\}$, but not defined at 0. Taking the power series for $\sin z$,

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

and dividing by z gives the Laurent series for $(\sin z)/z$ at 0:

$$\begin{aligned} \frac{\sin z}{z} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} \\ &= 1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - \frac{1}{7!} z^6 + \dots, \end{aligned}$$

for all $z \in \mathbb{C} \setminus \{0\}$. Thus g has a removable singularity at $z = 0$.

(iii) Consider the function

$$h(z) = \cos\left(\frac{1}{z}\right).$$

This is holomorphic on $\mathbb{C} \setminus \{0\}$. We find a Laurent series by substituting $1/z$ into the power series for $\cos z$:

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{z}\right)^{2n} \\ &= \dots - \frac{1}{6!} z^{-6} + \frac{1}{4!} z^{-4} - \frac{1}{2!} z^{-2} + 1, \end{aligned}$$

for all $z \in \mathbb{C} \setminus \{0\}$. Thus h has an essential isolated singularity at $z = 0$.

(iv) Consider

$$\cot z = \frac{\cos z}{\sin z}.$$

Then $\cot z$ is holomorphic at all points where $\sin z \neq 0$; that is, $\cot z$ is holomorphic on

$$\mathbb{C} \setminus \{\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, \dots\}.$$

In particular, $\cot z$ is holomorphic on the punctured disc $B'(0, \pi)$ of radius π about 0. To calculate a Laurent expansion, we expand the series for $\cos z$ and $\sin z$:

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \dots \\ \sin z &= z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right) \\ &= z(1 - r) \end{aligned}$$

where $r = \frac{z^2}{6} - \frac{z^4}{120} + \dots$. Note that if z is sufficiently close to 0, then $|r| < 1$ and then

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots.$$

Hence

$$\cot z = \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right)\left(1 + \left(\frac{z^2}{6} - \frac{z^4}{120} + \dots\right) + \left(\frac{z^4}{36} - \dots\right) + \dots\right)$$

$$= \frac{1}{z}\left(1 - \frac{z^2}{3} + (\text{terms of degree} \geqslant 4)\right)$$
$$= \frac{1}{z} - \frac{z}{3} + (\text{terms of degree} \geqslant 3).$$

This establishes the first two terms of the Laurent series for $\cot z$ at $z = 0$. Hence $\cot z$ has a simple pole at 0.

Removing a removable singularity: If f has a removable singularity at a point a, then there is some punctured open disc $B'(a, r)$ for which f is holomorphic on the punctured disc and has a Laurent series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

valid on this punctured disc. However, this series is actually a power series and converges not only on the punctured disc $B'(a, r)$ but also at the point $z = a$ (since all terms, except possibly the first, are zero). Hence the power series $\sum_{n=0}^{\infty} c_n (z - a)^n$ converges on the (non-punctured) open disc $B(a, r)$ and so defines a holomorphic function on this disc. As a consequence, if we *redefine* the original function f by specifying that $f(a) = c_0$ then

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

for all $z \in B(a, r)$ and now f is a holomorphic function on the whole open disc $B(a, r)$. This process is often called "removing the singularity" at a: it has the effect of taking a function f with a removable singularity at a point a, redefining the value $f(a)$ and obtaining a function that is now holomorphic in an open set containing a.

Cauchy's Residue Theorem

To understand how to integrate a function f around an isolated singularity, we need the following definition.

Definition 6.7 Suppose that the function f has an isolated singularity at a point a and Laurent series

$$\sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

at a. The coefficient c_{-1} appearing in this Laurent series is called the *residue* of f at a. We write

$$\operatorname{res}(f, a)$$

for this residue.

Example 6.8 If we manage to find the Laurent series, we can simply read off the residue from the coefficients:

$$\operatorname{res}\left(\frac{1}{(z-1)^3}, 1\right) = 0$$
$$\operatorname{res}\left(\cos\left(\frac{1}{z}\right), 0\right) = 0$$
$$\operatorname{res}(\cot z, 0) = 1$$

according to the calculations in Example 6.6.

It turns out that it is the residue of a function f that determines the value when we integrate f around a contour that encircles an isolated singularity.

Proposition 6.9 *Suppose that f is holomorphic on some punctured disc $B'(a,r)$, where $a \in \mathbb{C}$ and $r > 0$. Let γ be a positively oriented circular contour of radius R (where $0 < R < r$) about a. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}(f,a).$$

Sketch Proof: First note that, for any integer n,

$$\int_\gamma (z-a)^n\,\mathrm{d}z = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1, \end{cases}$$

using the Fundamental Theorem of Calculus for integrals along a curve for $n \neq -1$ (Theorem 3.12) and a generalisation of Example 3.9 (or Cauchy's Integral Formula if preferred) for $n = -1$. By Laurent's Theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

for some coefficients c_n. Then

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma \sum_{n=-\infty}^{\infty} c_n (z-a)^n\,\mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} c_n \int_\gamma (z-a)^n\,\mathrm{d}z \\ &= 2\pi i\, c_{-1} \\ &= 2\pi i \operatorname{res}(f,a). \end{aligned}$$

The interchange of summation and integral requires justification. It can be established using uniform convergence of the Laurent series on the image γ^* of the contour, but as before the details are omitted as uniform convergence is covered in another module. □

We have hidden the hard part of the most important result of this chapter within the omitted step of the above proposition. As a consequence, this theorem, as follows, is now easy to establish.

Theorem 6.10 (Cauchy's Residue Theorem) *Let f be holomorphic on an open set containing a positively oriented contour γ and its interior, except for finitely many isolated singularities $a_1, a_2, \dots, a_k$ in the interior of γ. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{k} \operatorname{res}(f,a_j).$$

Proof: Since the singularities $a_1, a_2, \dots, a_k$ are isolated, we can choose small radii $r_1, r_2, \dots, r_k$ such that the positively oriented circular contours γ_j of radius r_j about a_j are disjoint and contained inside γ. Insert curves that join these circular contours to the original contour γ (as shown in Figure 6.2) to create two new contours δ_1 and δ_2 that together involve all the pieces of γ and γ_j (for $1 \leqslant j \leqslant k$) and such that the singularities $a_1, a_2, \dots, a_k$ are in the exteriors of both δ_1 and δ_2. Hence, by Cauchy's Theorem,

$$\int_{\delta_1} f(z)\,\mathrm{d}z = \int_{\delta_2} f(z)\,\mathrm{d}z = 0.$$

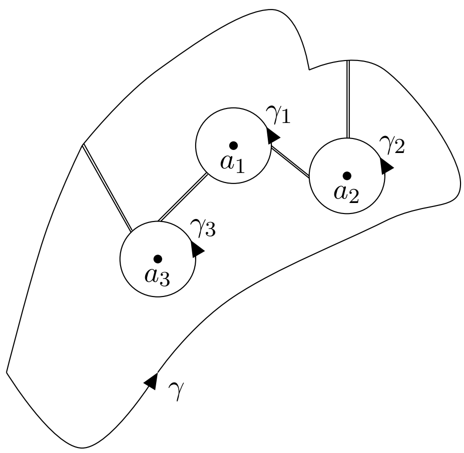


Figure 6.2: Proof of Cauchy's Residue Theorem

Note that the curves joining the contours are traced once in each direction, so upon adding, the integrals along these curves cancel and we obtain

$$\int_\gamma f(z)\,\mathrm{d}z - \sum_{j=1}^{k} \int_{\gamma_j} f(z)\,\mathrm{d}z = 0.$$

Hence

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} \int_{\gamma_j} f(z)\,\mathrm{d}z = \sum_{j=1}^{k} 2\pi i \operatorname{res}(f, a_j),$$

using Proposition 6.9. This establishes the theorem. □

Calculating residues

In view of Cauchy's Residue Theorem, one can see that calculating residues is important for determining integrals.

Method 0: Find a Laurent series for our function f about an isolated singularity a by brute force and then read off the residue from the coefficient of $(z-a)^{-1}$ in this series.

This basic method is how we found the residues in Example 6.8. However, in certain circumstances we can use other more straightforward methods.

Lemma 6.11 (Type I, Multiple Pole) *Suppose*

$$f(z) = \frac{g(z)}{(z-a)^m}$$

where g is holomorphic in some open disc $B(a,r)$ about a and $g(a) \neq 0$. Then f has a pole of order m at a and

$$\operatorname{res}(f, a) = \frac{1}{(m-1)!}\, g^{(m-1)}(a).$$

In particular, when $m = 1$, we see that $g(z)/(z-a)$ has a simple pole at a when $g(a) \neq 0$ and the residue in that case is

$$\operatorname{res}\left(\frac{g(z)}{z-a}, a\right) = g(a).$$

PROOF: Since g is holomorphic in $B(a,r)$, it has a power series representation

$$g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for $z \in B(a,r)$, by Taylor's Theorem (Theorem 4.9). Hence, upon dividing by $(z-a)^m$,

$$f(z) = c_0(z-a)^{-m} + c_1(z-a)^{-m+1} + \cdots + c_{m-1}(z-a)^{-1} \\ + c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \dots$$

for $z \in B'(a,r)$. Hence $c_0 = g(a) \neq 0$, so f has a pole of order m. Finally

$$\operatorname{res}(f,a) = c_{m-1} = \frac{1}{(m-1)!} g^{(m-1)}(a),$$

by the formula for the coefficients given in Taylor's Theorem. □

Example 6.12 (i) $\mathrm{e}^z/(z-1)$ has a simple pole at $z=1$ and residue

$$\operatorname{res}\left(\frac{\mathrm{e}^z}{z-1}, 1\right) = \mathrm{e}.$$

(ii) $\sin z/(z-\frac{1}{2}\pi)^3$ has a triple pole at $z = \frac{1}{2}\pi$ and residue

$$\begin{aligned} \operatorname{res}\left(\frac{\sin z}{(z-\frac{1}{2}\pi)^3}, \tfrac{1}{2}\pi\right) &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\sin z)\bigg|_{z=\frac{1}{2}\pi} \\ &= -\tfrac{1}{2}\sin\left(\tfrac{1}{2}\pi\right) = -\tfrac{1}{2}. \end{aligned}$$

Lemma 6.13 (Type II, Simple Pole) *Suppose that*

$$f(z) = \frac{g(z)}{h(z)}$$

where g and h are holomorphic in some open disc $B(a,r)$ about a, $g(a) \neq 0$, $h(a) = 0$ and $h'(a) \neq 0$. Then f has a simple pole at a and

$$\operatorname{res}(f,a) = \frac{g(a)}{h'(a)}.$$

PROOF: First note that the assumption on h ensures that its Taylor series at a has zero constant term and non-zero degree 1 term, say $h(z) = \sum_{n=1}^{\infty} c_n(z-a)^n$ for some coefficients $c_n \in \mathbb{C}$ with $c_1 \neq 0$. Therefore $h(z) = (z-a)\,k(z)$ where $k(z) = \sum_{n=0}^{\infty} c_{n+1}(z-a)^n$ is a power series that converges on $B(a,r)$ and consequently is a holomorphic function on this disc. Furthermore,

$$k(a) = c_1 = h'(a) \neq 0$$

(using Taylor's Theorem 4.9). Then

$$f(z) = \frac{g(z)/k(z)}{z-a} \tag{6.1}$$

where $g(z)/k(z)$ is holomorphic in some open disc around a. Hence, by Lemma 6.11, f has a simple pole at a.

To calculate the residue, we now use the formula in Lemma 6.11:

$$\operatorname{res}(f,a) = \frac{g(a)}{k(a)} = \frac{g(a)}{h'(a)},$$

as claimed. □

Example 6.14

$$\operatorname{res}(\cot z, 0) = \operatorname{res}\left(\frac{\cos z}{\sin z}, 0\right) = \frac{\cos 0}{\cos 0} = 1.$$

Chapter 7

Applications of Contour Integration

In this chapter, we demonstrate applications of the theory developed to the calculation of a variety of real integrals and summations. Very little new theory relating directly to complex analysis is presented in this chapter, though a few additional facts about inequalities will be required.

Evaluation of real integrals

Example 7.1 *Evaluate*

$$\int_0^\infty \frac{1}{x^4+1}\,\mathrm{d}x.$$

SOLUTION: Define

$$f(z) = \frac{1}{z^4+1}$$

and integrate f around the contour γ shown in Figure 7.1, where we assume $R > 1$.

The function f is holomorphic on an open set containing γ and its interior, except for simple poles inside γ at

$$z = \mathrm{e}^{\pi i/4} \quad \text{and} \quad \mathrm{e}^{3\pi i/4}.$$

The resides at these poles (by Lemma 6.13) are

$$\begin{aligned} \operatorname{res}(f, \mathrm{e}^{\pi i/4}) &= \left.\frac{1}{4z^3}\right|_{z=\mathrm{e}^{\pi i/4}} = \frac{1}{4}\,\mathrm{e}^{-3\pi i/4} \\ \operatorname{res}(f, \mathrm{e}^{3\pi i/4}) &= \left.\frac{1}{4z^3}\right|_{z=\mathrm{e}^{3\pi i/4}} = \frac{1}{4}\,\mathrm{e}^{-\pi i/4}. \end{aligned}$$

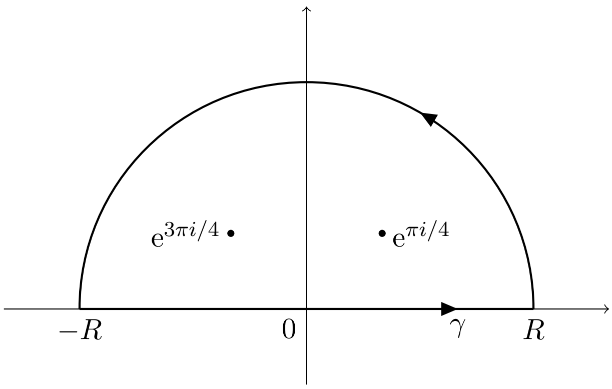


Figure 7.1: The contour γ consisting of the line segment $[-R, R]$ and a semicircular arc.

Hence, by Cauchy's Residue Theorem (Theorem 6.10),

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= 2\pi i\left(\frac{1}{4}\,\mathrm{e}^{-3\pi i/4}+\frac{1}{4}\,\mathrm{e}^{-\pi i/4}\right)\\
&= \frac{\pi i}{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)\\
&= -\frac{\pi i^2}{\sqrt{2}}\\
&= \frac{\pi}{\sqrt{2}}. \qquad (7.1)
\end{aligned}$$

Write Γ_R for the semicircular part of the contour γ. Note that $|z| = R$ if z lies on the image Γ_R^* of this semicircular arc, so

$$\left|z^4+1\right| \geqslant |z|^4 - 1 = R^4 - 1$$

and

$$|f(z)| = \left|\frac{1}{z^4+1}\right| \leqslant \frac{1}{R^4-1}$$

for such z. Hence

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &\leqslant \frac{1}{R^4-1}\cdot \pi R\\
&= \frac{\pi}{R^3-(1/R)} \to 0 \qquad \text{as } R\to\infty \qquad (7.2)
\end{aligned}$$

by the Crude Estimation Theorem (Theorem 3.14). Translating Equation (7.1) into its smooth pieces:

$$\int_{-R}^{R} \frac{1}{x^4+1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{\sqrt{2}};$$

that is,

$$2\int_0^R \frac{1}{x^4+1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{\sqrt{2}}$$

using the fact that $1/(x^4+1)$ is an even function of the real variable x. Let $R \to \infty$, using Equation (7.2), to conclude

$$\int_0^\infty \frac{1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}.$$

□

Our second example is similar, but will involve an integral over the whole interval $(-\infty,\infty) = \mathbb{R}$. In this context, we need to note the following.

Warning: Our method in Example 7.1 involved a function $f(x)$ of a real variable that is an even function, so we were permitted to write

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = 2\int_0^R f(x)\,\mathrm{d}x$$

and then let $R \to \infty$ to determine the value of

$$\int_0^\infty f(x)\,\mathrm{d}x.$$

When $f(x)$ is *not* an even function, we can apply a similar method to calculate the limit

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x.$$

This limit is denoted

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

and is called *Cauchy's principal value integral*, but it is *not necessarily* the integral

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

An example of the problem is

$$\int_{-R}^{R} x\,\mathrm{d}x = 0$$

for any choice of $R > 0$, so upon letting $R \to \infty$ we observe

$$\mathrm{PV}\int_{-\infty}^{\infty} x\,\mathrm{d}x = 0.$$

However the integral

$$\int_{-\infty}^{\infty} x\,\mathrm{d}x$$

does not exist, since if x were integrable on $(-\infty,\infty)$, then it would also have an integral on $[0,\infty)$ and yet

$$\int_{0}^{R} x\,\mathrm{d}x = \tfrac{1}{2}R^2 \to \infty \qquad \text{as } R \to \infty.$$

If, however, one can show that, given some function f,

$$\lim_{R\to\infty}\int_{0}^{R} |f(x)|\,\mathrm{d}x \qquad \text{and} \qquad \lim_{R\to\infty}\int_{-R}^{0} |f(x)|\,\mathrm{d}x$$

are both finite, then this will tell us that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

exists and will be equal to the principal value integral.

The full explanation for this belongs in a course in analysis that covers integration theory. In what follows, we shall use this last fact to justify dropping the "PV" from the integral.

Example 7.2 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,\mathrm{d}x.$$

SOLUTION: Define

$$f(z) = \frac{1}{z^2+z+1}$$

and integrate f around the contour γ shown in Figure 7.2, where $R > 1$, as was used in the previous example. Observe that f is holomorphic on an open set containing γ and its interior except for a simple pole at $z = \mathrm{e}^{2\pi i/3}$ inside γ. (Note that $(z-1)(z^2+z+1) = z^3 - 1$, so the zeros of z^2+z+1 are $z = \mathrm{e}^{2\pi i/3}$ and $\mathrm{e}^{4\pi i/3}$.)

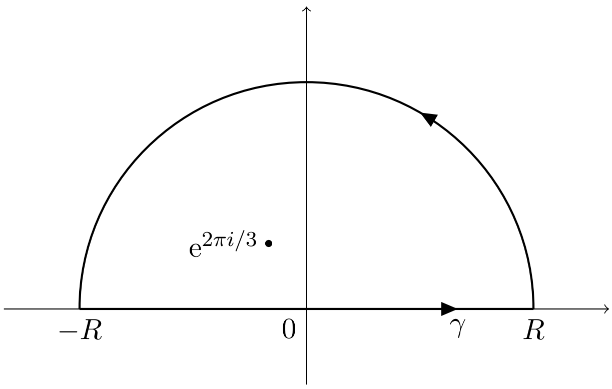


Figure 7.2: The contour γ consisting of the line segment $[-R, R]$ and a semicircular arc.

Here $e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so the residue of f at $e^{2\pi i/3}$ (by Lemma 6.13) is

$$\operatorname{res}(f, e^{2\pi i/3}) = \left.\frac{1}{2z+1}\right|_{z=e^{2\pi i/3}} = \frac{1}{\sqrt{3}\,i}$$

Hence by Cauchy's Residue Theorem (Theorem 6.10),

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{res}(f, e^{2\pi i/3}) = \frac{2\pi}{\sqrt{3}}. \tag{7.3}$$

As before we write Γ_R for the semicircular piece, of radius R, of the contour γ. Now if z lies on Γ_R^*, then $|z| = R$ and

$$|f(z)| = \frac{1}{|z^2+z+1|} \leqslant \frac{1}{|z|^2 - |z| - 1} = \frac{1}{R^2 - R - 1},$$

so, by the Crude Estimation Theorem (Theorem 3.14),

$$\begin{aligned} \left|\int_{\Gamma_R} f(z)\,dz\right| &\leqslant \frac{\pi R}{R^2 - R - 1} \\ &= \frac{\pi}{R - 1 - (1/R)} \to 0 \qquad \text{as } R \to \infty. \end{aligned}$$

Expanding Equation (7.3) gives

$$\int_{-R}^{R} \frac{1}{x^2+x+1}\,dx + \int_{\Gamma_R} f(z)\,dz = \frac{2\pi}{\sqrt{3}}.$$

Let $R \to \infty$ to conclude

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}.$$

Finally we verify that this is a genuine integral on $(-\infty, \infty)$, not just a PV-integral. Essentially, we observe that $f(x)$ behaves asymptotically like $1/x^2$. Indeed, note first note that if $x \geqslant 1$, then $x^2 + x + 1 \geqslant x^2$, so

$$\begin{aligned} \int_0^R |f(x)|\,dx &\leqslant \int_0^1 f(x)\,dx + \int_1^R f(x)\,dx \\ &= \int_0^1 f(x)\,dx - \left.\frac{1}{x}\right|_{x=1}^{R} \end{aligned}$$

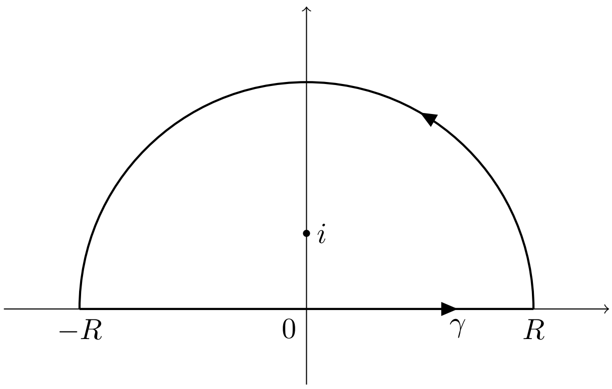


Figure 7.3: The contour γ consisting of the line segment $[-R, R]$ and a semicircular arc.

$$\begin{aligned} &= 1 - \frac{1}{R} + \int_0^1 f(x)\,\mathrm{d}x \\ &\leqslant 1 + \int_0^1 f(x)\,\mathrm{d}x. \end{aligned}$$

Therefore the sequence of integrals is bounded as R varies and so we conclude

$$\lim_{R\to\infty} \int_0^R |f(x)|\,\mathrm{d}x \quad \text{exists.}$$

A similar argument, with a little bit more care, shows that $\int_{-R}^0 |f(x)|\,\mathrm{d}x$ is also bounded as $R \to \infty$. We have now verified the required steps to permit us to remove the PV from the integral and we conclude

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,\mathrm{d}x = \frac{2\pi}{\sqrt{3}}.$$

□

Example 7.3 *Evaluate*

$$\int_0^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x.$$

One is tempted to integrate the function $(\cos z)/(z^2+1)$ about the contour γ we have been using before. The problem with that suggestion is that this function is not sufficiently bounded on the semicircular part of the contour. Instead, one proceeds as follows.

SOLUTION: Define

$$f(z) = \frac{\mathrm{e}^{iz}}{z^2+1}$$

and integrate f around the contour γ shown in Figure 7.3, where $R > 1$. Note that f is holomorphic on an open set containing γ and its interior except for a simple pole at $z = i$.

By Lemma 6.13, the residue is

$$\operatorname{res}(f, i) = \left.\frac{\mathrm{e}^{iz}}{2z}\right|_{z=i} = \frac{\mathrm{e}^{-1}}{2i},$$

so, by Cauchy's Residue Theorem (Theorem 6.10),

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}(f, i) = \frac{\pi}{\mathrm{e}}. \tag{7.4}$$

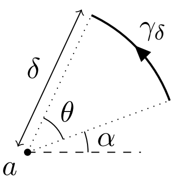


Figure 7.4: Indentation about a simple pole: the circular arc γ_δ about 0 of radius δ.

If z lies on the semicircular part Γ_R of the contour, then

$$\left|\mathrm{e}^{iz}\right| = \mathrm{e}^{\mathrm{Re}(iz)} = \mathrm{e}^{-\operatorname{Im} z} \leqslant 1$$

and

$$\left|z^2+1\right| \geqslant |z|^2 - 1 = R^2 - 1,$$

so

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &\leqslant \frac{\pi R}{R^2-1}\\ &= \frac{\pi}{R-(1/R)} \to 0 \qquad \text{as } R \to \infty\end{aligned}$$

using the Crude Estimation Theorem (Theorem 3.14). Now expand Equation (7.4):

$$\int_{-R}^{R} \frac{\mathrm{e}^{ix}}{x^2+1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{\mathrm{e}}.$$

Hence

$$\int_{-R}^{R} \frac{\cos x}{x^2+1}\,\mathrm{d}x + i\int_{-R}^{R} \frac{\sin x}{x^2+1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{\mathrm{e}},$$

so

$$2\int_{0}^{R} \frac{\cos x}{x^2+1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{\mathrm{e}},$$

since $(\cos x)/(x^2+1)$ is an even function and $(\sin x)/(x^2+1)$ is an odd function of the real number x. Now let $R \to \infty$ to conclude

$$\int_{0}^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2\mathrm{e}}.$$

□

For the next example, we need an additional fact about integration around an arc centred at a simple pole and a fact about a real inequality.

Lemma 7.4 (Indentation at a simple pole) *Suppose f has a simple pole at a point a and γ_δ is a positively oriented arc of a circle centred at a, radius $\delta > 0$ and subtending an angle θ at a. Then*

$$\lim_{\delta\to 0} \int_{\gamma_\delta} f(z)\,\mathrm{d}z = i\theta \operatorname{res}(f, a).$$

Proof: Exploiting the Laurent series of f, we can write

$$f(z) = \frac{b}{z-a} + g(z)$$

valid in some punctured disc $B'(a,r)$, where $b = \operatorname{res}(f,a)$ and $g(z)$, given by a convergent power series, is holomorphic in the open disc $B(a,r)$. Now g is, in particular, continuous on the closed and bounded set $\{ z \in \mathbb{C} \mid |z-a| \leqslant \frac{1}{2}r \}$, so is bounded, say

$$|g(z)| \leqslant M \qquad \text{if } |z-a| \leqslant \tfrac{1}{2}r.$$

Thus, if $\delta \leqslant \frac{1}{2}r$,

$$\left| \int_{\gamma_\delta} g(z)\,\mathrm{d}z \right| \leqslant M \cdot \delta\theta \to 0 \qquad \text{as } \delta \to 0,$$

with use of the Crude Estimation Theorem (Theorem 3.14), noting $L(\gamma_\delta) = \delta\theta$.

On the other hand, let us parametrise γ_δ, shown in Figure 7.4, as $\gamma_\delta(t) = a + \delta \mathrm{e}^{it}$ for $\alpha \leqslant t \leqslant \beta$ (so that $\theta = \beta - \alpha$). Then

$$\begin{aligned} \int_{\gamma_\delta} \frac{b}{z-a}\,\mathrm{d}z &= \int_\alpha^\beta \frac{b}{\delta \mathrm{e}^{it}}\,\delta i \mathrm{e}^{it}\,\mathrm{d}t \\ &= ib \int_\alpha^\beta \mathrm{d}t \\ &= ib(\beta - \alpha) = ib\theta. \end{aligned}$$

Hence

$$\int_{\gamma_\delta} f(z)\,\mathrm{d}z = ib\theta + \int_{\gamma_\delta} g(z)\,\mathrm{d}z \to ib\theta$$

as $\delta \to 0$. □

Lemma 7.5 (Jordan's Inequality) *If* $0 < t \leqslant \frac{1}{2}\pi$, *then*

$$\frac{2}{\pi} \leqslant \frac{\sin t}{t} \leqslant 1.$$

PROOF: Write $f(t) = (\sin t)/t$. Using the Taylor series expansion for $\sin t$, observe

$$\frac{\sin t}{t} \to 1 \qquad \text{as } t \to 0.$$

Also note that $f(\frac{1}{2}\pi) = 2/\pi$. Then we calculate

$$f'(t) = \frac{\cos t}{t} - \frac{\sin t}{t^2} = \frac{t\cos t - \sin t}{t^2}.$$

In view of this, we now consider $g(t) = t\cos t - \sin t$. Note $g(0) = 0$ and that

$$g'(t) = \cos t - t\sin t - \cos t = -t\sin t \leqslant 0$$

for $0 \leqslant t \leqslant \frac{1}{2}\pi$. Hence g is a decreasing function on $[0, \frac{1}{2}\pi]$, so

$$t\cos t - \sin t \leqslant 0 \qquad \text{for } 0 \leqslant t \leqslant \tfrac{1}{2}\pi.$$

Therefore

$$f'(t) = \frac{t\cos t - \sin t}{t^2} \leqslant 0 \qquad \text{for } 0 < t \leqslant \tfrac{1}{2}\pi,$$

so f is a decreasing function on $(0, \frac{1}{2}\pi]$. Putting this together we now conclude

$$\frac{2}{\pi} \leqslant f(t) \leqslant 1 \qquad \text{for } 0 < t \leqslant \tfrac{1}{2}\pi.$$

□

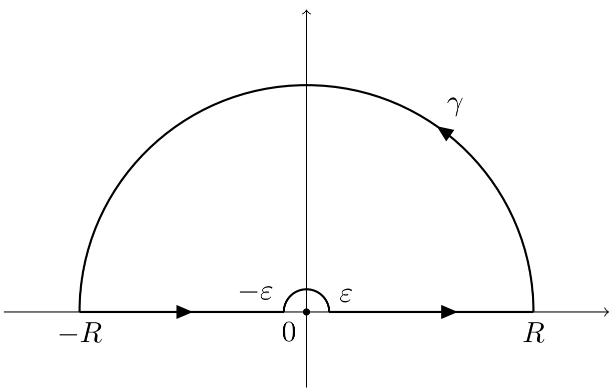


Figure 7.5: The contour γ with an indentation around a simple pole at $z = 0$.

We now turn to our example:

Example 7.6 *Evaluate*

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

SOLUTION: Define

$$f(z) = \frac{\mathrm{e}^{iz}}{z}$$

and integrate f around the contour γ shown in Figure 7.5 where $0 < \varepsilon < R$. Since f is holomorphic in an open set containing γ and its interior (it has an isolated singularity at 0),

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \tag{7.5}$$

by Cauchy's Theorem.

Write Γ_R and Γ_ε for the *positively oriented* contours of radii R and ε about 0 in the Figure. The residue of f at the simple pole is

$$\operatorname{res}(f, 0) = \mathrm{e}^0 = 1,$$

using Lemma 6.11, and hence use of Lemma 7.4 tells us

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z = i\pi \operatorname{res}(f, 0) = i\pi.$$

Parametrise Γ_R as $\Gamma_R(t) = R\,\mathrm{e}^{it}$ for $0 \leqslant t \leqslant \pi$. Then

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &= \left|\int_0^\pi \frac{\mathrm{e}^{iR(\cos t + i\sin t)}}{R\,\mathrm{e}^{it}} \cdot Ri\,\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&= \left|\int_0^\pi \mathrm{e}^{R(i\cos t - \sin t)}\,\mathrm{d}t\right| \\
&\leqslant \int_0^\pi |\mathrm{e}^{R(i\cos t - \sin t)}|\,\mathrm{d}t \\
&= \int_0^\pi \mathrm{e}^{-R\sin t}\,\mathrm{d}t \\
&= 2\int_0^{\pi/2} \mathrm{e}^{-R\sin t}\,\mathrm{d}t
\end{aligned}$$

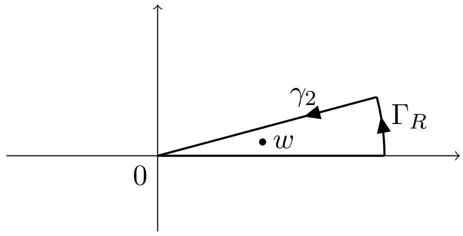


Figure 7.6: The wedge contour γ containing the pole $w = \mathrm{e}^{i\pi/100}$.

$$\begin{aligned}
&\leqslant 2\int_0^{\pi/2} \mathrm{e}^{-2Rt/\pi}\,\mathrm{d}t && \text{by Jordan's Inequality (Lemma 7.5)}\\
&= -\frac{\pi}{R}\,\mathrm{e}^{-2Rt/\pi}\Big|_{t=0}^{\pi/2}\\
&= \frac{\pi}{R}(1-\mathrm{e}^{-R})\\
&\to 0 \qquad \text{as } R\to\infty.
\end{aligned}$$

Finally expand Equation (7.5) as

$$\int_\varepsilon^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x - \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z = 0.$$

Take the imaginary part and use the fact that $(\sin x)/x$ is an even functions to give

$$2\int_\varepsilon^R \frac{\sin x}{x}\,\mathrm{d}x + \mathrm{Im}\int_{\Gamma_R} f(z)\,\mathrm{d}z - \mathrm{Im}\int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z = 0.$$

Let $\varepsilon \to 0$ and $R \to \infty$, using the observations above, to conclude

$$2\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - \pi = 0.$$

Hence

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

□

More specialised examples of real integrals

We have now covered the standard methods for evaluating real integrals using contour integration. The following examples of evaluating real integrals are a little more specialised.

Example 7.7 *Evaluate*

$$\int_0^\infty \frac{1}{x^{100}+1}\,\mathrm{d}x.$$

Solution: Define

$$f(z) = \frac{1}{z^{100}+1}$$

and integrate f around the "wedge" contour γ shown in Figure 7.6, where $R > 1$. The line segment γ_2 shown is at an angle of $\pi/50$ to the real axis, so that precisely one root of $z^{100}+1$, namely $w = \mathrm{e}^{i\pi/100}$, lies inside γ.

The residue of f at the simple pole $z = w$ is (by Lemma 6.13)

$$\operatorname{res}(f, w) = \frac{1}{100w^{99}}.$$

Note $w^{100} = -1$, so $1/w^{99} = -w$ and hence $\operatorname{res}(f, w) = -w/100$. Cauchy's Residue Theorem (Theorem 6.10) then tells us

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}(f, w) = -\frac{\pi i w}{50}. \tag{7.6}$$

Write Γ_R for the circular arc appearing as a piece of γ. If z lies on Γ_R^*,

$$\left|z^{100} + 1\right| \geqslant |z|^{100} - 1 = R^{100} - 1,$$

so

$$\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| \leqslant \frac{1}{R^{100} - 1} \cdot \frac{\pi R}{50} \to 0 \qquad \text{as } R \to \infty.$$

The line segment γ_2 is parametrised as $\gamma_2(t) = \mathrm{e}^{i\pi/50}(R - t)$ for $0 \leqslant t \leqslant R$, so

$$\begin{aligned}
\int_{\gamma_2} f(z)\,\mathrm{d}z &= \int_0^R \frac{1}{(R-t)^{100} + 1} \cdot (-\mathrm{e}^{i\pi/50})\,\mathrm{d}t \\
&= -\mathrm{e}^{i\pi/50} \int_0^R \frac{1}{(R-t)^{100} + 1}\,\mathrm{d}t \\
&= -\mathrm{e}^{i\pi/50} \int_0^R \frac{1}{x^{100} + 1}\,\mathrm{d}x
\end{aligned}$$

upon substituting $x = R - t$. Hence Equation (7.6) is

$$(1 - \mathrm{e}^{i\pi/50}) \int_0^R \frac{1}{x^{100} + 1}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z = -\frac{\pi i w}{50},$$

so, upon letting $R \to \infty$, we obtain

$$\begin{aligned}
\int_0^\infty \frac{1}{x^{100} + 1}\,\mathrm{d}x &= \frac{\pi i w}{50(\mathrm{e}^{i\pi/50} - 1)} \\
&= \frac{\pi i\, \mathrm{e}^{i\pi/100}}{50(\mathrm{e}^{i\pi/50} - 1)} \\
&= \frac{\pi i}{50(\mathrm{e}^{i\pi/100} - \mathrm{e}^{-i\pi/100})} \\
&= \frac{\pi}{100 \sin(\pi/100)} \\
&= \frac{\pi/100}{\sin(\pi/100)}.
\end{aligned}$$

□

Example 7.8 *Evaluate*

$$\int_0^\infty \cos x^2\,\mathrm{d}x.$$

Solution: Integrate $f(z) = \mathrm{e}^{iz^2}$ around the contour γ shown in Figure 7.7. Cauchy's Theorem tells us that

$$\int_\gamma f(z)\,\mathrm{d}z = 0;$$

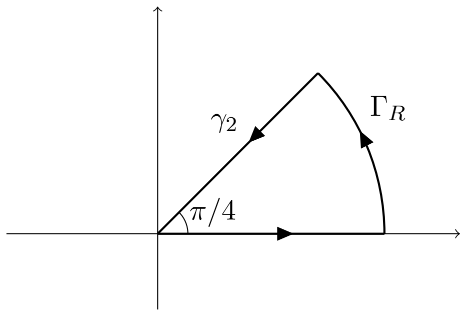


Figure 7.7: The "wedge" contour used in Example 7.8.

that is,

$$\int_0^R \mathrm{e}^{ix^2}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 0, \tag{7.7}$$

where Γ_R denotes the circular arc and γ_2 the line segment at an angle of $\pi/4$ to the real axis.

Note, upon parametrising Γ_R as $\Gamma_R(t) = R\,\mathrm{e}^{it}$ for $0 \leqslant t \leqslant \pi/4$, that

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &= \left|\int_0^{\pi/4} \mathrm{e}^{iR^2\mathrm{e}^{2it}} \cdot iR\,\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&\leqslant \int_0^{\pi/4} |\mathrm{e}^{iR^2\mathrm{e}^{2it}} \cdot iR\,\mathrm{e}^{it}|\,\mathrm{d}t \\
&= R\int_0^{\pi/4} e^{-R^2\sin 2t}\,\mathrm{d}t \\
&\leqslant R\int_0^{\pi/4} \mathrm{e}^{-4R^2t/\pi}\,\mathrm{d}t,
\end{aligned}$$

since by Jordan's Inequality (Lemma 7.5), $(\sin 2t)/2t \geqslant 2/\pi$ for $0 \leqslant t \leqslant \pi/4$. Hence

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &\leqslant -\frac{\pi}{4R}\,\mathrm{e}^{-4R^2t/\pi}\Big|_{t=0}^{\pi/4} \\
&= \frac{\pi}{4R}(1-\mathrm{e}^{-R^2}) \\
&\to 0
\end{aligned}$$

as $R \to \infty$. We parametrise γ_2 as $\gamma_2(t) = \mathrm{e}^{i\pi/4}(R-t)$ for $0 \leqslant t \leqslant R$, so that

$$\begin{aligned}
\int_{\gamma_2} f(z)\,\mathrm{d}z &= \int_0^R \mathrm{e}^{i\mathrm{e}^{i\pi/2}(R-t)^2} \cdot (-\mathrm{e}^{i\pi/4})\,\mathrm{d}t \\
&= -\mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-(R-t)^2}\,\mathrm{d}t.
\end{aligned}$$

Therefore Equation (7.7) becomes

$$\begin{aligned}
\int_0^R \mathrm{e}^{ix^2}\,\mathrm{d}x &= \mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-(R-t)^2}\,\mathrm{d}t - \int_{\Gamma_R} f(z)\,\mathrm{d}z \\
&= \mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-u^2}\,\mathrm{d}u - \int_{\Gamma_R} f(z)\,\mathrm{d}z,
\end{aligned}$$

upon substituting $u = R - t$. Let $R \to \infty$ to conclude

$$\int_0^\infty \mathrm{e}^{ix^2}\,\mathrm{d}x = \mathrm{e}^{i\pi/4}\int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u = \mathrm{e}^{i\pi/4}\cdot\frac{\sqrt{\pi}}{2},$$

as the latter integral is a standard known integral. (It arises in the context of probability and statistics in regard to the normal distribution. It can be verified by complex analysis with much ingenuity, but is most usually established by other methods.) Finally taking real parts gives

$$\int_0^\infty \cos x^2 \, \mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

□

Integrals of functions involving trigonometric functions

A different application of contour integration can be applied to evaluate integrals of the form

$$\int_0^{2\pi} f(\theta) \, d\theta$$

where $f(\theta)$ is a function expressed so as to involve trigonometric functions. The method is to write

$$\cos\theta = \tfrac{1}{2}(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}) \qquad \text{and} \qquad \sin\theta = \tfrac{1}{2i}(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta})$$

and to convert our integral into a contour integral involving a function of a complex variable about the positively oriented circular contour of radius 1 about 0.

Example 7.9 *Let a be a real number with $a > 1$. Evaluate*

$$\int_0^{2\pi} \frac{1}{a + \sin\theta} \, \mathrm{d}\theta.$$

Solution: Observe

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a+\sin\theta} \, \mathrm{d}\theta &= \int_0^{2\pi} \frac{2i}{2ai + \mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}} \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2}{(2ai + \mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta})\,\mathrm{e}^{i\theta}} \cdot i\mathrm{e}^{i\theta} \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2}{e^{2i\theta} + 2ai\,\mathrm{e}^{i\theta} - 1} \cdot i\mathrm{e}^{i\theta} \, \mathrm{d}\theta \\
&= \int_\gamma f(z) \, \mathrm{d}z
\end{aligned}$$

where γ denotes the positively oriented circular contour of radius 1 about 0 and

$$f(z) = \frac{2}{z^2 + 2aiz - 1}.$$

The roots of $z^2 + 2aiz - 1 = 0$ are

$$\begin{aligned}
\frac{-2ai \pm \sqrt{-4a^2 + 4}}{2} &= -ai \pm \sqrt{-(a^2 - 1)} \\
&= (-a \pm \sqrt{a^2 - 1})i.
\end{aligned}$$

Note that the root $b_1 = (-a + \sqrt{a^2 - 1})i$ lies inside the circle γ^*, but the other root $b_2 = (-a - \sqrt{a^2 - 1})i$ is in the exterior. As $f(z) = 2/(z - b_1)(z - b_2)$, the residue of f at b_1 is

$$\operatorname{res}(f, b_1) = \left.\frac{2}{z - b_2}\right|_{z = b_1} = \frac{2}{b_1 - b_2} = \frac{1}{i\sqrt{a^2 - 1}}.$$

Hence, by Cauchy's Residue Theorem (Theorem 6.10),

$$\begin{aligned}\int_0^{2\pi} \frac{1}{a+\sin\theta}\,\mathrm{d}\theta &= \int_\gamma f(z)\,\mathrm{d}z \\ &= 2\pi i \operatorname{res}(f, b_1) \\ &= \frac{2\pi}{\sqrt{a^2-1}}.\end{aligned}$$

□

Summation of series

For our final type of applications, we intend to evaluate the sum of an infinite series. In our example, we shall evaluate such a sum by integrating

$$f(z) = \frac{\cos z}{z^2 \sin z} = \frac{\cot z}{z^2}$$

around a square contour γ_N with vertices at the four points

$$\pm(N+\tfrac{1}{2})\pi \pm (N+\tfrac{1}{2})\pi i.$$

Accordingly, we need the following fact:

Lemma 7.10 *If z lies on the image of the square contour γ_N that has vertices at the points $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$, for some positive integer N, then*

$$|\cot z| \leqslant \coth\left(\frac{3\pi}{2}\right).$$

PROOF: If z lies on one of the horizontal sides of γ_N^*, then

$$z = x \pm (N+\tfrac{1}{2})\pi i$$

where $-(N+\frac{1}{2})\pi \leqslant x \leqslant (N+\frac{1}{2})\pi$. Then

$$\begin{aligned}|\cot z| = \frac{|\cos z|}{|\sin z|} &= \frac{|\mathrm{e}^{i(x\pm(N+\frac{1}{2})\pi i)} + \mathrm{e}^{-i(x\pm(N+\frac{1}{2})\pi i)}|}{|\mathrm{e}^{i(x\pm(N+\frac{1}{2})\pi i)} - \mathrm{e}^{-i(x\pm(N+\frac{1}{2})\pi i)}|} \\ &\leqslant \frac{\mathrm{e}^{(N+\frac{1}{2})\pi} + \mathrm{e}^{-(N+\frac{1}{2})\pi}}{\mathrm{e}^{(N+\frac{1}{2})\pi} - \mathrm{e}^{-(N+\frac{1}{2})\pi}} \\ &= \coth(N+\tfrac{1}{2})\pi \\ &\leqslant \coth\frac{3\pi}{2},\end{aligned}$$

since $f(x) = \coth x = \frac{\cosh x}{\sinh x}$ is decreasing:

$$f'(x) = \frac{\sinh x}{\sinh x} - \frac{\cosh^2 x}{\sinh^2 x} = 1 - \frac{\cosh^2 x}{\sinh^2 x} \leqslant 0.$$

On the other hand, if z lies on one of the vertical sides of γ_N^*, then

$$z = \pm(N+\tfrac{1}{2})\pi + xi$$

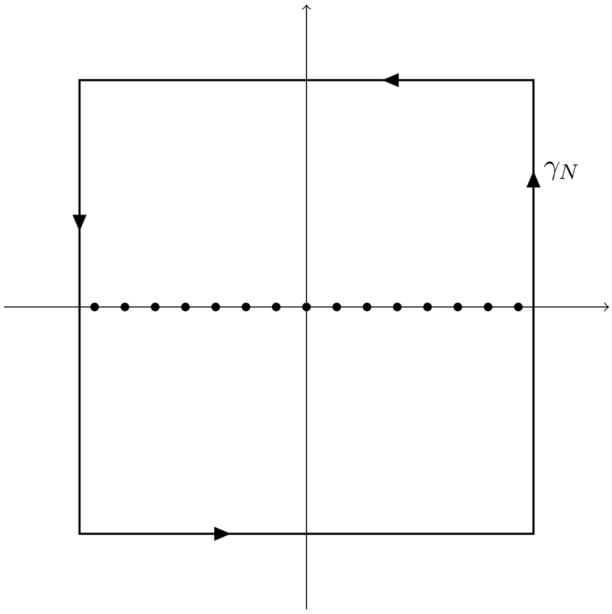


Figure 7.8: The square contour γ_N with vertices at $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})\pi i$.

where $-(N+\frac{1}{2})\pi \leqslant x \leqslant (N+\frac{1}{2})\pi$. Then

$$|\cos z| = \left|\cos(\pm(N+\tfrac{1}{2})\pi + xi)\right| = |\sin xi| = |\sinh x|$$

and

$$|\sin z| = \left|\sin(\pm(N+\tfrac{1}{2})\pi + xi)\right| = |\cos xi| = |\cosh x|,$$

so

$$|\cot z| = |\tanh x| \leqslant 1.$$

Note $\coth(3\pi/2) \geqslant 1$, so the result holds by combining the two inequalities. □

Example 7.11 *Evaluate the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

SOLUTION: We shall integrate the function $f(z) = (\cot z)/z^2$ around the square contour γ_N, shown in Figure 7.8, with vertices at $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$. Note that f has poles at $z = 0$, $\pm\pi$, $\pm 2\pi$, $\dots$, $\pm N\pi$ that all lie inside γ_N (the other poles are outside γ_N). Thus

$$\int_{\gamma_N} f(z)\,\mathrm{d}z = 2\pi i \sum_{n=-N}^{N} \operatorname{res}(f, n\pi)$$

by Cauchy's Residue Theorem (Theorem 6.10).

Since $f(z) = (\cos z)/(z \sin z)$, we have a simple pole (of Type II) at $z = n\pi$ (for $n \neq 0$) and, by Lemma 6.13,

$$\operatorname{res}(f, n\pi) = \left.\frac{\cos z}{z^2 \cos z}\right|_{z=n\pi} = \frac{1}{n^2\pi^2}.$$

To determine the residue of f at $z = 0$, we shall make use of the terms of the Laurent series for $\cot z$ that we found in Example 6.6(iv):

$$\cot z = \frac{1}{z} - \frac{z}{3} + (\text{terms of degree} \geqslant 3)$$

Hence the Laurent series of $f(z)$ around 0 is

$$f(z) = \frac{\cot z}{z^2} = z^{-3} - \frac{1}{3}z^{-1} + (\text{terms of degree} \geqslant 1)$$

and so

$$\operatorname{res}(f, 0) = -\frac{1}{3}.$$

Hence

$$\int_{\gamma_N} f(z)\,\mathrm{d}z = 2\pi i \left(-\frac{1}{3} + \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \right). \tag{7.8}$$

If z lies on γ_N^*, then

$$|z| \geqslant (N + \tfrac{1}{2})\pi \qquad \text{and} \qquad |\cot z| \leqslant \coth(3\pi/2)$$

by Lemma 7.10. Hence, using the Crude Estimation Theorem (Theorem 3.14),

$$\begin{aligned} \left| \int_{\gamma_N} f(z)\,\mathrm{d}z \right| &\leqslant \frac{\coth(3\pi/2)}{(N+\frac{1}{2})^2\,\pi^2} \cdot 4(2N+1)\pi \\ &= \frac{8\coth(3\pi/2)}{\pi} \cdot \frac{1}{N+\frac{1}{2}} \\ &\to 0 \qquad \text{as } N \to \infty. \end{aligned}$$

Hence letting $N \to \infty$ in Equation (7.8) gives

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{3}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Overview of summations: The method presented is useful for evaluating a sum of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

where k is *even*. It does not help when k is odd, since when we use the given method we observe

$$\operatorname{res}\left(\frac{\cot z}{z^k}, 0 \right) = 0$$

and

$$\operatorname{res}\left(\frac{\cot z}{z^k}, -n \right) = -\operatorname{res}\left(\frac{\cot z}{z^k}, n \right).$$

Consequently the sum of the residues, when k is odd, is 0 in this case we learn nothing by applying contour integration.

We can also apply contour integration to a function of the form

$$\frac{1}{z^k \sin z},$$

again when k is *even*, to determine the value of summations of the form

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^k}.$$

Chapter 8

Logarithms and Related Multifunctions

In this section, we seek to define what is mean by the logarithm of a complex number z. We require that the function $\log z$ has two principal properties:

1. It is the inverse of the exponential function e^z.

2. It is well-behaved (i.e., holomorphic) on a sufficiently large proportion of the complex plane that we can make use of it.

Let us explore Property 1 to see how we should define $\log z$. If $z \in \mathbb{C}$, let us suppose, for some $z \in \mathbb{C}$, we have defined

$$\log z = a + ib$$

with real part a and imaginary part b. We will require

$$z = \mathrm{e}^{\log z} = \mathrm{e}^{a+ib} = \mathrm{e}^{a}(\cos b + i \sin b).$$

Hence $\mathrm{e}^a = |z|$ and $b = \arg z$. In view of this, if one is to define logarithm as an inverse of the exponential function, then it would be given by

$$\log z = \log |z| + i \arg z, \tag{8.1}$$

where $\log |z|$ denotes the familiar *real-valued* logarithm of the real number $|z|$. This at least gives a value when $z \neq 0$.

Note then that if $z = x + iy$, then using the definition in Equation (8.1),

$$\log \mathrm{e}^z = \log \mathrm{e}^{x+iy} = \log \left|\mathrm{e}^{x+iy}\right| + i \arg \mathrm{e}^{x+iy} = \log \mathrm{e}^x + iy = x + iy = z.$$

Hence with this definition, $\log z$ would indeed be an inverse for the exponential function, since it satisfies both

$$\mathrm{e}^{\log z} = z \qquad \text{and} \qquad \log \mathrm{e}^z = z.$$

However, there are two obvious issues that arise:

1. The argument of a complex number is not uniquely specified.

2. It is not at all clear that Equation (8.1) actually defines a holomorphic function.

The first issue is the more profound: we cannot hope to show a function is holomorphic until after we have made sense of what the function even means.

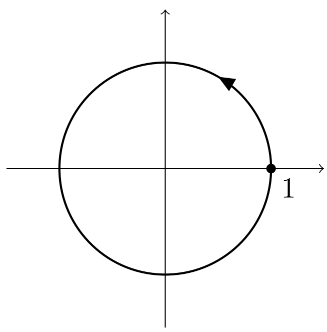


Figure 8.1: A singularity in $\log z$ as we trace a circle around 0

Up to this point, the main method we have used to get a unique value for the argument of a complex number is to prescribe that the argument is selected from a particular range of values. For example, one might require that

$$0 \leqslant \arg z < 2\pi \qquad \text{for all } z \in \mathbb{C}$$

(and we do not choose any value at all for the argument of 0). One could use argument defined in this way, via some restriction on the range of values, to then define logarithm via Equation (8.1).

This looks like a potential sensible way to define logarithm, but an issue arises if we examine the behaviour of logarithm as we follow the positively oriented contour of radius 1 about 0. We parametrise this contour as

$$\gamma(t) = \mathrm{e}^{it} \qquad \text{for } 0 \leqslant t \leqslant 2\pi.$$

Then

$$\begin{aligned} \log \gamma(t) &= \log \left|\mathrm{e}^{it}\right| + i \arg(\mathrm{e}^{it}) \\ &= i \arg(\mathrm{e}^{it}) \\ &= \begin{cases} it & \text{if } 0 \leqslant t < 2\pi \\ 0 & \text{if } t = 2\pi; \end{cases} \end{aligned}$$

that is, logarithm does not vary in a continous way as we trace this circle (as shown in Figure 8.1): there is a jump in the value as we complete the circle and return to 1.

We need to avoid such a discontinuity: once we have chosen a range of values for the argument, we need to prevent such a circling of the origin. The solution is the following action:

> Define a *branch cut*: select a part of the complex plane that we remove ("cut") from the plane in such a way to prevent circling the origin.

Example 8.1 To define the logarithm as

$$\log z = \log |z| + i \arg z$$

where

$$0 < \arg z < 2\pi,$$

cut the plane along the real axis. The resulting *cut plane* is

$$\mathbb{C}_{\text{cut}} = \mathbb{C} \setminus [0, \infty)$$

as illustrated in Figure 8.2.

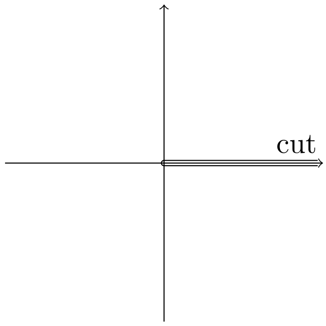


Figure 8.2: A cut plane $\mathbb{C}_{\text{cut}}$.

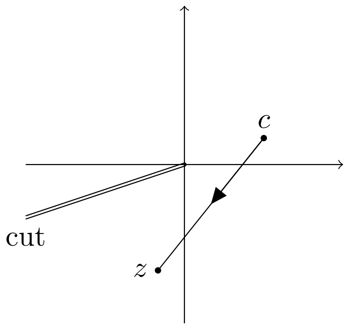


Figure 8.3: The cut plane and definition of $F(z)$ as an integral along $[c, z]$.

If we want to use a different range of argument, then we should use a different (appropriately selected) cut to the plane.

The point $z = 0$ where our branch cut begins (and which must be included in any cut used so as to avoid circling the origin) is called a *branch point* for the logarithm function. The function $\log z = \log|z| + i \arg z$ determined by the range of argument selected is then called a *branch* of the *multifunction* $\log z$.

Now that we have specified how to define (a branch of) logarithm with use of a branch cut, we need to show that the logarithm function so defined is actually holomorphic on the cut plane. This is what we now do.

Cut the plane with a straight line cut starting at the branch point $z = 0$. (The requirement that it be straight is not necessary for our argument that follows, but for convenience we shall use a straight line cut.) This choice of cut then defines, as described above, a range of values for the argument of a complex number in the resulting cut plane. Choose some point c that does not lie on the cut. For convenience we shall chooose c to be on the line that continues from the cut (see Figure 8.3). Now, if z lies in the cut plane (i.e., $z \in \mathbb{C}$ but not on the cut), define

$$F(z) = \int_{[c,z]} \frac{1}{w} \, \mathrm{d}w$$

where $[c, z]$ denotes the line segment from c to z. (Our choice of cut as a straight line together with the location of the point c ensures that the line segment $[c, z]$ is contained in the cut plane. If we chose a more complicated cut, for example, then we would need a more careful choice of path from c to z.)

Note that $1/w$ is a holomorphic function of w on the cut plane, since its pole at $w = 0$ is one of the points we have removed. Moreover, using Cauchy's Theorem, any integral of $1/w$ around a closed curve contained in the cut plane is zero, since the presence of the cut prevents us using a contour that contains 0 in its interior. We now use exactly the same method as used in Theorem 5.3 (also see Question 8(b) on Problem Sheet IV where a similar type of result is established) to conclude

the function F is holomorphic on $\mathbb{C}_{\mathrm{cut}}$ with derivative

$$F'(z) = \frac{1}{z}.$$

(The proof of this fact depends on the fact that the integral around a triangle is zero and then uses some careful ε–δ type work.)

Now consider the function $\mathrm{e}^{F(z)}/z$. This is also holomorphic on $\mathbb{C}_{\mathrm{cut}}$ (since 0 is not in this cut plane) and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{e}^{F(z)}}{z}\right) = \frac{F'(z)\,\mathrm{e}^{F(z)}}{z} - \frac{\mathrm{e}^{F(z)}}{z^2} = 0$$

since $F'(z) = 1/z$. Hence $\mathrm{e}^{F(z)}/z$ is constant on $\mathbb{C}_{\mathrm{cut}}$,so

$$\frac{\mathrm{e}^{F(z)}}{z} = \frac{\mathrm{e}^{F(c)}}{c} = \frac{\mathrm{e}^{0}}{c} = \frac{1}{c};$$

that is,

$$\mathrm{e}^{F(z)} = \tfrac{1}{c}z \qquad \text{for all } z \in \mathbb{C}_{\mathrm{cut}}.$$

This now enables us to determine the function $F(z)$. If z is some point in the cut plane, let us write $F(z) = a + ib$ for some $a, b \in \mathbb{R}$. Then

$$\mathrm{e}^{a+ib} = \tfrac{1}{c}z,$$

so $\mathrm{e}^{a} = |z/c|$ and therefore

$$a = \log|z/c| = \log|z| - \log|c|$$

and

$$b = \arg(z/c) = \arg z - \arg c$$

(the latter modulo addition or subtraction of 2π, depending upon our precise range of values for argument). Putting this together

$$\begin{aligned} F(z) &= \log|z| + i\arg z - k \\ &= \log z - k \end{aligned}$$

where k is a constant determined by the choice of c. (Indeed, if we had selected to cut along the negative real axis and had chosen $c = 1$, then we would have $k = 0$.)

We already know that F is holomorphic with derivative $1/z$, and now we can conclude the same is true for the complex logarithm, since it differs from $F(z)$ by a constant. Thus:

Theorem 8.2 *Define the complex logarithm as*

$$\log z = \log|z| + i\arg z$$

on a suitable cut plane using a branch cut starting at the branch point $z = 0$ and a suitable choice of argument. Then $\log z$ is holomorphic on $\mathbb{C}_{\mathrm{cut}}$ with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}(\log z) = \frac{1}{z}.$$

□

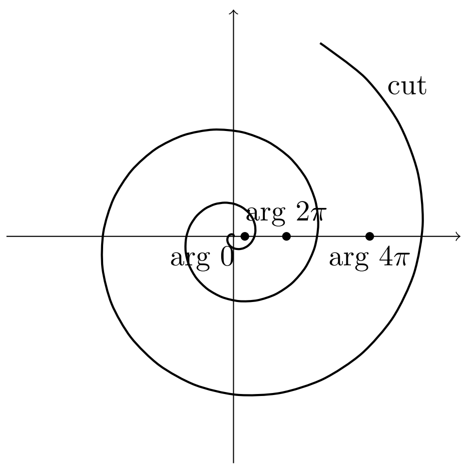


Figure 8.4: Values of argument when using a spiral-shaped cut.

The most common choices of cut are:

(i) along the positive real axis with argument range $0 < \arg z < 2\pi$;

(ii) along the negative real axis with argument range $-\pi < \arg z < \pi$;

(iii) along the negative imaginary axis with argument range $-\pi/2 < \arg z < 3\pi/2$.

This third option is actually the lecturer's favourite choice: it often has the advantage of avoiding the contour that we are attempting to work with in many examples.

One could also use more esoteric choices of cut. For example, there is nothing to stop us using a spiral cut as shown in Figure 8.4. For such a cut, the argument will vary as we wander around on the cut plane, so with a spiral we might have points on the positive real axis with argument 0, 2π, 4π, etc., though for any specific point the choice of argument is unique and the restriction of the cut ensures that argument is continuous on $\mathbb{C}_{\text{cut}}$. Having said that, I have never seen an example where such a spiral cut is either necessary nor particularly helpful.

Example 8.3 Consider a logarithmic function $\log(z^2 - 1)$. To define a suitable cut, we shall cut the plane so that we prevent $z^2 - 1$ from sitting on the positive real axis (that is, we seek to define a function that corresponds to the logarithm $\log z$ with cut along the positive real axis). With such a cut, $z^2 - 1$ will have a sufficiently well-behaved unique choice of argument as z varies on the cut plane $\mathbb{C}_{\text{cut}}$. Note

$$\begin{aligned} z^2 - 1 \in [0,\infty) \qquad &\text{if and only if} \qquad z^2 \in [1,\infty) \\ &\text{if and only if} \qquad z \in (-\infty,-1] \cup [1,\infty). \end{aligned}$$

Hence a suitable cut is as illustrated in Figure 8.5.

Example 8.4 *Let a be a real number with $a > 0$. Evaluate*

$$\int_0^\infty \frac{\log x}{x^2 + a^2} \, \mathrm{d}x.$$

Solution: To define the complex logarithm, we shall cut the plane along the negative imaginary axis. Thus for $z \in \mathbb{C}_{\text{cut}} = \mathbb{C} \setminus \{\, yi \mid y \leqslant 0 \,\}$, we define

$$\log z = \log |z| + i \arg z$$

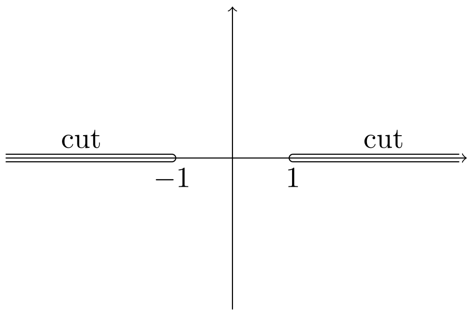


Figure 8.5: A suitable cut plane for the function $\log(z^2 - 1)$.

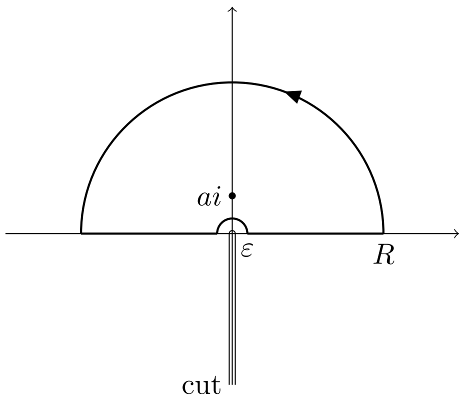


Figure 8.6: The contour γ used when integrating $(\log z)/(z^2 + a^2)$.

where argument is selected from the range $-\pi/2 < \arg z < 3\pi/2$. In terms of this logarithm, define

$$f(z) = \frac{\log z}{z^2 + a^2}$$

and integrate this f around the contour γ shown in Figure 8.6, where we choose $R > a$ and $0 < \varepsilon < \min\{1, a/2\}$.

Note that f is holomorphic on an open subset of $\mathbb{C}$ that avoids the cut and contains γ and its interior, except for a simple pole at $z = ai$ inside γ. By Lemma 6.13,

$$\operatorname{res}(f, ai) = \frac{\log(ai)}{2ai} = \frac{\log a + i\pi/2}{2ai}$$

so

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= 2\pi i \operatorname{res}(f, ai) \\ &= \frac{\pi}{a}\left(\log a + \frac{i\pi}{2}\right). \end{aligned} \tag{8.2}$$

Write Γ_R and Γ_ε, respectively, for the semicircular pieces of γ of radii R and ε (so the first is positively oriented and the second negatively oriented about 0). If z lies on Γ_R^*, then

$$|\log z| = |\log R + i \arg z| \leqslant \log R + \pi$$

and

$$\left|z^2 + a^2\right| \geqslant |z|^2 - a^2 = R^2 - a^2.$$

Hence

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| &\leqslant \frac{\log R+\pi}{R^2-a^2}\cdot \pi R\\ &= \frac{\pi(\log R+\pi)}{R-(a^2/R)} \to 0 \qquad \text{as } R\to\infty,\end{aligned}$$

since $(\log R)/R \to 0$ as $R\to\infty$.

If z lies on Γ_ε^*, then

$$|\log z| = |\log\varepsilon + i\arg z| \leqslant |\log\varepsilon| + \pi = \pi - \log\varepsilon$$

(since $\log\varepsilon < 0$ for $\varepsilon < 1$), and

$$\left|z^2+a^2\right| \geqslant a^2 - |z|^2 = a^2-\varepsilon^2 \geqslant \tfrac{3}{4}a^2 > \tfrac{1}{2}a^2.$$

Hence

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z\right| &\leqslant \frac{\pi-\log\varepsilon}{\frac{1}{2}a^2}\cdot \pi\varepsilon\\ &= \frac{2\pi}{a^2}(\pi\varepsilon - \varepsilon\log\varepsilon) \to 0 \qquad \text{as } \varepsilon\to 0\end{aligned}$$

(since $\varepsilon\log\varepsilon \to 0$ as $\varepsilon\to 0$).

Let us parametrise the line segment from $-\varepsilon$ to $-R$ by $\delta(t) = -t$ for $\varepsilon \leqslant t \leqslant R$. Note $\delta'(t) = -1$ and $\log(-t) = \log t + i\pi$ for $\varepsilon \leqslant t \leqslant R$ according to our definition of logarithm. Hence

$$\int_{[-R,-\varepsilon]} f(z)\,\mathrm{d}z = -\int_\delta f(z)\,\mathrm{d}z = \int_\varepsilon^R \frac{\log t + i\pi}{t^2+a^2}\,\mathrm{d}t.$$

Consequently Equation 8.2, when expanded, becomes

$$\int_\varepsilon^R \frac{\log x}{x^2+a^2}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_\varepsilon^R \frac{\log(t)+i\pi}{t^2+a^2}\,\mathrm{d}t + \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z = \frac{\pi}{a}\left(\log a + \frac{i\pi}{2}\right);$$

that is,

$$2\int_\varepsilon^R \frac{\log x}{x^2+a^2}\,\mathrm{d}x + i\pi\int_\varepsilon^R \frac{1}{t^2+a^2}\,\mathrm{d}t + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z = \frac{\pi}{a}\left(\log a + \frac{i\pi}{2}\right).$$

Take real parts and let $\varepsilon\to 0$ and $R\to\infty$. We conclude that

$$2\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{a}\log a;$$

that is,

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\log a}{2a}.$$

□

Functions defined as complex exponent powers

One can use logarithm to then define what we mean, given any complex number α, by a power z^α as a function of a complex variable z.

Definition 8.5 Let $\alpha\in\mathbb{C}$. To define z^α for z in an appropriate subset of $\mathbb{C}$, proceed as follows:

(i) Cut the plane to define a suitable branch of logarithm $\log z = \log |z| + i \arg z$.

(ii) Define

$$z^\alpha = \mathrm{e}^{\alpha \log z}.$$

Note that, as a composite of two holomorphic functions, z^α is then holomorphic on the cut plane $\mathbb{C}_{\text{cut}}$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(z^\alpha) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^{\alpha \log z}\right) \\
&= \alpha \cdot \frac{\mathrm{d}}{\mathrm{d}z}(\log z) \cdot \mathrm{e}^{\alpha \log z} \\
&= \alpha \cdot \frac{1}{z} \cdot z^\alpha \\
&= \alpha\, z^{\alpha-1}.
\end{aligned}$$

This function also satisfies other natural properties. For example, if $\alpha > 0$ (most likely non-integer since otherwise we would use the usual power function) then

$$|z^\alpha| = \mathrm{e}^{\mathrm{Re}(\alpha \log z)} = \mathrm{e}^{\alpha |\log z|} = |z|^\alpha .$$

If one uses such power functions, then one can obtain similar integrals to that calculated in Example 8.4 and the examples in the previous chapter. Due to current pressures of time, no example is included in these notes of such an integral.

Chapter 9

Locating and Counting Zeros and Poles

In this chapter, we shall discuss how the theory developed help us determine information about the location of the zeros and poles of some functions. We shall also be counting the number of zeros and poles, but when doing so we are counting these *including multiplicities*. Consequently, all though the function z^3 has only one location at which it has a zero, namely $z = 0$, this is a repeated zero and we will count this zero three times.

To make the concept of multiplicities precise, we first recall the concept of the *order* of a pole: A function f has a pole of order m at a point $a \in \mathbb{C}$ if its Laurent series has the form

$$f(z) = c_{-m}(z-a)^{-m} + c_{-m+1}(z-a)^{-m+1} + \dots,$$

valid in some punctured open disc $B'(a, r)$, where $c_{-m} \neq 0$. When counting a pole of order m, we shall count this pole *m times*.

A similar definition is made for zeros:

Definition 9.1 Let f be holomorphic in some open disc $B(a, r)$. We say f has a *zero of order m* at a if the Taylor series for f has the form

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \dots,$$

valid in $B(a, r)$, where $c_m \neq 0$.

Note that since the coefficients c_0, c_1, …, c_{m-1} in the Taylor series are zero in this definition, a zero of order m at a point $a \in \mathbb{C}$ means that

$$f(a) = f'(a) = f''(a) = \dots = f^{(m-1)}(a) = 0$$

by the formula for the coefficients appearing in Taylor's Theorem.

Theorem 9.2 (Argument Principle) *Let γ be a positively oriented contour and let f be a function that is holomorphic on an open set containing γ and its interior,* except *that f has P poles (including multiplicities) inside γ. Assume that f is non-zero on γ and has Z zeros (including multiplicities) inside γ. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = Z - P.$$

Proof: First note that f'/f is defined and holomorphic on γ and its interior, except that it has isolated singularities whenever f has a pole or a zero. Let a_1, a_2, …, a_k be the points inside γ

where f has either a pole or a zero. Then Cauchy's Residue Theorem (Theorem 6.10) says

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=1}^{k} \operatorname{res}\left(\frac{f'}{f}, a_j\right). \tag{9.1}$$

Consider a point $a = a_j$ where f has a pole or a zero. The Laurent series for f valid in a punctured disc around a has the form

$$f(z) = (z-a)^m\, g(z)$$

where g is holomorphic in an open disc around a, where *m is positive at a zero and is negative at a pole*, and where $g(a) \neq 0$. Then

$$f'(z) = m(z-a)^{m-1}\, g(z) + (z-a)^m\, g'(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{9.2}$$

Here $g'(z)/g(z)$ is holomorphic in some open disc around a (since $g(a) \neq 0$), so we conclude that Equation (9.2) is essentially the Laurent series for $f'(z)/f(z)$ about a (at least once we expand $g'(z)/g(z)$ as a power series). We can then extract the residue as the coefficient of $(z-a)^{-1}$:

$$\operatorname{res}\left(\frac{f'}{f}, a\right) = m.$$

Hence, in the sum appearing in Equation (9.1), every zero of order m contributes m to the sum and every pole of order m contributes $-m$ to the sum. We conclude

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=1}^{k} \operatorname{res}\left(\frac{f'}{f}, a_j\right) = Z - P,$$

as claimed. □

Rouché's Theorem

The above theorem gives a lot of information about the location and multiplicities of zeros and poles. We shall use this to establish the following result:

Theorem 9.3 (Rouché's Theorem) *Let f and g be holomorphic on an open set containing a contour γ and its interior. Suppose*

$$|f(z)| > |g(z)| \qquad \text{for } z \text{ on } \gamma^*.$$

Then f and $f+g$ have the same number of zeros inside γ.

PROOF: Let t be a real number with $t \in [0,1]$. Note that

$$|f(z) + t\,g(z)| \geqslant |f(z)| - t\,|g(z)| \geqslant |f(z)| - |g(z)| > 0$$

by our assumption, so

$$f(z) + t\,g(z) \neq 0 \qquad \text{for all } z \in \gamma^*.$$

Hence the function $f + tg$ satisfies the hypotheses of Theorem 9.2 and we conclude

$$\phi(t) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + t\,g'(z)}{f(z) + t\,g(z)}\,\mathrm{d}z$$

equals the number of zeros (up to multiplicity) of $f(z) + t\,g(z)$ inside γ.

We claim that ϕ is a continuous function of t. Indeed, observe

$$\begin{aligned}\phi(t) - \phi(s) &= \frac{1}{2\pi i}\int_\gamma \left(\frac{f'(z) + t\,g'(z)}{f(z) + t\,g(z)} - \frac{f'(z) + s\,g'(z)}{f(z) + s\,g(z)}\right) \mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma \frac{(f(z) + s\,g(z))(f'(z) + t\,g'(z)) - (f(z) + t\,g(z))(f'(z) + s\,g'(z))}{(f(z) + t\,g(z))(f(z) + s\,g(z))} \mathrm{d}z \\ &= \frac{t-s}{2\pi i}\int_\gamma \frac{f(z)\,g'(z) - f'(z)\,g(z)}{(f(z) + t\,g(z))(f(z) + s\,g(z))} \mathrm{d}z.\end{aligned}$$

Recall that $f(z) + t\,g(z)$ is non-zero for $t \in [0, 1]$ and $z \in \gamma^*$. Hence the integrand above is a continuous function of s, t and z. We have already noted that a continuous function defined on a closed and bounded subset of $\mathbb{C}$ is bounded. The same is true for a continuous function of (s, t, z) on the set $[0, 1] \times [0, 1] \times \gamma^*$ (basically because $[0, 1]$ is a closed and bounded subset of $\mathbb{R}$ and γ^* is a closed and bounded subset of $\mathbb{C}$). Hence, there exists some constant M such that

$$\left|\frac{f(z)\,g'(z) - f'(z)\,g(z)}{(f(z) + t\,g(z))(f(z) + s\,g(z))}\right| \leqslant M$$

for all $s, t \in [0, 1]$ and all $z \in \gamma^*$. Hence, by the Crude Estimation Theorem (Theorem 3.14),

$$|\phi(t) - \phi(s)| \leqslant \frac{M \cdot L(\gamma)}{2\pi}\,|t - s|\,,$$

which is enough to show ϕ is continuous. (Given $\varepsilon > 0$, take $\delta = \varepsilon/M \cdot L(\gamma)$. For such δ, $|t - s| < \delta$ implies $|\phi(t) - \phi(s)| < \varepsilon$.)

However, the function ϕ counts the number of zeros of $f + tg$ inside γ, so *only takes integer values*. Hence, if ϕ is also continuous, then we conclude ϕ is constant. Therefore $\phi(0) = \phi(1)$; that is $f(z) + g(z)$ has the same number of zeros (including multiplicity) as $f(z)$ inside γ. □

Example 9.4 *Show that all the solutions of*

$$z^5 + z^3 + 2z + 5 = 0$$

satisfy $|z| < 2$.

Solution: Take $f(z) = z^5$, $g(z) = z^3 + 2z + 5$ and γ be the positively oriented circular contour of radius 2 about 0. Note that $f(z)$ has zeros at 0 (of multiplicity 5), so all five of the zeros of $f(z)$ lie inside γ. Note that when z lies on γ^*,

$$|f(z)| = \left|z^5\right| = 2^5 = 32$$

and

$$|g(z)| = \left|z^3 + 2z + 5\right| \leqslant |z|^3 + 2\,|z| + 5 = 2^3 + 4 + 5 = 17.$$

Hence f and g satisfy the hypotheses of Rouché's Theorem, so

$$f(z) + g(z) = z^5 + z^3 + 2z + 5$$

has the same number of zeros (that is, five) inside γ as f. Thus all five solutions of $f(z)+g(z) = 0$ lie inside γ; that is, satisfy $|z| < 2$. □

Example 9.5 *Determine the number of solutions of*

$$z^5 + 3z^2 + 6z + 1 = 0$$

in the open annulus $A = \{\, z \in \mathbb{C} \mid 1 < |z| < 2 \,\}$.

SOLUTION: First take γ_1 to be the positively oriented circular contour of radius 1 about 0,

$$f_1(z) = 6z + 1 \qquad \text{and} \qquad g_1(z) = z^5 + 3z^2.$$

Note that f_1 has one zero inside γ_1, namely $z = -\frac{1}{6}$. If z lies on γ_1^*, then

$$|f_1(z)| = |6z + 1| \geqslant 6\,|z| - 1 = 5$$

and

$$|g_1(z)| = \left|z^5 + 3z^2\right| \leqslant \left|z^5\right| + 3\left|z^2\right| = 4.$$

Hence, by Rouché's Theorem (Theorem 9.3), $f_1(z) + g_1(z) = z^5 + 3z^2 + 6z + 1$ has the same number of zeros inside γ_1 as $f_1(z)$ does, namely one.

Also note, as observed in the proof of Rouché's Theorem, $f_1(z) + g_1(z)$ is non-zero on γ_1 since $|f_1(z) + g_1(z)| \geqslant |f_1(z)| - |g_1(z)| \geqslant 5 - 4 = 1$.

Now take γ_2 be the positively oriented circular contour of radius 2 about 0,

$$f_2(z) = z^5 \qquad \text{and} \qquad g_2(z) = 3z^2 + 6z + 1.$$

Note that $f(z)$ has five zeros inside γ_2, and that if z lies on γ_2^*, then

$$|f_2(z)| = |z|^5 = 2^5 = 32$$

and

$$|g_2(z)| = \left|3z^2 + 6z + 1\right| \leqslant 3\,|z|^2 + 6\,|z| + 1 = 25.$$

Hence, by Rouché's Theorem, $f_2(z) + g_2(z) = z^5 + 3z^2 + 6z + 1$ has the same number of zeros inside γ_2 as $f_2(z)$ does, namely five.

Putting this together, we conclude that $z^5 + 3z^2 + 6z + 1 = 0$ has *four* solutions in the annulus $A = \{\, z \in \mathbb{C} \mid 1 < |z| < 2 \,\}$. □

The Argument Principle

Let us return to the conclusion of Theorem 9.2. Consider a function that is holomorphic on and inside a contour γ and suppose f is non-zero on the image of γ. Then the theorem says that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = Z, \tag{9.3}$$

the number of zeros inside γ.

Now if we cut the plane, we can define a branch of logarithm that is holomorphic on the resulting cut plane. This will mean that on suitable open subsets of $\mathbb{C}$, namely those on which both logarithm and f are holomorphic,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\log f(z)) = \frac{f'(z)}{f(z)}.$$

Consequently, for suitable curves $\gamma\colon [a,b] \to \mathbb{C}$ that, in particular, do not cross the cut

$$\int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = \log f(\gamma(b)) - \log f(\gamma(a)).$$

This suggests that the integral in Equation (9.3) should actually give the answer zero, since we are integrating around a closed contour, whereas we know it actually equals the number of zeros inside γ. What is happening here? The answer relates to the behaviour of the logarithm

$$\log f(z) = \log |f(z)| + i \arg f(z)$$

and the fact that every time that $f(z)$ crosses the branch cut, the argument jumps by 2π. Hence the reason why we get the answer Z, the number of zeros, in Equation (9.3) is that $f(z)$ has travelled Z times around the origin 0 as we traced γ and, each time we cross the branch cut, $2\pi i$ is added to the value of the integral.

In view of this we make the following definition:

Definition 9.6 Write $\Delta_\gamma(\arg f)$ for the overall change in the argument of $f(z)$ as z traces the curve γ; that is,

$$\Delta_\gamma(\arg f) = \frac{1}{i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

The interpretation of the Argument Principle, that we shall treat as a method in what follows, is that Theorem 9.2 tells us that, if f is holomorphic on and inside a contour γ with f non-zero on the image γ^*, then

$$\frac{1}{2\pi}\,\Delta_\gamma(\arg f) = Z,$$

the number of zeros of f inside γ. All we need to do, therefore, is keep track of the change in the argument of $f(z)$ as we follow the contour γ.

Example 9.7 Consider the function $f(z) = z^2$ and let γ be the positively oriented circular contour of radius 1 about 0. Let us compare the three we values we are interested in.

(i) The function f has a repeated zero at $z = 0$ inside γ, so the number of zeros is $Z = 2$.

(ii) The derivative is $f'(z) = 2z$, so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{2}{z}\,\mathrm{d}z = 2$$

by Cauchy's Integral Formula, Theorem 4.2 (or by Example 3.9).

(iii) The contour γ is parametrised as $z = \gamma(t) = \mathrm{e}^{it}$ and as t increases from 0 to 2π, the argument of e^{it} correspondingly increases. At the same time, $z^2 = \mathrm{e}^{2it}$ and as t increases from 0 to 2π, the argument of z^2 increases (continuously) from 0 to 4π as we circle the origin twice. Hence

$$\Delta_\gamma(\arg z^2) = 4\pi,$$

so that

$$\frac{1}{2\pi}\Delta_\gamma(\arg z^2) = 2,$$

as expected.

Example 9.8 *Determine the number of solutions of the equation $z^3+1 = 0$ in the first quadrant of the complex plane.*

In some ways this question is easy: The solutions of $z^3+1 = 0$ are $\mathrm{e}^{\pi i/3}$, $\mathrm{e}^{\pi i} = -1$ and $\mathrm{e}^{5\pi i/3}$. Of these, only the first lies in the first quadrant, so the answer is one. We shall use this example to illustrate how to apply the Argument Principle to solve the problem so as to consider a more challenging example next.

SOLUTION: Define $f(z) = z^3 + 1$.

First note that when x is a real number with $x \geqslant 0$, then $f(x) = x^3 + 1 \geqslant 1$, while if y is a real number, then $f(iy) = (iy)^3 + 1 = 1 - iy^3 \neq 0$ (as its real part is non-zero). Hence there are no zeros of $f(z)$ on the positive real or imaginary axes. There are three zeros of $f(z) = z^3 + 1$ in $\mathbb{C}$, so we can take some radius $R > 0$ sufficiently large such that all zeros of $z^3 + 1$ that lie in the first quadrant are in the interior of the contour γ shown in Figure 9.1.

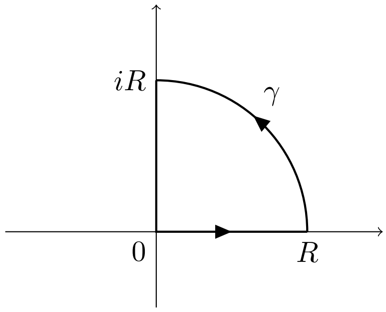


Figure 9.1: Application of the Argument Principle

As z travels along the real axis, $f(z)$ remains real along the line segment from 0 to R; that is, the argument of $f(z)$ takes the value 0 along this part of the contour. Thus

$$\Delta_{[0,R]}(\arg f(z)) = 0.$$

The quarter circle piece Γ_R of the contour can be parametrised as $\Gamma_R(t) = R\,\mathrm{e}^{it}$ for $0 \leqslant t \leqslant \pi/2$. Observe that for $z = \Gamma_R(t)$,

$$f(z) = R^3\,\mathrm{e}^{3it} + 1 = (R^3 \cos 3t + 1) + i\,R^3 \sin 3t.$$

Thus the argument ϕ of $f(z)$ is given by

$$\phi = \tan^{-1}\left(\frac{R^3 \sin 3t}{R^3 \cos 3t + 1}\right) \approx 3t,$$

when R is large. (The approximation is valid here, because our final answer will be an integer so we lose nothing by dropping very small terms.) Since t ranges from 0 to $\pi/2$, the argument of $f(z)$ ranges from 0 to (approximately) $3\pi/2$. We conclude that the increase in argument of $f(z)$ as z ranges along the curve Γ_R is

$$\Delta_{\Gamma_R}(\arg f(z)) \approx \frac{3\pi}{2}.$$

Finally, as z travels along the line segment from iR to 0 along the imaginary axis, $z = iy$. For such z,

$$f(z) = (iy)^3 + 1 = 1 - iy^3$$

and the argument is given by

$$\phi = \tan^{-1}(-y^3).$$

As we travel along the line segment, y varies from a large value of R to 0, so that ϕ varies from a starting value of $\tan^{-1}(-R^3) \approx -\pi/2$ and increases to $\tan^{-1}(0) = 0$. Hence

$$\Delta_{[Ri,0]}(\arg f(z)) \approx \frac{\pi}{2}.$$

Putting this together,

$$\frac{1}{2\pi}\Delta_\gamma(\arg f(z)) = \frac{1}{2\pi}\left(0 + \frac{3\pi}{2} + \frac{\pi}{2}\right) = 1.$$

Thus f has one zero inside the first quadrant, as predicted. $\square$

Note that the approximations made when calculating $\Delta_{\Gamma_R}(\arg f(z))$ and $\Delta_{[iR,0]}(\arg f(z)$ will cancel. Whatever the final answer actually is, it must be an integer, and the *precise* value of $\Delta_{\Gamma_R}(\arg f(z))$ will be a little bit bigger than the value $3\pi/2$ used (since $R^3\sin 3t/(R^3\cos 3t + 1) > \tan 3t$) while the value of $\Delta_{[iR,0]}(\arg f(z))$ will be a little bit less than the value $\pi/2$ (since $\tan^{-1}(-R^3) > -\pi/2$).

We now turn to an example where the method can be used, but where the answer is not obvious.

Example 9.9 *Determine the number of solutions of the equation*

$$z^3 - iz - i = 0$$

in the first quadrant of the complex plane.

SOLUTION: Define $f(z) = z^3 - iz - i$. Note that when x is real, $f(x) = x^3 - i(x+1)$ is non-zero as its real and imaginary parts cannot be simultaneously be zero. Similarly, when y is real, $f(iy) = y - i(y^3+1)$ is non-zero. Hence there are no zeros of $f(z)$ on the positive real or imaginary axes. There are at most three zeros of $f(z)$ in the first quadrant, so we can take a sufficiently large radius $R > 0$ such that all the zeros of $f(z)$ in the first quadrant lie inside the contour γ shown in Figure 9.1 used above. We shall consider the change of argument $\phi = \arg f(z)$ as z travels along the contour γ.

On the line segment $[0, R]$: Here $z = x$ with $0 \leqslant x \leqslant R$ and

$$f(z) = x^3 - i(x+1).$$

Hence

$$\phi = \arg f(z) = \tan^{-1}\left(-\frac{x+1}{x^3}\right).$$

As x increases from 0 to R, the fraction $-(x+1)/x^3$ varies from $-\infty$ to $-(R+1)/R^3 \approx 0$, and the value of ϕ varies from $-\pi/2$ to (approximately) 0. Thus

$$\Delta_{[0,R]}(\arg f(z)) \approx \frac{\pi}{2}.$$

On the quarter circle Γ_R: Here $z = R\,\mathrm{e}^{it}$ with $0 \leqslant t \leqslant \pi/2$ and

$$f(z) = R^3\mathrm{e}^{3it} - iR\,\mathrm{e}^{it} - i = (R^3\cos 3t + R\sin t) + (R^3\sin 3t - R\cos t - 1)i.$$

Hence

$$\phi = \arg f(z) = \tan^{-1}\left(\frac{R^3\sin 3t - R\cos t - 1}{R^3\cos 3t + R\sin t}\right) \approx 3t$$

when R is large. As t increases from 0 to $\pi/2$, the value of ϕ increases from 0 to $3\pi/2$, so

$$\Delta_{\Gamma_R}(\arg f(z)) \approx \frac{3\pi}{2}.$$

On the line segment $[0, iR]$: Here $z = iy$ where y decreases from R to 0 as we travel along the line segment. Then

$$f(z) = y - i(y^3+1)$$

and

$$\phi = \arg f(z) = \tan^{-1}\left(-\frac{y^3+1}{y}\right).$$

Here we need to analyse the behaviour of $-(y^3+1)/y$ as y ranges from R down to 0. Note that

$$\frac{\mathrm{d}}{\mathrm{d}y}(-y^2 - y^{-1}) = -2y + y^{-2} = \frac{1-2y^3}{y^2},$$

which vanish when $y^3 = 1/2$ and is positive when $0 < y < 1/\sqrt[3]{2}$ and negative for $y > 1/\sqrt[3]{2}$. Hence, when we allow y to *decrease* from R to 0, the fraction $-(y^3+1)/y$ initially increases from its initial value $-(R^3+1)/R \approx -R^2$, to a maximum (albeit still negative) value at $y = -1/\sqrt[3]{2}$, and then decreases towards $-\infty$ (which it approaches as $y \to 0$).

The corresponding effect on the argument is that ϕ starts starts at the value of (approximately) $-\pi/2$ as we begin tracing the line segment, increases in value, but returns to the value $-\pi/2$ as we complete the segment. In summary,

$$\Delta_{[iR,0]}(\arg f(z)) \approx 0.$$

Hence

$$\frac{1}{2\pi}\Delta_\gamma(\arg f(z)) = \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{3\pi}{2} + 0\right) = 1.$$

We conclude that f has one zero in the first quadrant of the complex plane. $\square$

Bibliography

[1] John M. Howie, *Complex Analysis*, Springer Undergraduate Mathematics Series, Springer-Verlag, London 2003.

[2] H. A. Priestley, *Introduction to Complex Analysis, Second Edition*, OUP, 2003.