Heron Triangles

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Introduction

The problem that I have been given is to explore Heron triangles. A Heron triangle is a triangle whose sides are all integers. It also has an area that is an integer as well. An integer is a whole number that can be written without a fraction or a decimal. Heron, a Greek mathematician, discovered a formula for finding the areas of such triangles. The lengths of the three sides are a, b, and c:

Area = $\sqrt{(s(s - a)(s - b)(s - c))}$ where s equals ½ the perimeter:

$s = \frac{1}{2}(a + b + c)$

For example: the 3, 4, 5 triangle is a Heron triangle. Let us find out what s is:

$s = \frac{1}{2}(a + b + c)$

$s = \frac{1}{2}(3 + 4 + 5)$

$s = \frac{1}{2}(12)$

$s = 6$

Now we will plug 6 into the formula for s in order to find the area:

Area = $\sqrt{(s(s - a)(s - b)(s - c))}$

Area = $\sqrt{(6(6 - 3)(6 - 4)(6 - 5))}$

Area = $\sqrt{(6(3)(2)(1))}$

Area = $\sqrt{(36)}$

Area = 6

Since all three sides are integers, and the area is an integer, a 3, 4, 5 triangle is a Heron triangle.

With Heron triangles, you can multiply the sides by the same number in order to create other Heron triangles. The area of that triangle will be that number squared times the original area. For example, the sides of the 3, 4, 5 triangle can be doubled to create a 6, 8, 10 triangle whose area is 2^2 x 6, which is 24. Or you can multiply all 3 sides by 3 to get a 9, 12, 15 triangle with an area of 3^2 x 6 which is 54. All the relatives of the 3, 4, 5 triangle are Heron triangles.

Primitive Heron triangles have side lengths that share no common factors. For example, the 3, 4, 5 triangle is a primitive triangle whose area is 6 (note that 3, 4, and 5 share no common factors other than 1). A 9, 12, 15 Heron triangle is not primitive as all three sides can be divided by 3.

I have found 20 primitive Heron triangles with areas less than 100. The areas are included in the list in parenthesis:

- 3,4,5 (6) - Pythagorean
- 3, 25, 26 (36)
- 4, 13, 15 (24)
- 4, 51, 53 (90)
- 5, 5, 6 (12) - Isosceles
- 5, 5, 8 (12) - Isosceles
- 5, 12, 13 (30) - Pythagorean
- 5, 29, 30 (72)
- 6, 25, 29 (60)

- 7, 15, 20 (42)
- 7, 24, 25 (84) - Pythagorean
- 8, 15, 17 (60) - Pythagorean
- 8, 29, 35 (84)
- 9, 10, 17 (36)
- 10, 13, 13 (60) - Isosceles
- 10, 17, 21 (84)
- 11, 13, 20 (66)
- 12, 17, 25 (90)
- 13, 13, 24 (60) - Isosceles
- 13, 14, 15 (84)

Section 1: Pythagorean Triangles

There are many interesting facts about Heron triangles that I will address. The first one is that all Pythagorean triangles are Heron triangles. That is, all right triangles whose sides (a, b, c) follow the equation $a^2 + b^2 = c^2$, and that have all their sides as integers then have areas that are integers. Sides a and b are the shorter sides that meet at a 90° angle and side c is the hypotenuse. This formula $a^2 + b^2 = c^2$ is the Pythagorean formula. To find the area of such a triangle, you multiply side a by side b and take half of this. This formula is written as $A = \frac{1}{2}bh$ in which A = area, b = base or side b, and h = height or side a. Thus, $A = \frac{1}{2}ab$.

A Pythagorean triangle is made up of sides that have two odd numbers and one even number. This is so because of modulus 2. Let us take a look at how this works and note that an explanation of remainders upon division 2 will follow the chart:

Modulus 2: Multiplication

x	0	1
0	0	0
1	0	1

The 0 represents even numbers because in terms of modulus 2, the 0 represents 0 remainders when dividing by 2. The 1 represents odd numbers because in terms of modulus 2 , the 1 represents a remainder of 1 when dividing by 2. If we look at all the possibilities of odds and evens multiplied together, they follow the pattern of modulus 2.

Even x even = even yes, this works: For example, 2 x 2 = 4 and even x even ≠ odd. (This appears as 0 x 0 = 0 in the given chart.)

Even x odd = even yes, this works: For example, 4 x 3 = 12 and even x odd ≠ odd. (This appears as 0 x 1 = 0 in the given chart.)

Odd x odd = odd yes this works: For example, 3 x 5 = 15 and odd x odd ≠even. (This appears as 1 x 1 = 1 in the given chart.)

Odd x even = even yes this works: For example, 5 x 4 = 20 and odd x even ≠ odd. (This appears as 1 x 0 = 0 in the given chart.)

Let us look at the addition in terms of modulus 2:

Modulus 2: Addition

+	**0**	**1**
0	**0**	**1**
1	**1**	**0**

As stated before, 0 represents even numbers and 1 represents odd numbers (only in terms of modulus 2). The chart shows us that an even number plus an even number is an even number; for example 2 + 2 = 4. An even number plus an odd number is an odd number; for example 2 + 3 = 5. An odd number plus an even number is an odd number; for example 3 + 4 = 7. And an odd number plus an odd number is an even number; for example 5 + 5 = 10.

It is given that all three sides of a Pythagorean triangle are integers. It is also given that the area of a Pythagorean triangle is an integer. To find the area of a Pythagorean triangle, one must multiply the two shorter sides together and then divide the product by two. Because the product of the two sides is divided by two and the product is also an integer, then this product must be an even number. If we look at the chart for multiplication modulus 2, we see that there are only two possibilities to arrive at a product that is an even number: an even number times an even number, or an even number times an odd number (odd times even is the same). When both sides are even, the third side must be even because of Addition modulus 2. Even plus even = even, so the first 2 eve numbers are even and must be added to an even number to get a sum that is even. If all three sides are even, then we are looking at a triangle that is not a primitive triangle because all three sides can be divided by two. I want to focus on primitive triangles, so this means that in a Pythagorean triangle, there must be one even numbered side and one odd numbered side. I believe that the hypotenuse must be an odd number as we will see below.

Now we know that with Pythagorean triangles, $a^2 + b^2 = c^2$. So we are looking at the sum of two numbers: $a^2 + b^2$. Note that an even number squared is an even number and an odd number squared is an odd number (see multiplication modulus 2). So then we are looking at the sum of an even number and an odd number. This sum has to be odd, according to the Addition modulus 2 chart above. This means that the hypotenuse must therefore be an odd number. It should also be noted that there are possibilities of an even number² + an even number² which is an even number. But we have already addressed the fact that we want to look at primitive triangles. An odd number² + an even number² is an odd number, and therefore the hypotenuse is an odd number.

Heon triangles also must have 2 sides odd and one side even. To find the area of a Heron triangle (recall that Area = $\sqrt{s(s-a)(s-b)(s-c)}$), you must take half the perimeter to make s (remember that $s = \frac{1}{2}(a + b + c)$). So I will now work with the formula for area substituting s for $\frac{1}{2}(a + b + c)$.

$$A = \sqrt{(\tfrac{1}{2}(a + b + c))(\tfrac{1}{2}(a + b + c) - a)(\tfrac{1}{2}(a + b + c) - b)(\tfrac{1}{2}(a + b + c) - c)}$$

$$A^2 = (\tfrac{1}{2}(a + b + c))(\tfrac{1}{2}(a + b + c) - a)(\tfrac{1}{2}(a + b + c) - b)(\tfrac{1}{2}(a + b + c) - c)$$

$$A^2 = ((a + b + c)/2)((a + b + c)/2 - (2a/2))((a + b + c)/2 - (2b/2))((a + b + c)/2 - (2c/2))$$

$A^2 = ((a + b + c)/2)(-a + b + c)/2)(a - b + c)/2)(a + b - c)/2))$

$A^2 = ((a + b + c)(-a + b + c)(a - b + c)(a + b - c)/16)$

$16A^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$

We know that the area, A, is an integer. We also know that because it is multiplied by 16, the product of $(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$ must be even. Let us look at modulus 2. If A is odd, then $odd^2 = odd$. Then we multiply the odd number by 16. An odd number times an even number is even. So we know that $16A^2$ = even number.

Remember that (a, b, c) are the sides of a Heron triangle. I will use both addition and multiplication modulus 2 to multiply this out in terms of even (ie; 0) sides and odd (ie; 1) sides. $16A^2$ is even and therefore we will call it 0 in modulus 2.

When all sides are even, a = 0, b = 0, c = 0:

$0 = (0 + 0 + 0)(-0 + 0 + 0)(0 - 0 + 0)(0 + 0 - 0)$

$0 = 0$

So therefore in a Heron triangle, all three sides can be even. But remember that we are looking at primitive Heron triangles and if all three sides are even, they can be divided by two.

When one side is odd and two sides are even, a = 1, b = 0, c = 0:

$0 = (1 + 0 + 0)(-1 + 0 + 0)(1 - 0 + 0)(1 + 0 - 0)$

$0 = (1)(-1)(1)(1)$

$0 = -1$

This does not work because 0 does not equal 1. A Heron triangle cannot have one odd side with two even sides.

When two sides are odd and one side is even, a = 1, b = 1, c = 0:

$0 = (1 + 1 + 0)(-1 + 1 = 0\ )(1 – 1 + 0)(1 + 1 – 0)$

$0 = (0)(0)(0)(0)$

$0 = 0$

Note that when you add 1 + 1 you get 2 which becomes 0 in modulus 2.

Now let us look at the possibility of all three sides being odd. Then a = 1, b = 1, and c = 1:

$0 = (1 + 1 + 1)(-1 + 1 + 1)(1 – 1 + 1)(1 + 1 – 1)$

$0 = (1)(1)(1)(1)$

$0 = 1$

This does not work because 0 does not equal 1. Note that 1 + 1 + 1 = 3 which is 1 in modulus 2.

Therefore you cannot have a Heron triangle in which all three sides are odd. The only possibility that leads to a primitive Heron triangle is when there is one even side and two odd sides.

Another valuable fact about Pythagorean triangles is that one can create Pythagorean triangles from two odd numbers. Choose two odd numbers, m and n. Then the sides a, b, c are equal to the formulas below. I do not know where these formulas come from but I do know that the early Greeks used these formulas to generate Pythagorean triangles.

$a = (m^2 - n^2)/2$

$b = mn$

$c = (m^2 + n^2)/2$

Note that according to modulus 2, these three numbers a, b, and c must be integers. For a, let us consider 2 odd numbers m and n. Since they are both odd, their squares are also odd (see multiplication, modulus 2). Now, subtraction is the inverse operation of addition so these two operations are closely related. If we subtract two odd numbers, according to addition modulus 2, we should get an even number. 5 – 3 = 2. Where there are two odd numbers on the chart, there is an even number. And the difference of these two odd numbers is divided by two. Since the difference of these two odd numbers is an even number, it is indeed divisible by two and therefore, a is an integer.

Now, let us look at b. b is the product of two odd numbers. According to multiplication modulus 2, the product of two odd numbers is an odd number, so therefore b is an integer.

With regards to c, it is similar to a. An odd number squared is an odd number, so we are looking at the sum of two odd numbers: $m^2 + n^2$. According to addition modulus 2, the sum of two odd numbers is an even number. 25 + 9 = 24. This sum is then divided by two and since the sum of these two numbers is even, it is indeed divisible by two and therefore c is an integer.

Now let us look at these formulas to see if they make sense. The proof is pretty much straight forward. Since $a^2 + b^2 = c^2$, I will show that $((m^2 - n^2)/2)^2 + (mn)^2 = ((m^2 + n^2)/2)^2$.

$a^2 = ((m^2 - n^2)/2)^2 =$

$(m^2 - n^2)/2 \text{ x } (m^2 - n^2)/2 =$

$(m^2 - n^2)(m^2 - n^2)/4 =$

$(m^2(m^2 - n^2) - n^2(m^2 - n^2))/4 =$

$(m^4 - m^2n^2 - (m^2n^2 - n^4))/4 =$

$(m^4 - m^2n^2 - m^2n^2 + n^4)/4 =$

$(m^4 - 2m^2n^2 + n^4)/4$

So $a^2 = ((m^2 - n^2)/2)^2 = (m^4 - 2m^2n^2 + n^4)/4$

$b^2 = (mn)^2 = m^2n^2$

$a^2 + b^2 = (m^4 - 2m^2n^2 + n^4)/4 + m^2n^2 =$

$(m^4 - 2m^2n^2 + n^4)/4 + 4m^2n^2/4=$

$(m^4 - 2m^2n^2 + n^4 + 4m^2n^2)/4 =$

$(m^4 + 2m^2n^2 + n^4)/4 =$

$((m^2 + n^2)/2)^2$ and this equals c^2

So $a^2 + b^2 = c^2$, and $((m^2 - n^2)/2)^2 + (mn)^2 = ((m^2 + n^2)/2)^2$.

I will try these formulas on 2 odd numbers to see if it really works. I pick 9 and 7.

$a = (m^2 - n^2)/2$ So $a = (9^2 - 7^2)/2 = (81 - 49)/2 = 32/2 = 16$

$b = mn$ So $b = 9 \times 7 = 63$

$c = (m^2 + n^2)/2$ So $c = (9^2 + 7^2)/2 = (81 + 49)/2 = 130/2 = 65$

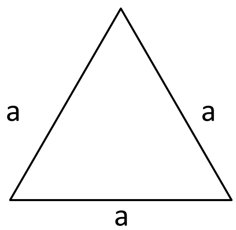
This yields a Pytharogean triangle of 16, 63, 65. To verify this, $16^2 + 63^2$ must equal 65^2. $16^2 + 63^2 = 256 + 3969 = 4225$ and $\sqrt{(4225)} = 65$.

Section 2: Equilateral Triangles

Equilateral triangles can never be Heron triangles. First of all, as stated above, a primitive Heron triangle must have two odd sides and one even side. This automatically means that all three numbers cannot be the same number. An equilateral triangle has three equal sides by definition.

The other reason why equilateral triangles cannot be Heron triangles is because when you calculate the area of an equilateral triangle in terms of a = one of the sides, you end up with a number that cannot be an integer.

Start with an equilateral triangle whose sides are of the length a.



If we divide the triangle in half, we will now have a right triangle with the sides of ½ a, b (the new side whose quantity is unknown), and the hypotenuse a. Because this is a right triangle, we can square both sides (b and ½ a) to get the hypotenuse a squared. Then we find out what b is in terms of a:

$b^2 + (\frac{1}{2} a)^2 = a^2$

$b^2 + \frac{1}{4} a^2 = a^2$

$b^2 = a^2 - \frac{1}{4} a^2$

$b^2 = 4a^2/4 - \frac{1}{4} a^2$

$b^2 = (4a^2 - 1a^2)/4$

$b^2 = (3a^2)/4$

$b = (a\sqrt{3})/2$

Now we have a right triangle with the two short sides $(a\sqrt{3})/2$ and ½ a. If we multiply them together, they give us half the area. But we do not need to divide the area in half because we have 2 identical triangles with these measurements.

$(a\sqrt{3})/2 \times \frac{1}{2} a = (a^2\sqrt{3})/4.$

So area = $(a^2\sqrt{3})/4$

This area is clearly not an integer. The square root of 3 is an irrational number and cannot be expressed as any fraction or whole number. Therefore $(a^2\sqrt{3})/4$ is not the area of a Heron triangle. Consequently, equilateral triangles cannot be Heron triangles.

Section 3: Isosceles Triangles

Some isosceles triangles can be Heron triangles. Let us look at the algebra behind the isosceles triangle. If we have a triangle whose equal sides are a with a base b and we cut the triangle down the middle, now we have two right triangles with hypotenuses a, bases ½ b and a shared side c. Because these are right triangles, we know that $a^2 = c^2 + (\frac{1}{2}b)^2$. Let us solve for c so we can find the area in terms of a and b:

$a^2 = c^2 + (\frac{1}{2}b)^2$

$a^2 = c^2 + \frac{1}{4} b^2$

$c^2 = a^2 - \frac{1}{4} b^2$

$c^2 = (4a^2)/4 - (b^2)/4$

$c^2 = (4a^2 - b^2)/4$

$c = \sqrt{(4a^2 - b^2)}/\sqrt{4}$

$c = \sqrt{(4a^2 - b^2)}/2$

The area of the original isosceles triangle would then be:

$\text{Area} = c \times \frac{1}{2} b$

$\text{Area} = \sqrt{(4a^2 - b^2)}/2 \ \times \ b/2$

$\text{Area} = (b\sqrt{(4a^2 - b^2)})/4$

Because Heron triangles must have the area as an integer, therefore we want the numbers under the radical sign to be integers and to be perfect squares. In other words, $4a^2 - b^2$ needs to be a perfect square. Let us make $4a^2 - b^2 = d^2$ and we will solve for a because a is the hypotenuse:

$4a^2 - b^2 = d^2$

$4a^2 = d^2 + b^2$

$(2a)^2 = d^2 + b^2$

We are looking for right triangles where two sides are d and b and the hypotenuse is 2a. From what we said above about Pythagorean triangles, we know that it cannot be primitive. Let us try one of the isosceles Heron triangles that we already know: 5, 5, 6 where a = 5 and b = 6:

$(2a)^2 = d^2 + b^2$

$(2 \times 5)^2 = d^2 + 6^2$

$10^2 = d^2 + 36$

$100 = d^2 + 36$

$d^2 = 100 - 36$

$d^2 = 64$

$d = 8$

Now let us find out what c is. Remember that $c = \sqrt{(4a^2 - b^2)}/2$ and we can substitute and values of $a = 5$ and $b = 6$.

$c = \sqrt{(4 \times 5^2 - 6^2)}/2$

$c = \sqrt{(4 \times 25 - 36)}/2$

$c = \sqrt{(100 - 36)}/2$

$c = \sqrt{(64)}/2$

$c = 8/2$

$c = 4$

We know that the area = c x ½ b, so the area = 4 x ½ 6 = 4 x 3 = 12. The area of the 5, 5 ,6 triangle is an integer and its sides are integers. Therefore the 5, 5, 6 triangle is a Heron triangle as well as an isosceles triangle.

Now, because c = 4 and c is the height of the triangle, we see that this isosceles triangle is made up of two identical right triangles with sides 3, 4, 5. In fact, all Heron isosceles triangles are made up of two right triangles. Because all right triangles are Heron triangles, therefore all Heron isosceles triangles are made up of two identical Heron triangles.

I found four primitive isosceles Heron triangles: 5, 5, 6; 5, 5, 8; 13, 13, 10; and 13, 13, 24. Each of these triangles can be manipulated by the formulas above to show what right triangles they are made from and to show that they all have areas that are integers.

I decided to test the theory that one can construct an Isosceles Heron triangle from a Heron Pythagorean triangle. Let us start with two odd numbers, m and n. Let us make m = 7 and n = 3. Earlier in the paper I discussed that one can use the formulas below to create a Heron Pythagorean triangle:

$a = (m^2 - n^2)/2$

$b = mn$

$c = (m^2 + n^2)/2$

$a = (7^2 - 3^2)/2 = (49 - 9)/2 = 40/2 = 20$

$b = m \times n = 7 \times 3 = 21$

$c = (7^2 + 3^2)/2 = (49 + 9)/2 = 58/2 = 29$

So now I have a Pythagorean Heron triangle with sides 20, 21 and 29. I was able to create an isosceles Heron triangle of 29, 29, 40 by using the hypotenuse (the longest side) and doubling one of the bases. This triangle, 29, 29, 40 is one possibility if you make a = 29 and b = 40. This is two triangles of 20, 21, 29 connected by the side of 21 to make a Heron Isosceles triangle with a base of 40. One can also connect these 2 Pythagorean Heron triangles by the side of 20 to make a Heron Isosceles triangle of 29, 29, 41. Because $(2a)^2 = d^2 + b^2$, either number for the base can be exchanged (in this case 20 and 21 to make bases 40 and 42) so two different Heron Isosceles triangles can be created by a Pythagorean triangle. Some Heron triangles can indeed be isosceles triangles and some isosceles triangles can be Heron triangles.

Section 4: Scalene Triangles

The majority of Heron triangles are scalene triangles. A scalene triangle must have 3 side lengths that are not equal.

Method 1

One can create scalene Heron triangles three different ways. One way is by finding two different Pythagorean triangles that share one common side. If we use the formulas mentioned in section 1, we can say that one Pythagorean triangle will have the sides of $(m^2 - n^2)/2$, mn, and $(m^2 + n^2)/2$ where m and n are two odd numbers. Likewise, we can create another Pythagorean triangle with the formula $(p^2 - q^2)/2$, pq, and $(p^2 + q^2)/2$ where p and q are two odd numbers. The trick is to find four odd numbers in which $mn = pq$.

In order to find these four odd numbers, one must find an odd number made up of the product of three odd numbers, in which two of the numbers must be different. That is, say I take the product of 3 x 3 x 5 = 45. Then I can make 45 two different ways: 3 x 15 and 9 x 5. I can also find the product of 3 x 5 x 5 = 75. 75 can be made two different ways: 3 x 25 and 15 x 5.

Now, let us look at the example of 45: since $mn = pq$ we can use $15 \times 3 = 9 \times 5$ where $m = 15$, $n = 3$, $p = 9$ and $q = 5$. This will create two different triangles. I am creating a triangle a, b, c out of m and n. Next I will create triangle d, e, f out of p and q:

$a = (m^2 - n^2)/2 = (15^2 - 3^2)/2 = (225 – 9)/2 = 216/2 = 108$

$b = mn = 15 \times 3 = 45$

$c = (m^2 + n^2)/2 = (15^2 + 3^2)/2 = (225 + 9)/2 = 234/2 = 117$

Triangle a, b, c (108, 45, 117) is a right triangle and also a Heron triangle.

Since $a^2 + b^2 = c^2$, then $108^2 + 45^2 = 117^2$ which is $11{,}664 + 2025 = 13{,}689$ and this is correct.

Since the area $= \sqrt{(s(s – a)(s – b)(s - c))}$ when $s = ½ (a + b + c)$, then $s = ½(108 + 45 + 117) = ½(270) = 135$. Then area $= \sqrt{(135(135 – 108)(135 – 45)(135 – 117))} = \sqrt{((135)(27)(90)(18))} = \sqrt{(5904900)} = 2430$. The area is also half the two shorter sides multiplied together which is $(108 \times 45)/2 = 4860/2 = 2430$. So this triangle is indeed a Pythagorean triangle as well as a Heron triangle. All its sides are integers and its area is an integer.

Now let us look at the other triangle formed by p and q with the formulas: $(p^2 - q^2)/2$, pq, and $(p^2 + q^2)/2$ where $p = 9$ and $q = 5$, which we will call triangle d, e, f.

$d = (p^2 - q^2)/2 = (9^2 - 5^2)/2 = (81 – 25)/2 = 56/2 = 28$

$e = pq = 9 \times 5 = 45$

$f = (p^2 + q^2)/2 = (9^2 + 5^2)/2 = (81 + 25)/2 = 106/2 = 53$

Triangle d, e, f (28,45,53) is a Pythagorean triangle. I will test it using the Pythagorean formula: $d^2 + e^2 = f^2$. $28^2 + 45^2 = 784 + 2025 = 2809$ and $53^2 = 2809$, so this triangle is indeed a Pythagorean triangle. It is also a Heron triangle because the two shorter sides, 28 and 45, when multiplied together and then divided in half equal an integer: $(28 \times 45)/2 = 1260/2 = 630$.

We can combine these two triangles together to create a new scalene Heron triangle. To place these two triangles together, I line them up so they share the side of 45. Now I have a triangle whose sides are 117, 53 and 136 (which comes from adding the two sides of 108 and 28).

Let me test it using the Heron formula for area to see if the area is an integer. Recall that the Heron formula is: $\sqrt{s(s-a)(s-b)(s-c)}$ where s equals ½ the perimeter: $s = \frac{1}{2}(a + b + c)$.

So $s = \frac{1}{2}(117 + 53 + 136) = \frac{1}{2}(306) = 153$.

Area $= \sqrt{(153)(153-117)(153-53)(153-136)} = \sqrt{(153)(36)(100)(17)} = \sqrt{9363600} = 3060$.

An easier way to find the area is to add the area of the two smaller triangles: 630 + 2430 = 3060. Thus, one way to create a scalene Heron triangle is to find four odd numbers m, n, p, q where mn = pq.

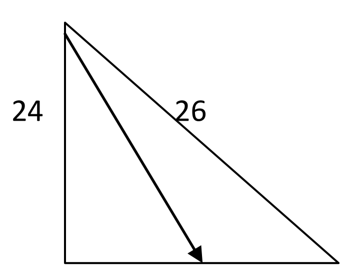
Method 2

When I went back over the list of the 20 primitive Heron triangles that have areas less than 100, I found something very interesting. All of the Heron scalene triangles come from Pythagorean triangles. That is, all Heron scalene triangles can be constructed from either combining two Pythagorean triangles or by subtracting two Pythagorean triangles. The trick is to find a side of a Heron scalene triangle that can be divided into twice the area of the Heron scalene triangle.

This is method number two. There is an exception to this rule that I will address later (method #3 for creating Heron scalene triangles).

Let me show an example. Let me take the scalene Heron triangle of 3, 25, 26 whose area is 36. If I draw this triangle as an acute scalene triangle where the shortest side is 3 and the longest side is 26. I will draw a line coming from the point between 3 and 25 landing 90^0 on the base of 26. This line is the height of the triangle. Since area = half base x height: A = ½bh, I can manipulate this formula by multiplying both sides of the equation by 2 and dividing both sides of the equation by b, the base, to get the formula in terms of h, the height: h = (2A)/b . The height of the 3, 25, 26 triangle is unknown. But I do know the area, 36 and the base, 26. Then we get h = (2 x 36)/26 = 72/26. This is not an integer. I cannot use this number to generate more Heron triangles or Pythagorean triangles.

If I draw the height to land on the base of 25, I get a similar situation. h = (2 x 36)/25 = 72/25. I must use the base of 3 because 3 can divide twice the area to give me an integer. So we have h = (2 x 36)/3 = 72/3 = 24. Since this height that I have found has a shorter length than the other two sides of 25 and 26, the original scalene triangle of 3, 25, 26 is therefore an obtuse triangle with the obtuse angle being between the sides of 25 and 3. Let me call this triangle, triangle G:



? 3

I want to find the base of the bigger Pythagorean triangle (which I will call triangle H) that has been created. I know two of the sides 24 and 26. I also know that the side of 26 is the hypotenuse. So I can use the formula $a^2 + b^2 = c^2$ in terms of b^2: $b^2 = c^2 - a^2$. So $b^2 = 26^2 - 24^2 = 676 - 576 = 100$. If $b^2 = 100$, then $b = 10$. So triangle H has the sides 10, 24 and 26.

There is also another triangle that shows itself in the picture: triangle J. This is the smaller right triangle with sides of 24 and 25. I can figure out the base by subtracting the base of triangle G from the base of triangle H: $10 - 3 = 7$. So triangle J has the sides of 7, 24 and 25. This is a Pythagorean triangle and also a Heron triangle. Triangle J has an area of 84 (I multiplied the base, 7 by the height, 24 and took half of this). Triangle H has an area of 120. The area of triangle G is the same as the area of triangle H minus the area of triangle J: $120 - 84 = 36$.

Therefore the scalene Heron triangle, 3, 25, 26 is formed by subtracting two Pythagorean triangles: triangle 10, 24, 26 and triangle 7, 24, 25.

I can even create another Heron scalene triangle by combining the two right triangles. Since they both share the side of 24 and they both have a right angle at one of the points on side 24, I can bring them together to form triangle K which has the sides of 25, 26 and 17. I know this triangle is a Heron triangle because the area is an integer. The area is equal to the areas of triangle H, 120, and triangle J, 84, added together which gives an area of 204.

I can also use the Heron formula to show that this new triangle (triangle K) is a Heron triangle: $\sqrt{s(s - a)(s - b)(s - c)}$ where s equals ½ the perimeter: $s = \frac{1}{2}(a + b + c)$.

$s = ½ (17 +25 + 26) = ½ (68) = 34$

$\text{Area} = \sqrt{(34)(34 - 17)(34 – 25)(34 – 26)} = \sqrt{(34)(17)(9)(8)} = \sqrt{41616} = 204$

This area of 204 is the same as when I added the areas of triangle H and J together. So from that one scalene Heron triangle of 3, 25, 26 I was able to discover 3 other Heron triangles, two of which are also Pythagorean triangles.

Method 3

Of the original 20 primitive Heron triangles that I listed in section 1, four of these are isosceles triangles as I mentioned in section 3. Another four of these triangles are Pythagorean triangles; 3, 4, 5 ; 5, 12, 13; 7, 24, 25; and 8, 15, 17. One can use the Pythagorean formula $a^2 + b^2 = c^2$ on any of these four to verify them: $8^2 + 15^2 = 17^2$ so $64 + 225 = 289$ and $289 = 289$. That leaves 12 scalene triangles, each which can be created from two Pythagorean triangles using the method that I used for the 3, 25, 26 triangle.

There is one exception, the 5, 29, 30 scalene triangle with the area of 72. This triangle must be treated differently because none of the sides divide twice the area. This is the only triangle of the 12 scalene Heron triangles with areas less than 100 that has no sides that divide twice the area.

Method #3: For this triangle (I will call it triangle L), I must manipulate the situation in order to show that it is indeed made up from two Pythagorean triangles. Now, given the fact that we can equally and proportionally increase the sides of any Heron triangle to form a new (and non primitive) Heron triangle, we will use the base of 5 to multiply each of the sides of triangle L to get sides of 25, 145, and 150 (I will call this triangle M). It should be noted that when you increase the sides of a Heron triangle, the area is increased by the square of the increased number. Because I am increasing the sides by 5, the area will increase by 25. So the new area of triangle M is 72 x 25 which is 1800.

Now we can drop the height from the point between 145 and 150 to the base of 25. Since the height, h is equal to twice the area divided by the base, b we get: $h = (2A)/b = (2 \times 1800)/25 = 3600/25 = 144$. So now we have three triangles: triangle L with sides 25, 145 and 150; triangle M with sides 144, 150 and unknown side, and triangle N (the smaller right triangle) with sides 144, 145, and an unknown side. To figure out the unknown side of triangle M, we will use the Pythagorean formula: $c^2 - b^2 = a^2$. $150^2 - 144^2 = 22500 – 20736 = 1764 = a^2$. So $a = 42$. Thus the sides of triangle M are 42, 144, and 150. (By the way, this triangle can be reduce to a more primitive state as the 7, 24, 25 Pythagorean triangle). I can also figure out the area of triangle M by taking half the product of 144 and 42 = ½ (144 x 42) = ½ (6048) = 3024.

To figure out the unknown side of triangle N, I simply subtract the base of triangle L from the base of triangle M: 42 – 25 = 17. Triangle N has the sides of 17, 144 and 145. And the area of triangle N is half the product of 144 x 17 = ½ (144 x 17) = ½ (2448) = 1224.

Both these new triangles (triangle M and triangle N) are Pythagorean triangles and when one subtracts the smaller one (triangle N) from the larger one (triangle M), you end up with our initial scalene Heron triangle, triangle L, triangle 25, 145, 150. I can also subtract the areas of triangles M and N to get the area of triangle L: 3024 – 1224 = 1800. Triangle L can then be reduced to create the original primitive scalene Heron triangle of 5, 29, 30.

It should also be noted that I can create another triangle by combining triangles M and N. This will create a new scalene Heron triangle with sides 59, 145, and 150. The area will be the combined areas of triangles M and N which is 4248. This can always be checked with the Heron formula.

I found out where the other 10 scalene Heron triangles come from. The 4, 13, 15 triangle comes from subtracting the Pythagorean triangle 5, 12, 13 from the Pythagorean triangle 9, 12, 15. By adding these two triangles, scalene Heron triangle 13, 14, 15 is born. The 4, 51, 53 scalene Heron triangle is formed by subtracting Pythagorean triangle 24, 45, 51 from Pythagorean triangle 28, 45, 53. By combining these two triangles, the scalene Heron triangle 51, 52, 53 is created. The scalene Heron triangle 6, 25, 29 is formed by subtracting two Pythagorean triangles; the 15, 20, 25 from the 20, 21, 29. These two triangles can then be combined to form the scalene Heron triangle 25, 29, 36. There are five more scalene triangles (from the list in section 1) that can be formed by subtracting two Pythagorean triangles.

I found one scalene Heron triangle, the 10, 17, 21 triangle that is formed by adding two Pythagorean triangles together; the 6, 8, 10 triangle and the 8, 15, 17 triangle. Thus, all Heron scalene triangles come from adding or subtracting two Pythagorean triangles.