Probabilistic value-distribution theory of zeta-functions

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Abstract

The starting point of the value-distribution theory of zeta-functions is Bohr's achievement in the first half of the 20th century, who proved the denseness results and probabilistic limit theorems on the values of zeta-functions. Later in 1970s, Voronin discovered the universality theorem. All of those theorems were first proved for the Riemann zeta-function, but now, similar results are known for many other zeta and L-functions. In this article the present stage of this value-distribution theory is surveyed.

1 The work of Bohr

We begin with the value-distribution theory of the most classical zeta-function, the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

Here $s = \sigma + it$ is a complex variable. The series (1.1) is convergent only in the region $\sigma = \Re(s) > 1$, but it is well known that $\zeta(s)$ can be continued meromorphically to the whole complex plane. If one knows the value of $\zeta(s)$ then one can also know the value of $\zeta(1-s)$ by the functional equation. Therefore we can restrict our consideration to the half plane $\sigma \geq 1/2$.

When $\sigma > 1$, then (1.1) is convergent absolutely, hence the values $\zeta(\sigma + it)$ for any t are in the disc of radius $\zeta(\sigma)$ whose center is the origin. However this value of radius tends to infinity when $\sigma \to 1$. H. Bohr [5] studied this situation and proved that $\zeta(s)$ takes any non-zero value infinitely many times in the half plane $\sigma \geq 1$ (1911). Next Bohr proceeded to the study of the behaviour of $\zeta(s)$ in the strip $1/2 < \sigma \leq 1$, and published many results in the first half of 1910s. Here we mention the following two results.

Theorem 1 (Bohr-Courant [7]) *For any fixed σ satisfying $1/2 < \sigma \leq 1$, the set $\{\zeta(\sigma + it) \mid t \in \mathbf{R}\}$ is dense in $\mathbf{C}$.*

Theorem 2 (Bohr [6]) *For any fixed σ satisfying $1/2 < \sigma \leq 1$, the set $\{\log\zeta(\sigma + it) \mid t \in \mathbf{R}, \sigma + it \in G\}$ is dense in $\mathbf{C}$.*

We should decide the branch of $\log\zeta(\sigma+it)$ in Theorem 2. From the half plane $\sigma > 1/2$, we remove all the points which have the same imaginary part as, and smaller real part than, one of the possible zeros or the pole of $\zeta(s)$ in this region, and the remaining part we denote by G. (We cannot exclude the possibility of existence of zeros because we do not assume the Riemann hypothesis.) And we determine the value of $\log\zeta(\sigma + it)$ for $\sigma + it \in G$ by the analytic continuation from the region of absolute convergence along the line which fixes the imaginary part. Note that Theorem 1 follows immediately from Theorem 2.

The value-distribution theory of the Riemann zeta-function is the lifework of Bohr. His famous theory of almost periodic functions is also motivated by the value-distribution theory. In 1930s, Bohr arrived at the following result. Fix a rectangle R in the complex plane whose edges are parallel to the axes. Denote by $\mu_1(X)$ the one dimensional Lebesgue measure of the set X. For any fixed $\sigma > 1/2$ and $T > 2$, let

$$V(T, R, \sigma; \zeta) = \mu_1(\{t \in [0, T] \mid \sigma + it \in G,\ \log\zeta(\sigma + it) \in R\}). \tag{1.2}$$

Then

Theorem 3 (The limit theorem of Bohr-Jessen [9]) *For any $\sigma > 1/2$, the limit value*

$$W(R, \sigma; \zeta) = \lim_{T\to\infty} T^{-1} V(T, R, \sigma; \zeta) \tag{1.3}$$

exists.

The limit value $W(R, \sigma; \zeta)$ may be regarded as the probability how many values of $\log\zeta(s)$ on the line $\Re s = \sigma$ belong to the rectangle R. Since this theorem is really fundamental, we sketch the original proof of Bohr-Jessen. First assume $\sigma > 1$. Then $\zeta(s)$ has the Euler product expression

$$\prod_{n=1}^{\infty} (1 - p_n^{-s})^{-1},$$

where p_n denotes the nth prime number. Hence

$$\log \zeta(s) = -\sum_{n=1}^{\infty} \log(1 - p_n^{-\sigma} \exp(-it \log p_n)). \tag{1.4}$$

This is approximated by the finite sum

$$f_N(s) = -\sum_{n=1}^{N} \log(1 - p_n^{-\sigma} \exp(-it \log p_n)). \tag{1.5}$$

First we prove the analogue of (1.3) for

$$V_N(T, R, \sigma; \zeta) = \mu_1(\{t \in [0, T] \mid f_N(\sigma + it) \in R\}). \tag{1.6}$$

For this purpose, put

$$z_n(\theta_n; \zeta) = -\log(1 - p_n^{-\sigma} \exp(2\pi i \theta_n))$$

for $0 \leq \theta_n < 1$, and define the mapping S_N from the N-dimensional torus $[0, 1)^N$ to $\mathbf{C}$ by

$$S_N(\theta_1, \ldots, \theta_N; \zeta) = \sum_{n=1}^{N} z_n(\theta_n; \zeta). \tag{1.7}$$

The important point here is that the quantities $\log p_n$ $(n = 1, 2, \ldots, N)$, appearing on the right-hand side of (1.5), are linearly independent over the rational number field $\mathbf{Q}$. This fact, which is equivalent to the uniqueness of decomposition of integers into the product of prime numbers, is the essential reason why probabilistic statements are valid for $\zeta(s)$. Indeed this fact assures that we can use the Kronecker-Weyl theorem on the theory of uniform distribution to obtain that there exists the limit value

$$W_N(R, \sigma; \zeta) = \lim_{T \to \infty} T^{-1} V_N(T, R, \sigma; \zeta), \tag{1.8}$$

and that it coincides with the N-dimensional Lebesgue measure $\mu_N(S_N^{-1}(R))$ of the set $S_N^{-1}(R)$.

The next step is the proof of the existence of the limit value

$$W(R, \sigma; \zeta) = \lim_{N \to \infty} W_N(R, \sigma; \zeta). \tag{1.9}$$

To prove (1.9), Bohr-Jessen used the fact that the points $z_n(\theta_n; \zeta)$ describes a closed convex curve when θ_n moves from 0 to 1. In their days, under the influence of Blaschke, the theory of plane convex curves was studied actively; Bohr-Jessen themselves developed an exquisite geometric theory of sums (in the sense of (1.7)) of closed convex curves ([8]), whose results they used in the proof of (1.9).

The last step is the proof that the limit value (1.9) is equal to the right-hand side of (1.3). When $\sigma > 1$ this is clear because the limit of $f_N(s)$ when $N \to \infty$ is $\log \zeta(s)$. When $1/2 < \sigma \leq 1$ such a simple relation does not hold, but in this case we can use the fact that $f_N(s)$ approximates $\log \zeta(s)$ in a certain mean value sense. Bohr had already proved a suitable mean value theorem in a joint work [10] with Landau. (It is also possible now to quote Carleson's theorem.) Hence Bohr-Jessen's proof of Theorem 3 is finished.

Theorem 3 is the final reaching point of Bohr's research on the value-distribution theory, and at the same time, the starting point of many mathematicians after him.

It is to be noted here that the situation of value-distribution is completely different on the line $\sigma = 1/2$. Concerning this matter, there is an important theory originated by Selberg, but we do not discuss this theory in the present article. Also we do not mention many other aspects of probabilistic value-distribution theory; this article only treats the value-distribution theory of the style of Bohr.

The author expresses his sincere gratitude to Professors A. Laurinčikas, Shin-ya Koyama, Jun-ichi Tanaka, Hidehiko Mishou, and Hirofumi Nagoshi for valuable comments.

2 Limit theorems for general zeta-functions

When one thinks of the Bohr-Jessen limit theorem, one natural question is how general this kind of results holds. Look back the sketch of the proof in the preceding section. It is based on the existence of the Euler product, but this is not so a serious problem, because many important zeta-functions have Euler products. A more serious obstacle to generalization is the fact that the curve $z_n(\theta_n; \zeta)$ is a closed convex curve, which we used in the course of the proof.

For any positive integer n, we attach positive integers $g(n)$ and $f(j,n)$ $(1 \leq j \leq g(n))$. Let $a_n^{(j)}$ be complex numbers satisfying

$$|g(n)| \leq C p_n^{\alpha}, \quad |a_n^{(j)}| \leq p_n^{\beta}, \tag{2.1}$$

where $C > 0,\ \alpha \geq 0,\ \beta \geq 0$. Define polynomials $\Phi_n(X)$ in X by

$$\Phi_n(X) = \prod_{j=1}^{g(n)} \left(1 - a_n^{(j)} X^{f(j,n)}\right), \tag{2.2}$$

and consider a general class of zeta-functions which are defined by the Euler product of the form

$$\varphi(s) = \prod_{n=1}^{\infty} \Phi_n(p_n^{-s})^{-1}. \tag{2.3}$$

If we try to apply Bohr-Jessen's argument presented in the preceding section to $\varphi(s)$, we find that the quantity corresponding to $z_n(\theta_n;\zeta)$ is

$$z_n(\theta_n;\varphi) = -\sum_{j=1}^{g(n)} \log\left(1 - a_n^{(j)} p_n^{-f(j,n)\sigma} \exp(2\pi i f(j,n)\theta_n)\right). \tag{2.4}$$

However this is not always a closed convex curve. In the case of the Riemann zeta-function $\zeta(s)$ or the Dirichlet L-function $L(s,\chi)$, the corresponding polynomial (2.2) is linear, hence (2.4) is a closed convex curve and Bohr-Jessen's argument works. In the case of the Dedekind zeta-function $\zeta_F(s)$ attached to an algebraic number field F of finite degree, the corresponding polynomial is not linear, but (2.4) is closed convex if F is a Galois number field. The author [60] called $\varphi(s)$ the convex Euler product if the corresponding (2.4) is a closed convex curve. We can apply Bohr-Jessen's argument directly to convex Euler products to prove the limit theorem.

However, Dedekind zeta-functions attached to non-Galois number fields are already non-convex. In general, for many other important classes of zeta-functions which have Euler products, such as automorphic L-functions, we cannot expect at all that they are convex Euler products. Can we find some alternative approach to limit theorems which does not use convexity?

Alternative proofs of Theorem 3 were already published by Jessen-Wintner [27] and Borchsenius-Jessen [11], but their proofs also depend on convexity. The first proof of a limit theorem without using convexity is, as far as the author knows, due to Nikishin [73]. Nikishin considered not the probability of $\log\zeta(\sigma + it) \in R$, but the probability of $|\zeta(\sigma+it)| < C$, where C is a positive constant. Nikishin's proof is based on the theory of almost periodic functions. As mentioned before, the theory of almost periodic functions was also cultivated by Bohr. Bohr himself did not use the theory of almost periodic functions in his proof of Theorem 3. But limit theorems on almost periodic functions were already discussed by Wintner [90], and the theory of Borchsenius-Jessen [11] is also based on almost periodic functions. Therefore we can say, as Bohr expected, that the theory of almost periodic functions is now a useful tool in value-distribution theory. Note that there is an incomplete point in Nikishin's paper, but it can be amended. See Laurinčikas [43].

Laurinčikas [35] discussed, using the idea of Nikishin [73], limit theorems for Dirichlet L-functions. Moreover, he obtained the limit theorem of the following form in [40]. Define the probability measure P_T on $\mathbf{C}$ by

$$P_T(A;\sigma,\zeta) = T^{-1}\mu_1(\{t \in [0,T] \mid \zeta(\sigma+it) \in A\}),$$

where A is any Borel subset of $\mathbf{C}$. Then

Theorem 4 (Laurinčikas [40]) *For any $\sigma > 1/2$, $P_T(A;\sigma,\zeta)$ is convergent weakly to a certain probability measure $Q(A;\sigma,\zeta)$ as $T \to \infty$.*

Around 1988, without having known the results mentioned above, the author searched for an alternative proof of Theorem 3, which can be generalized to some class of zeta-functions which have non-convex Euler products. The author noticed that, replacing the original proof of (1.9) by an argument based on Prokhorov's theorem in probability theory, one can obtain a proof of Theorem 3 without using convexity. The author first published this proof for automorphic L-functions attached to cusp forms with respect to $SL(2, \mathbf{Z})$ and its congruence subgroups ([59]), and then wrote down more simplified proof for more general zeta-functions ([60]).

The zeta-function $\varphi(s)$ defined by the Euler product (2.3) is, under the condition (2.1), convergent absolutely in the half plane $\sigma > \alpha + \beta + 1$. Hence $\varphi(s)$ is holomorphic and non-vanishing in this region. Let ρ be a number satisfying $\rho \geq \alpha + \beta + 1/2$, and assume that, if $\varphi(s)$ can be continued meromorphically to $\sigma \geq \rho$, then the following conditions hold:

(i) All poles of $\varphi(s)$ in the half plane $\sigma \geq \rho$ are contained in a certain compact set;

(ii) There exists a positive constant C for which $|\varphi(\sigma + it)| = O((|t| + 1)^C)$ holds in the same half plane.

(iii) There is no pole on the line $\sigma = \rho$, and

$$\int_0^T |\varphi(\rho + it)|^2 dt = O(T) \tag{2.5}$$

holds. (These conditions trivially hold if $\rho > \alpha + \beta + 1$.)

Denote by $\mathcal{M}$ the family of all functions defined by the Euler product (2.3) and satisfy the above conditions if continued to $\sigma \geq \rho$. Then the limit theorem the author proved in [60] is as follows.

Theorem 5 ([60]) *For any zeta-function $\varphi(s)$ belonging to the family $\mathcal{M}$, define $V(T, R, \sigma; \varphi)$ similarly to (1.2). Then the limit value*

$$W(R, \sigma; \varphi) = \lim_{T \to \infty} T^{-1} V(T, R, \sigma; \varphi) \tag{2.6}$$

exists for any $\sigma > \rho$.

Among the above conditions, (i) and (ii) are usually satisfied for typical examples of zeta-functions. The estimate (2.5) in (iii) is an essential condition; for instance, the Riemann zeta-function does not satisfy (2.5) when $\sigma = 1/2$, hence on this line the behaviour is different as was mentioned in the preceding section. To check (iii), it is sufficient to apply Potter's general result [75] when $\varphi(s)$ satisfies a functional equation. Therefore $\mathcal{M}$ is a wide class which includes rather general zeta and L-functions having Euler products and functional equations.

The author [62] gave another proof of Theorem 5 by using Lévy's convergence theorem. This is also a proof without using convexity. In [62] only Dedekind zeta-functions (of arbitrary number fields of finite order) are treated, but the proof given there can be applied to any $\varphi(s)$ belonging to $\mathcal{M}$, as was remarked in [63].

Thus now, we can prove limit theorems for fairly general class of zeta-functions without using convexity. However we are still not completely released from convexity. For instance, the author [58] [62] studied the rate of convergence in (1.3), and the best estimate at present is given in [23]; a lemma of Jessen-Wintner [27] based on convexity is used in those papers, hence such kind of results has been obtained only for convex Euler products.

3 Quantitative results

Now let us remind Bohr-Jessen's limit theorem again. It is a beautiful result, but it is just an existence theorem. To obtain real information on the value-distribution of zeta-functions, we should refine this theorem quantitatively. With this motivation in mind, the author had thought about Bohr-Jessen's theorem repeatedly from the early 1980s. The theory of uniform distribution is used effectively in the proof of Bohr-Jessen's theorem, while now we have the quantitative theory which estimates the discrepancy from uniform distribution. What can we say if we apply the theory of discrepancy to Bohr-Jessen's theorem? This is the initial idea of the author which produced the study of the rate of convergence mentioned at the end of the preceding section.

More important is the quantitative study of the limit value $W(R,\sigma;\zeta)$ itself. To formulate the problem more definitely, we specialize the rectangle R to the square

$$R(\ell)=\{z\in\mathbf{C}\mid -\ell\leq\Re z\leq\ell, -\ell\leq\Im z\leq\ell\},$$

centered at the origin. How is the behaviour of $W(R(\ell),\sigma;\zeta)$ when $\ell\to\infty$? In the region of absolute convergence $\sigma>1$, clearly $W(R(\ell),\sigma;\zeta)=1$ for sufficiently large ℓ, so the problem is not interesting. But the situation is different in the strip $1/2<\sigma\leq 1$, as we can see from Theorem 2. The only obvious fact in this case is that $W(R(\ell),\sigma;\zeta)\to 1$ when $\ell\to\infty$. How is the rate of convergence here? Thus we can formulate a quantitative problem.

Concerning this problem, already Jessen-Wintner [27] obtained a result. Their result says that for any positive a, there exists a positice constant $C=C(a,\sigma)$ for which

$$1-W(R(\ell),\sigma;\zeta)\leq C\exp(-a\ell^2) \tag{3.1}$$

holds. Nikishin [73] also treated this type of problems. In 1986, Joyner arrived at the following result.

Theorem 6 (Joyner [28]) *Let $1/2 < \sigma < 1$. Then there exist positive constants c_1 and c_2 with $c_1 > c_2$, depend only on σ, for which the inequality*

$$\begin{aligned}\exp\left(-c_1\ell^{1/(1-\sigma)}(\log \ell)^{\sigma/(1-\sigma)}\right) &\leq 1 - W(R(\ell), \sigma; \zeta) \\ &\leq \exp\left(-c_2\ell^{1/(1-\sigma)}(\log \ell)^{\sigma/(1-\sigma)}\right) \qquad (3.2)\end{aligned}$$

holds for any sufficiently large ℓ.

This is the first work which gives a lower bound for $1-W(R(\ell), \sigma; \zeta)$. Applying Joyner's method to the case $\sigma = 1$, we obtain

Theorem 7 ([61]) *There exist positive constants c_3 and c_4 with $c_3 > c_4$ for which*

$$\begin{aligned}\exp\left(-c_3 \exp\exp(2\ell(1 + o(1)))\right) &\leq 1 - W(R(\ell), 1; \zeta) \\ &\leq \exp\left(-c_4 \exp\exp(2^{-1}\ell(1 + o(1)))\right) \qquad (3.3)\end{aligned}$$

holds.

A basic tool of Joyner's method is a probabilistic lemma due to Montgomery [68] (see also Montgomery-Odlyzko [69]). Let $\mathbf{r} = \{r_n\}$ be a sequence of non-negative numbers, including infinitely many non-zero terms, and satisfying

$$\sum_{n=1}^{\infty} r_n^2 < \infty.$$

For any positive integer N, put

$$A_N(\mathbf{r}) = \sum_{n=N+1}^{\infty} r_n^2, \qquad B_N(\mathbf{r}) = \sum_{n=1}^{N} r_n.$$

Let $\{\theta_n\}$ be a sequence of independent random variables on a certain probability space $(\Omega, \mathcal{P})$, uniformly distributed in $[0, 1]$, and $X_n = \cos(2\pi\theta_n)$ (or $\sin(2\pi\theta_n)$). Define

$$X = \sum_{n=1}^{\infty} r_n X_n.$$

By Kolmogorov's theorem X is convergent almost surely, and the following inequality holds.

Lemma (Montgomery [68]) (i) *For any N, the inequality*

$$\mathcal{P}(X \geq 2B_N(\mathbf{r})) \leq \exp\left(-\frac{3}{4}B_N(\mathbf{r})^2 A_N(\mathbf{r})^{-1}\right) \qquad (3.4)$$

holds. (ii) *Moreover, if the sequence* $\mathbf{r} = \{r_n\}$ *is monotonically decreasing, then there exist positive constants* $M_1,\ M_2,\ M_3$ *for which*

$$\mathcal{P}(X \geq M_1 B_N(\mathbf{r})) \geq M_2 \exp\left(-M_3 B_N(\mathbf{r})^2 A_N(\mathbf{r})^{-1}\right) \tag{3.5}$$

holds.

In the case of the Riemann zeta-function, we take $\mathbf{r} = \{p_n^{-\sigma}\}$. This is monotonically decreasing, hence we can apply the lemma, and the problem is reduced to the evaluation of $A_N(\mathbf{r})$, $B_N(\mathbf{r})$ appearing on the right-hand sides of (3.4), (3.5). It can be easily done by using the prime number theorem, and Theorems 6 and 7 follow.

Now recall Theorem 5 in the preceding section. It asserts the existence of the limit value $W(R, \sigma; \varphi)$ for any $\varphi(s) \in \mathcal{M}$. Therefore it is a natural question to ask: can we generalize the above quantitative results to such a general class of zeta-functions? Since it is easy to generalize the inequality (3.1) of Jessen-Wintner to the case of any $\varphi(s) \in \mathcal{M}$ ([61]), the main problem is how to generalize Joyner's inequality. For Dirichlet L-functions, Joyner himself proved such an inequality in [28].

When we try to generalize Joyner's argument to general $\varphi(s) \in \mathcal{M}$, we encounter the problem of applying Montgomery's lemma to the sequence

$$\mathbf{r} = \mathbf{r}(\varphi) = \{r_n(\varphi)\}, \qquad r_n(\varphi) = \Bigg| \sum_{\substack{1 \leq j \leq g(n) \\ f(j,n)=1}} a_n^{(j)} \Bigg| \, p_n^{-\sigma}.$$

Since the upper bound part (3.4) of the lemma has no additional assumption, it can be applied to the general case (as was remarked in [63]), and hence the problem is, similarly to the above, reduced to the evaluation of $A_N(\mathbf{r})$, $B_N(\mathbf{r})$ on the right-hand side of (3.4). Therefore we can show an upper bound inequality of Joyner's type if there is some result corresponding to the prime number theorem which we have used in the case of the Riemann zeta-function. And indeed, by the prime ideal theorem in the case of Dedekind zeta-functions of algebraic number fields, or by Rankin's results [76] in the case of the automorphic L-function φ_f attached to a holomorphic cusp form f with respect to $SL(2, \mathbf{Z})$, we can proceed along this line to obtain upper bound inequalities ([61][63]). Note that in the latter case, we assume f is a normalized Hecke eigenform of weight κ, because we need the existence of the Euler product expansion of φ_f. Hereafter in this article we always keep the above assumptions for f and φ_f. Rankin's results mentioned above can be stated as follows. Denote by $\tilde{a}(n)$ the nth Fourier coefficient of f, and put $a(n) = \tilde{a}(n) n^{(1-\kappa)/2}$. Then

$$\sum_{p_n \leq x} |a(p_n)|^2 = \frac{x}{\log x}(1 + o(1)) \tag{3.6}$$

and

$$\frac{1}{\sqrt{2}}\frac{x}{\log x}(1+o(1)) \leq \sum_{p_n\leq x}|a(p_n)| \leq \frac{2+3\sqrt{6}}{10}\frac{x}{\log x}(1+o(1)). \tag{3.7}$$

However, the situation is quite different if we try to prove lower bound inequalities. It seldom happens that the sequence $\mathbf{r}(\varphi)$ completely satisfies the monotonic decreasing property, which is the assumption of part (ii) of Montgomery's lemma. Joyner succeeded in proving upper and lower bounds for Dirichlet L-functions, because in this case $\mathbf{r}$ is monotonically decreasing except for finitely many primes which are divisors of the modulus of the character, hence we can apply the lemma. As for the case of Dedekind zeta-functions $\zeta_F(s)$, if F is Galois, then the lemma can be applied because $\mathbf{r}(\zeta_F)$ is monotonically decreasing except for finitely many primes which are ramified in F. Even in the case of a non-Galois field, we can lift the argument to its Galois closure, and we can prove upper and lower bounds ([61]). But such techniques cannot be always useful. In more general cases, such as the case of automorphic L-functions, it is almost impossible to reduce the problem to the monotonically decreasing case.

However, it is the inequality (3.5) which is really necessary, and the condition of monotonic decreasing is just a sufficient condition of (3.5). Actually this is a too strong condition. In [24] Hattori and the author gave a necessary and sufficient condition for the validity of (3.5), and using this result, proved upper and lower bounds of Joyner's type for the automorphic L-function φ_f. A key fact here is the result of Ram Murty [70] which asserts that there are sufficiently many $|\tilde{a}(p_n)|$ whose values are not so smaller than Deligne's upper bound $|\tilde{a}(p_n)| \leq 2p_n^{(\kappa-1)/2}$.

This shows that, different from the results mentioned in the preceding section which can be treated in a general framework, it is necessary to consider arithmetic properties of each zeta-function to obtain quantitative results. This makes the problem more difficult but more interesting.

Now we return to the inequality (3.2) for the Riemann zeta-function. One may guess from this inequality that there will be a constant A, between c_1 and c_2, which will give the real order. Recently Hattori and the author [25] proved the existence of such a constant $A = A(\sigma)$.

Theorem 8 ([25]) *For any $1/2 < \sigma < 1$, we have*

$$1 - W(R(\ell), \sigma; \zeta) = \exp\left(-A(\sigma)\ell^{1/(1-\sigma)}(\log \ell)^{\sigma/(1-\sigma)}(1+o(1))\right), \tag{3.8}$$

where

$$A(\sigma) = (1-\sigma)\left\{\frac{1-\sigma}{\sigma}\int_0^\infty \log I_0(y^{-\sigma})dy\right\}^{-\frac{\sigma}{1-\sigma}}$$

and I_0 is the modified Bessel function.

The main tool used in the proof of this theorem is the theory of regularly varying functions, originated by J. Karamata (see Seneta [81]), hence is completely different from that of Joyner. This result has settled the original problem of determining the order of $1 - W(R(\ell), \sigma; \zeta)$ in the case of the Riemann zeta-function.

However, the argument in [25] heavily depends on the properties of regularly varying functions, hence it is not easy to generalize it. An analogous argument is possible for Dedekind zeta-functions of Galois fields, and this is discussed in [25]. But for more general zeta-functions, at present we have no idea how to extend the argument in [25]. Furthermore, even in the case of the Riemann zeta-function, the argument in [25] fails for $\sigma = 1$, hence Theorem 7 has not yet been refined to an asymptotic equality similar to (3.8).

4 The universality theorem of Voronin

In the rest of this article we discuss another direction of development of Bohr's theory, that is the universality theory of zeta-functions. First, let us go back half a century.

H. Bohr, the originator of the value-distribution theory of zeta-functions, died in 1951. This is just the year when the famous textbook of Titchmarsh [82] on the Riemann zeta-function was published. Bohr's theory is explained in this book by using a whole chapter.

However, after those days, there came a long inactive period in the study of the Riemann zeta-function. During about twenty years, we got no harvest except a very few papers. It was only in 1970s when the research interest in the Riemann zeta-function came to revive. Many mathematicians contributed this revival, but we should first nominate the name of S. M. Voronin as a mathematician who revitalized the value-distribution theory.

Voronin, in [83], extended Bohr's Theorem 1 to the n-dimensional space.

Theorem 9 (Voronin [83]) *For any $1/2 < \Re s \leq 1$, any positive integer n, and any positive number h, the set*

$$\left\{(\zeta(s + imh), \zeta'(s + imh), \ldots, \zeta^{(n-1)}(s + imh)) \mid m = 1, 2, \ldots\right\}$$

is dense in $\mathbf{C}^n$.

In the same paper Voronin also proved that, if $s_1, \ldots, s_n$ are complex numbers with $1/2 < \Re s_j \leq 1$ $(1 \leq j \leq n)$ and different from each other, then the set

$$\{(\zeta(s_1 + imh), \zeta(s_2 + imh), \ldots, \zeta(s_n + imh)) \mid m = 1, 2, \ldots\}$$

is dense in $\mathbf{C}^n$.

A natural next step of research is to study the situation on infinite dimensional spaces, that is on function spaces. Concerning this problem, Voronin [84] discovered the following really surprising theorem.

Theorem 10 (Voronin's universality theorem [84]) *Let K be a compact subset of the strip $1/2 < \sigma < 1$ with connected complement. Fix a function $h(s)$ which is continuous, non-vanishing on K and holomorphic in the interior of K. For any $\varepsilon > 0$, put*

$$U(T;\zeta) = U(T,K,h,\varepsilon;\zeta) = \mu_1 \left\{ \tau \in [0,T] \mid \sup_{s\in K} |\zeta(s+i\tau) - h(s)| < \varepsilon \right\}. \tag{4.1}$$

Then we have

$$\liminf_{T\to\infty} T^{-1} U(T;\zeta) > 0. \tag{4.2}$$

This theorem implies roughly that any non-vanishing holomorphic function $h(s)$ can be approximated uniformly by some translation, parallel to the imaginary axis, of the Riemann zeta-function. Note that the statement in [84] is actually weaker. The above is the formulation of Reich [77].

The word "universality" of this theorem is coming from the similarity with Fekete's old result which asserts the existence of a series such that any continuous function on the interval $[-1,1]$ can be approximated uniformly by some suitable partial sum of that series. On the other hand, readers will recognize the similarity between the above theorem and the famous polynomial approximation theorem of Weierstrass. In fact, there is a theorem of Mergelyan [65], a complex analogue of Weierstrass' theorem. Mergelyan's theorem asserts that if K is a compact subset of $\mathbf{C}$ with connected complement and $h(s)$ is a function continuous on $K$and holomorphic in the interior of K, then $h(s)$ can be approximated uniformly on K by polynomials. The similarity between this theorem and Theorem 10 is clear. Moreover, Mergelyan's theorem plays an important role in the proof of Theorem 10. But of course, in the theorems of Weierstrass and Mergelyan, the choice of polynomials depends on the rate of approximation; while in Theorem 10, translations of a single function, the Riemann zeta-function, can approximate the target function $h(s)$ as close as possible.

Recall that, in the first stage of Bohr-Jessen's proof of Theorem 3, the logarithm of Euler product expression of $\zeta(s)$ was approximated by a finite sum (1.5). Similarly, to prove Theorem 10, Voronin first considered a finite product which approximate the Euler product in some sense, hence reduced the problem to the proof of a certain kind of universality of that finite product. And he proved the latter universality by using a result of Pecherskiĭ on rearrangements of sequences in Hilbert spaces.

In a few years after Voronin [84], the universality theorem was extended variously; in particular, the universality of the Dirichlet L-function $L(s,\chi)$, of the Dedekind zeta-function $\zeta_F(s)$, and of the Hurwitz zeta-function $\zeta(s,\alpha)$ were shown. However, since all of the three important papers written in this period, Voronin [87], Gonek [20], Bagchi [2], are Dr. Sci. theses and remain unpublished, it is not always easy to say who first proved the obtained results. Voronin [84] already mentioned that similar universality also holds for any $L(s,\chi)$. Moreover, the following result holds.

Theorem 11 *Let $K_1,\ldots,K_m$ be compact subsets of the strip $1/2<\sigma<1$ with connected complements. For each K_j, fix a function $h_j(s)$ continuous and non-vanishing on K_j and holomorphic in the interior of K_j ($1\leq j\leq m$). Let $\chi_1,\ldots,\chi_m$ be pairwise non-equivalent Dirichlet characters, and $\varepsilon>0$. Denote by $U(T;L(\ ,\chi_1),\ldots,L(\ ,\chi_m))$ the Lebesgue measure of the set of all $\tau\in[0,T]$ for which*

$$\sup_{s\in K_j}|L(s+i\tau,\chi_j)-h_j(s)|<\varepsilon$$

holds simultaneously for $1\leq j\leq m$. Then we have

$$\liminf_{T\in\infty}T^{-1}U(T;L(\ ,\chi_1),\ldots,L(\ ,\chi_m))>0. \tag{4.3}$$

We call this type of results the joint universality theorem. Theorem 11 was first published by Bagchi [3] (in the case when $\chi_1,\ldots,\chi_m$ have the same modulus), but similar results are also given in Voronin [87] and Gonek [20]. See Section 3, Chapter 7 of the textbook of Karatsuba-Voronin [32].

Concerning Dirichlet L-functions, we also know the χ-universality. In theorems 10 and 11, we fix the zeta (or L)-function and move its imaginary part; but this time we fix the variable s and move characters (among, for example, all characters whose muduli are powers of a certain fixed prime), then we find that the corresponding family of L-functions approximate uniformly any given holomorphic function. This kind of results is treated by Bagchi [2], Gonek [20] and Eminyan [14].

On the universality of Dedekind zeta-functions, there are publications of Reich [78][80]. Gonek [20] also treated this matter, and Voronin [87] discussed the joint universality of Dedekind zeta-functions.

Some mathematicians tried to find some general class of zeta-functions for which universality holds. Reich [77] proved the universality of Euler products satisfying certain conditions, while Laurinčikas [33][34][36][37] studied the universality of Dirichlet series with multiplicative coefficients.

Another interesting result obtained in this period is the universality of the Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}\quad(0<\alpha\leq 1)$$

(Bagchi [2], Gonek [20]). In this case it is not necessary to assume that $h(s)$ is non-vanishing. An interesting point of this result is that Hurwitz zeta-functions do not have Euler products in general (except for the cases $\alpha=1,\ 1/2$). This fact tells us that the existence of the Euler product, used in many papers including the paper of Voronin himself, is not an essential condition. To prove the universality of Hurwitz zeta-functions, when $\alpha=a/b$ is rational, we use the relation

$$\zeta(s,a/b)=\frac{b^s}{\phi(b)}\sum_{\chi(\mathrm{mod}\ b)}\bar{\chi}(a)L(s,\chi)$$

(where $\phi(b)$ is Euler's function) to reduce the problem to the joint universality of Dirichlet L-functions, and apply Theorem 11. On the other hand, when α is transcendental, we use the fact that $\log(n+\alpha)$ $(n=0,1,2,...)$ are linearly independent over $\mathbf{Q}$. Recalling that the proof of the universality of $\zeta(s)$ presented in Section 1 also depends on the linear independence of $\log p_n$ over $\mathbf{Q}$, we can see that the essentially important fact is the linear independence of certain quantities over $\mathbf{Q}$. In fact, when α is algebraic irrational the universality of $\zeta(s,\alpha)$ is still unproved, because the way of reducing the problem to some linear independence property has not yet been found in this case.

Recently, the universality of Lerch's zeta-function

$$L(s,\alpha,\lambda)=\sum_{n=0}^{\infty}\frac{\exp(2\pi i\lambda n)}{(n+\alpha)^s},\tag{4.4}$$

which is a generalization of $\zeta(s,\alpha)$, has also been studied. Here λ is a real number. We can prove the universality of (4.4) if α is transcendental (Laurinčikas [48]) or if α is rational and λ is also rational (Laurinčikas [49]). The joint universality of several Lerch zeta-functions $L(s,\alpha_1,\lambda_1),\ldots,L(s,\alpha_m,\lambda_m)$ is treated in [54] when $\alpha_1,\ldots,\alpha_m$ are transcendental and $\lambda_1,\ldots,\lambda_m$ are rational.

Various applications of universality theorems are known. Voronin himself studied the problems of independence (Voronin [83][85]) and of the distribution of zeros (Voronin [86][88][89]). The former is a classical problem from the days of Hilbert; in the case of the Riemann zeta-function, from the universality (or rather from Theorem 9) it immediately follows that, if F_j $(j=0,1,\ldots,n)$ are continuous functions of m-variables and

$$\sum_{j=0}^{n}s^jF_j\left(\zeta(s),\zeta'(s),\ldots,\zeta^{(m-1)}(s)\right)=0$$

holds for all s, then F_j $(j = 0, 1, \ldots, n)$ are identically 0. See Section 6.6 of Laurinčikas [43]. Generalization to Dedekind zeta-functions are discussed by Reich [79][80]. The case of Lerch zeta-functions is treated in [19] and [54].

On the distribution of zeros of Hurwitz zeta-functions, there is the following application.

Theorem 12 *If α $(\neq 1/2, 1)$ is rational or transcendental, then for any σ_1, σ_2 with $1/2 < \sigma_1 < \sigma_2 < 1$, the function $\zeta(s, \alpha)$ has infinitely many zeros in the strip $\sigma_1 < \sigma < \sigma_2$.*

It is a classical theorem of Davenport-Heilbronn [13] that $\zeta(s, \alpha)$ for rational or transcendental α $(\neq 1/2, 1)$ has infinitely many zeros in the half plane $\sigma > 1$. The same result is also known to be true even when α is algebraic irrational (Cassels [12]). However, Theorem 12 is based on the universality theorem, hence has been only proved for rational or transcendental α. This theorem was first announced by Voronin [88] when α is rational, but the proof (including the transcendental case) was given by Bagchi [2] and Gonek [20]. A quantitative result was shown by Laurinčikas [38] for certain type of Lerch zeta-functions. See also Garunkštis [16] for further study in this direction. Voronin [89] obtained a result on the distribution of zeros of a linear combination of Dirichlet L-functions, and this topic has been further developed by Laurinčikas [39][51].

Recently Andersson [1] used the universality theorem to disprove a conjecture of Ramachandra on Dirichlet polynomials. The connection between the universality and the quantum chaos is discussed by Gutzwiller [22]. The universality theorem is also used successfully in the evaluation of certain integrals appearing in quantum mechanics (Bitar-Khuri-Ren [4]). Dr. H. Nagoshi proposed to investigate the connection between the universality and random matrix theory. These directions of research will become more important in the future.

5 Limit theorems and the universality

A feature of Voronin's original proof of the universality theorem on $\zeta(s)$ is a successful application of the theory of Hilbert spaces. Bagchi [2][3] also used the theory of Hilbert spaces, and added the idea of Reich [77], to develop a method of deducing the universality theorem from a kind of limit theorem. The details [2] of Bagchi's theory had been unpublished for a long time, but now we can easily learn this theory by a detailed account in the book of Laurinčikas [43].

We state Bagchi's limit theorem in the case of the Riemann zeta-function. Denote by D the strip $1/2 < \sigma < 1$, and by $H(D)$ the set of all holomorphic functions on D, equipped with the topology of uniform convergence on compact subsets. The space $H(D)$ is metrizable in a standard way. Let $\mathcal{B}(H(D))$ be the family of all Borel subsets of $H(D)$, and put

$$\tilde{P}_T(\tilde{A};\zeta) = T^{-1}\mu_1(\{t \in [0,T] \mid \zeta(s+it) \in \tilde{A}\}) \tag{5.1}$$

for $\tilde{A} \in \mathcal{B}(H(D))$. Then $\tilde{P}_T$ is a probability measure on $(H(D), \mathcal{B}(H(D)))$, and

Theorem 13 (Bagchi [2]) *When $T \to \infty$, the measure $\tilde{P}_T(\tilde{A};\zeta)$ is convergent weakly to a certain probability measure $\tilde{Q}(\tilde{A};\zeta)$.*

Comparing this theorem with Theorem 4 we can readily see that this theorem is an analogue of Bohr-Jessen's limit theorem on function spaces.

Moreover Bagchi constructed the limit measure $\tilde{Q}$ explicitly. Let γ be the unit circle in the complex plane, and consider the infinite product

$$\Omega = \prod_p \gamma_p,$$

where the product runs over all primes and $\gamma_p = \gamma$ for any p. By the product topology and coordinatewise multiplication, Ω becomes a compact Abelian group, hence there exists a Haar measure m_H with $m_H(\Omega) = 1$. By $\mathcal{B}(\Omega)$ we denote the family of all Borel subsets of Ω. Then $(\Omega, \mathcal{B}(\Omega), m_H)$ is a probability space. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p, and for any positive integer k we define

$$\omega(k) = \prod_p \omega(p)^{\alpha(p)}, \tag{5.2}$$

where

$$\prod_p p^{\alpha(p)}$$

is the factorization of k into prime divisors (hence only finitely many $\alpha(p)$s are not zero). And define

$$\xi(s,\omega) = \sum_{k=1}^{\infty} \omega(k) k^{-s}.$$

This is an $H(D)$-valued random element defined on Ω. Bagchi proved that the distribution of this random element is just the limit measure $\tilde{Q}$, that is

$$\tilde{Q}(\tilde{A};\zeta) = m_H(\{\omega \in \Omega \mid \xi(s,\omega) \in \tilde{A}\}).$$

Bagchi deduced the universality theorem from his Theorem 13 and the above explicit expression of $\tilde{Q}$. After his work, this method of Bagchi has been repeatedly used by Laurinčikas, and now it is the most standard method of proving universality theorems.

For instance, Laurinčikas-Misevičius [56][57] proved a limit theorem of Bagchi's type with weight and the explicit expression of the limit measure, and by using them, Laurinčikas [42] obtained the universality theorem with weight for the Riemann zeta-function. A limit theorem in the space of continuous functions was proved, under the assumption of the Riemann hypothesis, by Laurinčikas [41]. This article [41] is also a very nice survey on various limit theorems known before the publication of it.

Results of Bagchi's type can be extended to the case of a fairly general class of zeta-functions. Laurinčikas [44][45] extended all of Theorem 4, Theorem 13 and the explicit expression of the limit measure in the latter theorem to any elements of the family $\mathcal{M}$ introduced in Section 2, in a generalized form with weight. The result in [45] can be stated, in the case of the trivial weight, as follows. Let $\varphi(s)$ be an element of $\mathcal{M}$, and by $D(\varphi)$ denote the half plane $\sigma > \rho$. Let $M(D(\varphi))$ be the space of all meromorphic functions on $D(\varphi)$, and $\mathcal{B}(M(D(\varphi)))$ the family of all Borel subsets of $M(D(\varphi))$. For $\tilde{A} \in \mathcal{B}(M(D(\varphi)))$, define $\tilde{P}_T(\tilde{A};\varphi)$ similarly to (5.1). Also we define

$$\xi_\varphi(s,\omega) = \sum_{k=1}^{\infty} b(k)\omega(k)k^{-s},$$

where

$$\sum_{k=1}^{\infty} b(k)k^{-s}$$

is the Dirichlet series expansion of $\varphi(s)$. Then $\xi_\varphi(s,\omega)$ is an $H(D(\varphi))$-valued random element defined on Ω. Denote its distribution by $\tilde{Q}(\tilde{A};\varphi)$. Then we have

Theorem 14 (Laurinčikas [45]) *When $T \to \infty$, the measure $\tilde{P}_T(\tilde{A};\varphi)$ is convergent weakly to $\tilde{Q}(\tilde{A};\varphi)$.*

Recently R. Kačinskaitė obtained discrete limit theorems for elements of $\mathcal{M}$.

Limit theorems of Bagchi's type can be proved for some kind of zeta-functions without Euler products. Limit theorems for Lerch zeta-functions are treated by Garunkštis and Laurinčikas. Laurinčikas [52] studied limit theorems for more general Dirichlet series.

(Note added in the English translation. Kačinskaitė's results are in [30][31] and her subsequent papers. Limit theorems for Lerch zeta-functions proved by Garunkštis and Laurinčikas can be seen in [15][17][18][46][47]; a book of them on Lerch zeta-functions is now in preparation. See also Ignatavičiūtė [26].)

Now we can say that Bagchi's limit theorem, that is the first half of Bagchi's theory, has been generalized very nicely. Is it also possible to generalize the second half of his theory, the deduction of the universality theorem from the limit theorem? Unfortunately the situation here is not so simple. Laurinčikas [50][51] tried to construct such a general theory, but he obtained a proof of universality for $\varphi(s) \in \mathcal{M}$ only under the assumption that $\varphi(s)$ further satisfies some strong condition. It is possible to construct explicitly such a $\varphi(s)$; however, for many zeta-functions important in number theory, it is very difficult to verify that condition. For example, for the automorphic L-function φ_f introduced in Section 3, that condition is almost impossible to verify. There is a paper of Kačėnas-Laurinčikas [29] on the universality of automorphic L-functions, but this paper also requires a strong condition. On the other hand, Yu. V. Linnik and I. A. Ibragimov conjectured that all Dirichlet series, except for trivial exceptions, would have the universality property. To check this conjecture, it is desirable to remove the conditions mentioned above, and to prove the universality theorem unconditionally for as wide class of zeta-functions as possible. We conclude this article with reporting recent developments in this direction in the next section.

6 The positive density method in the theory of universality

In the case of the Riemann zeta-function, an important point in Bagchi's argument of deducing the universality theorem from the limit theorem is the fact that the formula

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + a_1 + O(\exp(-a_2\sqrt{\log x})) \tag{6.1}$$

is used essentially. Here $x > 1$, a_1, a_2 are constants, $a_2 > 0$, and the summation runs over all primes up to x. (See Lemma 4.8 of Bagchi [3], or Theorem 4.14 of Chapter 6 of [43].) The formula (6.1) is a well-known result in the theory of the distribution of primes, and is actually equivalent with the prime number theorem with a remainder term. To carry out the proof in the style of Bagchi, such a rather sharp asymptotic formula as (6.1) is indispensable.

The automorphic L-function φ_f is an element of $\mathcal{M}$, hence we can apply Theorem 14. Therefore it is natural to expect that we may apply Bagchi's argument in this case and deduce the universality theorem from Theorem 14. For this purpose, corresponding to (6.1), a sharp asymptotic formula for

$$\sum_{p \leq x} |a(p)|/p$$

is necessary. However such an asymptotic formula is not known, because it is very difficult to treat sums involving Fourier coefficients of cusp forms. Formula (3.7) is insufficient at all. This situation implies that the simple analogy of Bagchi's argument collapses, hence some new idea is required. One possibility is to use

(3.6) instead. The conclusion is that it is still insufficient to use only (3.6), but we can go through the obstacle by using (3.6) combined with (6.1). This is the method developed in the paper [55] of Laurinčikas and the author, and the result is the following

Theorem 15 ([55]) *Let K be a compact subset of the strip $\kappa/2 < \sigma < (\kappa+1)/2$ with connected complement, and $h(s)$ be a function continuous and non-vanishing on K and holomorphic in the interior of K. Define*

$$\begin{aligned} U(T;\varphi_f) &= U(T,K,h,\varepsilon;\varphi_f) \\ &= \mu_1\left\{\tau\in[0,T] \mid \sup_{s\in K}|\varphi_f(s+i\tau)-h(s)|<\varepsilon\right\} \end{aligned} \tag{6.2}$$

for any $\varepsilon > 0$. Then we have

$$\liminf_{T\to\infty} T^{-1}U(T;\varphi_f) > 0. \tag{6.3}$$

That is, the universality property holds for the automorphic L-function φ_f. The details how to deduce this theorem from (3.6) and (6.1) are rather technical, but the basic idea is to use the fact that, for any fixed μ with $0 < \mu < 1$, there are sufficiently many primes p ("positive density" in the set of all primes) for which $|a(p)| > \mu$ holds. As explained in Section 3, this idea has already appeared in the proof of the inequality of Joyner's type for automorphic L-functions, due to Hattori and the author ([24]). We call this idea "the positive density method" in this article.

A similar argument is also performed in [54], mentioned in Section 4, on the joint universality of Lerch zeta-functions. In this paper a set of integers, which has a positive density in the set of all positive integers, is used. This paper treats the case when $\alpha_1,\ldots,\alpha_m$ are transcendental and $\lambda_1,\ldots,\lambda_m$ are rational. In the case $m = 1$, if α_1 is transcendental, we can show the universality theorem for any λ_1 (Laurinčikas [48]). However for $m \geq 2$, we cannot prove the universality at present if we do not assume the rationality of $\lambda_1,\ldots,\lambda_m$, and moreover, the positive density method is necessary in the proof. Therefore in this case the joint universality is essentially more difficult than the universality of a single function.

As for Hecke L-functions for algebraic number fields, when the field is quadratic, a joint universality theorem is announced in Voronin [89]. Joint value-distribution of Hecke L-functions attached to an imaginary quadratic field is also discussed in Section 4, Chapter 7 of Karatsuba-Voronin [32]. They apply the results proved there to the study of the distribution of zeros of zeta-functions of quadratic forms (see also Voronin [86][88]).

Recently Mishou [66] proved the universality of Hecke L-functions attached to any class character of any algebraic field F of finite degree. In Mishou's paper the universality is proved only in the region $1-\max\{d,2\}^{-1} < \sigma < 1$, where d is the degree of F over $\mathbf{Q}$, because the mean value theorem of the form (2.5) is known only in the region $\sigma > 1 - \max\{d, 2\}^{-1}$. A set of primes of positive density again plays an important role in Mishou's proof. In this case it is the set of primes which split completely in a certain Galois field, hence is of positive density by the Artin-Chebotarev theorem. Mishou used this fact and the class field theory skillfully to obtain the proof. Then in [67], Mishou proceeded further to obtain the proof of the universality of Hecke L-functions with any Grössencharakter of any algebraic field of finite degree. This time the set of primes of positive density, used in the proof, is constructed in a more complicated way. Moreover, a delicate argument based on a result of Mitsui is necessary in the proof; the argument is divided into three cases according as the field is totally real Galois, totally imaginary Galois, or non-Galois.

Now return to automorphic L-functions. It is known that the error term on the right-hand side of (3.6) can be improved (Perelli [74]). By using this refinement, we can show

$$\sum_{p \le x} \frac{c_p}{p} = \log\log x + a_3 + O(\exp(-a_4\sqrt{\log x})), \tag{6.4}$$

where c_ns are the coefficients of the Rankin-Selberg L-function

$$Z(s) = \zeta(2s)\sum_{n=1}^{\infty} a(n)n^{-s} = \sum_{n=1}^{\infty} c_n n^{-s}$$

attached to f. The asymptotic formula (6.4) is exactly an analogue of (6.1). By using (6.4), the universality of Rankin-Selberg L-functions can be shown (for $3/4 < \sigma < 1$) just analogously to the case of the Riemann zeta-function, without using the positive density method ([64]).

In [55] only holomorphic cusp forms with respect to $SL(2, \mathbf{Z})$ have been treated, but it is not necessary to restrict our consideration to the case of $SL(2, \mathbf{Z})$, and it is desirable to prove the joint universality of more general automorphic L-functions. In this direction, a joint research of Laurinčikas and the author is now going on.

We mentioned the χ-universality of Dirichlet L-functions in Section 4. Recently Nagoshi [71][72] studied universality properties of automorphic L-functions with respect to other parameters. Let f be the same as in Section 3, and put

$$\varphi_f^*(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

Then, a result of Nagoshi asserts that, for any $h(s)$ satisfying some conditions similar to those in Theorem 10, if we search cusp forms of up to sufficiently large weights, we can always find an f such that $\varphi_f^*(s)$ approximates $h(s)$ uniformly, and moreover, the set of such fs has a positive density in some sense. Nagoshi also proved similar kind of universality when we consider a family of normalized newforms with respect to the congruence subgroup $\Gamma_0(N)$ of level N, fix a weight and move levels; and also, when consider a family of L-functions attached to Maass forms and move eigenvalues of the Laplacian. Nagoshi used the results of Sarnak and of Serre based on the trace formula, and also the spectral large sieve and the Paley-Wiener theorem in his proof.

We can say from the above results that the range of research of unversality theorems is now spreading wider and wider. However, many fundamental problems are still unsolved. For instance, it is a natural hope to refine the universality theorems to more quantitative forms, and indeed there are attempts of Good [21] and Laurinčikas [53], but still this direction is almost uncultivated.

Universality theorems assert that zeta-functions have a kind of ergodic properties on function spaces. So far all proofs of universality theorems are based on some arithmetical properties. However it is not known whether universality is a really arithmetical phenomenon or not. It might be the right way of understanding to discuss universality in the framework of complex function theory; or, universality might always appear as a natural property associated with ergodic dynamical systems. A lot of further research is necessary to answer these questions in the future.

References

[1] J. Andersson, Power sum theory and zeta-functions, Filosofie Licentiatavhandling, Dept. of Math., Stockholm Univ., 1998.

[2] B. Bagchi, The statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series, Thesis, Indian Statistical Institute, Calcutta, 1981.

[3] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z. **181** (1982), 319-334.

[4] K. M. Bitar, N. N. Khuri and H. C. Ren, Path integrals and Voronin's theorem on the universality of the Riemann zeta function, Ann. Phys. **211** (1991), 172-196.

[5] H. Bohr, Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$, Nachr. Akad. Wiss. Göttingen II Math.-phys. Kl. (1911), 409-428.

[6] H. Bohr, Zur Theorie der Riemann'schen Zetafunktion im kritischen Streifen, Acta Math. **40** (1915), 67-100.

[7] H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, J. Reine Angew. Math. **144** (1914), 249-274.

[8] H. Bohr and B. Jessen, Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver, Dan. Vid. Selsk. Skr. Nat. Math. Afd. (8) **12** (1929), 1-82.

[9] H. Bohr and B. Jessen, Über die Wertverteilung der Riemannschen Zetafunktion, Erste Mitteilung, Acta Math. **54** (1930), 1-35; Zweite Mitteilung, ibid. **58** (1932), 1-55.

[10] H. Bohr and E. Landau, Ein Satz über Dirichlet'sche Reihen mit Anwendung auf die ζ-Funktion und die L-Funktionen, Rend. Circ. Mat. Palermo **37** (1914), 269-272.

[11] V. Borchsenius and B. Jessen, Mean motions and values of the Riemann zeta function, Acta Math. **80** (1948), 97-166.

[12] J. W. S. Cassels, Footnote to a note of Davenport and Heilbronn, J. London Math. Soc. **36** (1961), 177-184.

[13] H. Davenport and H. Heilbronn, On the zeros of certain Dirichlet series I, II, J. London Math. Soc. **11** (1936), 181-185, 307-312.

[14] K. M. Eminyan, χ-universality of the Dirichlet L-function, Mat. Zametki **47** (1990), 132-137 (in Russian); Math. Notes **47** (1990), 618-622.

[15] R. Garunkštis, An explicit form of limit distribution with weight for the Lerch zeta-function in the space of analytic functions, Liet. Mat. Rink. **37** (1997), 309-326 (in Russian); Lith. Math. J. **37** (1997), 230-242.

[16] R. Garunkštis, On the zeros of the Lerch zeta-function II, in "Probability Theory and Mathematical Statistics", Proc. 7th Vilnius Conf., B. Grigelionis et al. (eds.), VSP/TEV, 1999, pp. 267-276.

[17] R. Garunkštis and A. Laurinčikas, On the Lerch zeta-function, Liet. Mat. Rink. **36** (1996), 423-434 (in Russian); Lith. Math. J. **36** (1996), 337-346.

[18] R. Garunkštis and A. Laurinčikas, A limit theorem with weight for the Lerch zeta function in the space of analytic functions, Trudy Mat. Inst. Steklov. **218** (1997), 109-121 (in Russian); Proc. Steklov Inst. Math. **218** (1997), 104-116.

[19] R. Garunkštis and A. Laurinčikas, On one Hilbert's problem for the Lerch zeta-function, Publ. Inst. Math. Beograd **65**(79) (1999), 63-68.

[20] S. M. Gonek, Analytic properties of zeta and L-functions, Thesis, Univ. of Michigan, 1979.

[21] A. Good, On the distribution of the values of Riemann's zeta-function, Acta Arith. **38** (1981), 347-388.

[22] M. C. Gutzwiller, Stochastic behavior in quantum scattering, Physica **7D** (1983), 341-355.

[23] G. Harman and K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function IV, J. London Math. Soc. (2) **50** (1994), 17-24.

[24] T. Hattori and K. Matsumoto, Large deviations of Montgomery type and its application to the theory of zeta-functions, Acta Arith. **71** (1995), 79-94.

[25] T. Hattori and K. Matsumoto, A limit theorem for Bohr-Jessen's probability measures of the Riemann zeta-function, J. Reine Angew. Math. **507** (1999), 219-232.

[26] J. Ignatavičiūtė, On statistical properties of the Lerch zeta-function, Liet. Mat. Rink. **41** (2001), 424-440 (in Russian); Lith. Math. J. **41** (2001), 330-343.

[27] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. **38** (1935), 48-88.

[28] D. Joyner, Distribution Theorems of L-functions, Pitman Res. Notes in Math. Ser. Vol. 142, Longman Sci. & Tech., 1986.

[29] A. Kačėnas and A. Laurinčikas, On Dirichlet series related to certain cusp forms, Liet. Mat. Rink. **38** (1998), 82-97 (in Russian); Lith. Math. J. **38** (1998), 64-76.

[30] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function on the complex plane, Liet. Mat. Rink. **40** (2000), 475-492 (in Russian); Lith. Math. J. **40** (2000), 364-378.

[31] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function in the space of analytic functions, Liet. Mat. Rink. **41** (2001), 441-448 (in Russian); Lith. Math. J. **41** (2001), 344-350.

[32] A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function, Walter de Gruyter, 1992.

[33] A. Laurinčikas, Sur les séries de Dirichlet et les polynômes trigonométriques, Sém. Théorie des Nombres, Année 1978/79, exposé no. 24, 13pp.

[34] A. Laurinčikas, Distribution des valeurs de certaines séries de Dirichlet, C. R. Acad. Sci. Paris **289** (1979), 43-45.

[35] A. Laurinčikas, A limit theorem for Dirichlet L-series, Mat. Zametki **25** (1979), 481-485 (in Russian); Math. Notes **25** (1979), 251-253.

[36] A. Laurinčikas, Distribution of values of generating Dirichlet series of multiplicative functions, Liet. Mat. Rink. **22** (1982), 101-111 (in Russian); Lith. Math. J. **22** (1982), 56-63.

[37] A. Laurinčikas, Universality theorem, Liet. Mat. Rink. **23** (1983), 53-62 (in Russian); Lith. Math. J. **23** (1983), 283-289; II, Liet. Mat. Rink. **24** (1984), 113-121 (in Russian); Lith. Math. J. **24** (1984), 143-149.

[38] A. Laurinčikas, Zeros of certain Dirichlet series, Liet. Mat. Rink. **24** (1984), 116-126 (in Russian); Lith. Math. J. **24** (1984), 358-365.

[39] A. Laurinčikas, Zeros of linear combinations of Dirichlet series, Liet. Mat. Rink. **26** (1986), 468-477 (in Russian); Lith. Math. J. **26** (1986), 244-251.

[40] A. Laurinčikas, Limit theorems for the Riemann zeta-function in the complex space, in "Probability Theory and Mathematical Statistics, Vol. 2", Proc. 5th Vilnius Conf., B. Grigelionis et al. (eds.), VSP/Mosklas, 1990, pp. 59-69.

[41] A. Laurinčikas, On limit theorems for the Riemann zeta-function in some spaces, in "Probability Theory and Mathematical Statistics", Proc. 6th Vilnius Conf., B. Grigelionis et al. (eds.), VSP/TEV, 1994, pp. 457-483.

[42] A. Laurinčikas, On the universality of the Riemann zeta-function, Liet. Mat. Rink. **35** (1995), 502-507 (in Russian); Lith. Math. J. **35** (1995), 399-402.

[43] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer, 1996.

[44] A. Laurinčikas, Limit theorems for the Matsumoto zeta-function, J. Théorie des Nombres de Borbeaux **8** (1996), 143-158.

[45] A. Laurinčikas, On limit distribution of the Matsumoto zeta-function, Acta Arith. **79** (1997), 31-39; II, Liet. Mat. Rink. **36** (1996), 464-485 (in Russian); Lith. Math. J. **36** (1996), 371-387.

[46] A. Laurinčikas, A limit theorem for the Lerch zeta-function in the space of analytic functions, Liet. Mat. Rink. **37** (1997), 191-203 (in Russian); Lith. Math. J. **37** (1997), 146-155.

[47] A. Laurinčikas, On limit distribution of the Lerch zeta-function, in "Analytic and Probabilistic Methods in Number Theory", Proc. 2nd Palanga Conf., A. Laurinčikas et al. (eds.), VSP/TEV, 1997, pp. 135-148.

[48] A. Laurinčikas, The universality of the Lerch zeta-function, Liet. Mat. Rink. **37** (1997), 367-375 (in Russian); Lith. Math. J. **37** (1997), 275-280.

[49] A. Laurinčikas, On the Lerch zeta-function with rational parameters, Liet. Mat. Rink. **38** (1998), 113-124 (in Russian); Lith. Math. J. **38** (1998), 89-97.

[50] A. Laurinčikas, On the Matsumoto zeta-function, Acta Arith. **84** (1998), 1-16.

[51] A. Laurinčikas, On the zeros of linear combinations of the Matsumoto zeta-functions, Liet. Mat. Rink. **38** (1998), 185-204 (in Russian); Lith. Math. J. **38** (1998), 144-159.

[52] A. Laurinčikas, Value-distribution of general Dirichlet series, in "Probability Theory and Mathematical Statistics", Proc. 7th Vilnius Conf., B. Grigelionis et al. (eds.), VSP/TEV, 1999, pp. 405-414; II, Liet. Mat. Rink. **41** (2001), 449-460 (in Russian); Lith. Math. J. **41** (2001), 351-360; III, (with W. Schwarz and J. Steuding) preprint.

[53] A. Laurinčikas, On the effectivization of the universality theorem for the Lerch zeta-function, Liet. Mat. Rink. **40** (2000), 172-178 (in Russian); Lith. Math. J. **40** (2000), 135-139.

[54] A. Laurinčikas and K. Matsumoto, The joint universality and the functional independence for Lerch zeta-functions, Nagoya Math. J. **157** (2000), 211-227.

[55] A. Laurinčikas and K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, Acta Arith. **98**(2001), 345-359.

[56] A. Laurinčikas and G. Misevičius, Weighted limit theorem for the Riemann zeta function in the space of analytic functions, Liet. Mat. Rink. **34** (1994), 211-225 (in Russian); Lith. Math. J. **34** (1994), 171-182.

[57] A. Laurinčikas and G. Misevičius, On limit distribution of the Riemann zeta-function, Acta Arith. **76** (1996), 317-334.

[58] K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function I, Acta Arith. **48** (1987), 167-190; III, ibid. **50** (1988), 315-337.

[59] K. Matsumoto, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms, Nagoya Math. J. **116** (1989), 123-138.

[60] K. Matsumoto, Value-distribution of zeta-functions, in "Analytic Number Theory", K. Nagasaka and E. Fouvry (eds.), Lecture Notes in Math. Vol. 1434, Springer, 1990, pp. 178-187.

[61] K. Matsumoto, On the magnitude of asymptotic probability measures of Dedekind zeta-functions and other Euler products, Acta Arith. **60** (1991), 125-147.

[62] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory **40** (1992), 187-210.

[63] K. Matsumoto, Asymptotic probability measures of Euler products, in "Proceedings of the Amalfi Conference on Analytic Number Theory", E. Bombieri et al. (eds.), Univ. di Salerno, 1992, pp. 295-313.

[64] K. Matsumoto, The mean values and the universality of Rankin-Selberg L-functions, in "Number Theory", M. Jutila and T. Metsänkylä (eds.), Walter de Gruyter, 2001, pp. 201-221.

[65] S. N. Mergelyan, On the representation of functions by series of polynomials on closed sets, Dokl. Akad. Nauk SSSR (N.S.) **78** (1951), 405-408 (in Russian); in "Amer. Math. Soc. Transl. Ser. 1, Vol. 3, Series and Approximation", Amer. Math. Soc., 1962, pp. 287-293.

[66] H. Mishou, The universality theorem for L-functions associated with ideal class characters, Acta Arith. **98** (2001), 395-410.

[67] H. Mishou, The universality theorem for Hecke L-functions, preprint.

[68] H. L. Montgomery, The zeta function and prime numbers, in "Proceedings of the Queen's Number Theory Conference, 1979" P. Ribenboim (ed.), Queen's Papers in Pure and Appl. Math. Vol. 54, Queen's Univ., Kingston, Ontario, Canada, 1980, pp. 1-31.

[69] H. L. Montgomery and A. M. Odlyzko, Large deviations of sums of independent random variables, Acta Arith. **49** (1988), 427-434.

[70] M. Ram Murty, Oscillations of Fourier coefficients of modular forms, Math. Ann. **262** (1983), 431-446.

[71] H. Nagoshi, The universality of families of automorphic L-functions, preprint.

[72] H. Nagoshi, The universality of L-functions attached to eigenfunctions for $SL(2, \mathbf{Z})$, in preparation.

[73] E. M. Nikishin, Dirichlet series with independent exponents and some of their applications, Mat. Sb. **96**(138) (1975), 3-40 (in Russian); Math. USSR Sb. **25** (1975), 1-36.

[74] A. Perelli, On the prime number theorem for the coefficients of certain modular forms, in "Elementary and Analytic Theory of Numbers", H. Iwaniec (ed.), Banach Center Publ. Vol. 17, PWN-Polish Sci. Publ., 1985, pp. 405-410.

[75] H. S. A. Potter, The mean values of certain Dirichlet series I, Proc. London Math. Soc (2) **46** (1940), 467-478.

[76] R. A. Rankin, Sums of powers of cusp form coefficients II, Math. Ann. **272** (1985), 593-600.

[77] A. Reich, Universalle Wertverteilung von Eulerprodukten, Nachr. Akad. Wiss. Göttingen II Math.-phys. Kl. (1977), Nr.1, 1-17.

[78] A. Reich, Wertverteilung von Zetafunktionen, Arch. Math. **34** (1980), 440-451.

[79] A. Reich, Zetafunktionen und Differenzen-Differentialgleichungen, ibid. **38** (1982), 226-235.

[80] A. Reich, Zur Universalität und Hypertranszendenz der Dedekindschen Zetafunktion, Abh. Braunschweig. Wiss. Ges. **33** (1982), 197-203.

[81] E. Seneta, Regularly Varying Functions, Lecture Notes in Math. Vol. 508, Springer, 1976.

[82] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford, 1951.

[83] S. M. Voronin, On the distribution of nonzero values of the Riemann ζ-function, Trudy Mat. Inst. Steklov. **128** (1972), 131-150 (in Russian); Proc. Steklov Inst. Math. **128** (1972), 153-175.

[84] S. M. Voronin, Theorem on the "universality" of the Riemann zeta-function, Izv. Akad. Nauk. SSSR Ser. Mat. **39** (1975), 475-486 (in Russian); Math. USSR Izv. **9** (1975), 443-453.

[85] S. M. Voronin, On the functional independence of Dirichlet L-functions, Acta Arith. **27** (1975), 493-503 (in Russian).

[86] S. M. Voronin, On the zeros of zeta-functions of quadratic forms, Trudy Mat. Inst. Steklov. **142** (1976), 135-147 (in Russian); Proc. Steklov Inst. Math. (1979), Issue 3, 143-155.

[87] S. M. Voronin, Analytic properties of generating Dirichlet functions of arithmetical objects, Thesis Doctor Phys.-Mat. Nauk, Moscow, 1977 (in Russian).

[88] S. M. Voronin, On the zeros of zeta-functions of quadratic forms, Dokl. Acad. Nauk SSSR **235** (1977), 257-258 (in Russian); Soviet Math. Dokl. **18** (1977), 910-911.

[89] S. M. Voronin, Distribution of zeros of some Dirichlet series, Trudy Mat. Inst. Steklov. **163** (1984), 74-77 (in Russian); Proc. Steklov Inst. Math. (1985), Issue 4, 89-92.

[90] A. Wintner, On the asymptotic repartition of the values of real almost periodic functions, Amer. J. Math. **54** (1932), 339-345.