

Chapter 9

Analytic Continuation of the Riemann Zeta Function

9.1 Integral Representation

We have taken as the definition of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1. \quad (9.1)$$

Our purpose in this chapter is to extend this definition to the entire complex s -plane, and show that the Riemann zeta function is analytic everywhere except at $s = 1$, where it has a simple pole of residue 1.

First, consider the integral

$$I = \int_0^{\infty} dx \frac{x^{s-1}}{e^x - 1} = \int_0^{\infty} dx \frac{x^{s-1} e^{-x}}{1 - e^{-x}}. \quad (9.2)$$

We expand the denominator in the last form in a geometric series, and interchange sum and integral to obtain

$$I = \sum_{n=0}^{\infty} \int_0^{\infty} dx x^{s-1} e^{-(n+1)x} = \sum_{n=0}^{\infty} \frac{\Gamma(s)}{(n+1)^s} = \Gamma(s)\zeta(s), \quad (9.3)$$

which uses (9.1) and Euler's representation of the Gamma function,

$$\Gamma(s) = \int_0^{\infty} \frac{dt}{t} t^s e^{-t}, \quad \operatorname{Re} s > 0. \quad (9.4)$$

Thus we have the alternative representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}, \quad \operatorname{Re} s > 1. \quad (9.5)$$

9.2 Riemann's Representation

Consider the contour integral

$$K = \int_C dz \frac{(-z)^{s-1}}{e^z - 1}, \quad (9.6)$$

where we take the branch line to lie along the positive real z axis, and C is a contour that comes from $+\infty$ just below the branch line, encircles the branch point at $z = 0$, and returns to $+\infty$ just above the branch line. It is easy to check that, if $\operatorname{Re} s > 1$, the portion of the integral coming from the encirclement of the branch point is zero, and if we choose the phases so that $(-z)^{s-1} = e^{-i\pi(s-1)}x^{s-1}$ just above the branch line, and $(-z)^{s-1} = e^{+i\pi(s-1)}x^{s-1}$ just below, we have

$$K = \left(e^{-i\pi(s-1)} - e^{i\pi(s-1)} \right) I = -2i \sin \pi(s-1) I, \quad (9.7)$$

where I is the integral (9.2). Therefore we have the result

$$\zeta(s) = \frac{1}{2i \sin \pi s} \frac{1}{\Gamma(s)} \int_C dz \frac{(-z)^{s-1}}{e^z - 1}. \quad (9.8)$$

This can be given a bit simpler form by using the property of the Gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \quad (9.9)$$

Then,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C dz \frac{(-z)^{s-1}}{e^z - 1}, \quad (9.10)$$

which has the opposite sign to the formula in Whittaker and Watson (p. 266) because the sense of the contour of integration is reversed.

Note now that the contour integral is well-defined for all s , so that $\zeta(s)$ can have singularities only that the poles of $\Gamma(1-s)$, which occur at $s = 1, 2, 3, \dots$. However, we know that $\zeta(s)$ is regular for $s > 1$, so the poles of the gamma function at $s = 2, 3, \dots$ must be cancelled by zeros of the contour integral. Thus, we have proved that the only pole the Riemann zeta function occurs at $s = 1$. In the vicinity of $s = 1$, $\Gamma(1-s) \sim 1/(1-s)$, and since for $s = 1$ the contour integral is

$$\int_C \frac{dz}{e^z - 1} = \oint_{\gamma} \frac{dz}{e^z - 1} = -2\pi i, \quad (9.11)$$

where γ is a contour enclosing only the pole at $z = 0$, conducted in a negative sense, we see that the residue of the pole of the zeta function is unity:

$$\zeta(s) \sim \frac{1}{s-1}, \quad s \rightarrow 1. \quad (9.12)$$

9.3 Reflection Property

To obtain the reflection property, we consider a closed contour integral

$$J = \oint_{\Gamma} dz \frac{(-z)^{s-1}}{e^z - 1}, \quad (9.13)$$

where Γ is a circle of radius $(2N+1)\pi$, joined to the integral K , (9.6),

$$J = K + L, \quad (9.14)$$

where L corresponds to the circular part of the contour Γ , where with $R = (2N+1)\pi$,

$$L = i \int_0^{2\pi} d\theta \operatorname{Re} e^{i\theta} \frac{R^{s-1} e^{i(s-1)(\theta-\pi)}}{e^{R \cos \theta + iR \sin \theta} - 1}. \quad (9.15)$$

Now when $\cos \theta < 0$, $e^{R \cos \theta} \rightarrow 0$ as $R \rightarrow \infty$, so the denominator in the integrand in (9.15) tends to -1 . When $\cos \theta > 0$, $e^{R \cos \theta} \rightarrow \infty$, so the integrand in (9.15) vanishes. At $\cos \theta = 0$, $e^{\pm iR} = e^{\pm i\pi(2N+1)} = -1$, so the denominator is -2 . Thus we conclude that for large R , $|L| \propto R^s \rightarrow 0$ if $\operatorname{Re} s < 0$.

On the other hand, J is equal to $2\pi i$ times the sum of the residues of the poles of the integrand. The poles occur at $z = 2\pi in$, $n = \pm 1, \pm 2, \pm 3, \dots$ ($z = 0$ is outside the contour Γ .) It is easy to check that the residue of the pole at $z = 2\pi in$ is $(e^{\mp i\pi/2} 2\pi |n|)^{s-1}$, if n is positive or negative, respectively. Therefore, for $\operatorname{Re} s < 0$,

$$K = 2\pi i 2 \cos \frac{\pi}{2} (s-1) \sum_{n=1}^{\infty} (2\pi n)^{s-1}, \quad (9.16)$$

so from (9.10)

$$\zeta(s) = \Gamma(1-s) (2\pi)^{s-1} 2 \cos \frac{\pi}{2} (s-1) \zeta(1-s). \quad (9.17)$$

Alternatively, call $1-s = x$, so for $\operatorname{Re} x > 1$,

$$\zeta(1-x) = 2^{1-x} \pi^{-x} \cos \frac{\pi x}{2} \Gamma(x) \zeta(x). \quad (9.18)$$

Because both sides of this equation are analytic save where poles occur, we conclude that this formula holds true for all x .

From this formula, we can deduce special values of $\zeta(s)$ for nonpositive values of s :

$$\zeta(0) = \frac{1}{\pi} \lim_{x \rightarrow 1} \cos \frac{\pi}{2} x \zeta(x) = -\frac{1}{2}, \quad (9.19a)$$

$$\zeta(-1) = -\frac{1}{2\pi^2} \zeta(2) = -\frac{1}{12}, \quad (9.19b)$$

$$\zeta(-2m) = 0, \quad m \text{ a positive integer}, \quad (9.19c)$$

$$\zeta(-3) = \frac{3}{4\pi^4} \zeta(4) = \frac{1}{120}. \quad (9.19d)$$

In fact, for positive integer m we have the following relation to Bernoulli numbers:

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!} |B_{2m}|, \quad (9.20a)$$

$$\zeta(1 - 2m) = -\frac{B_{2m}}{2m}. \quad (9.20b)$$