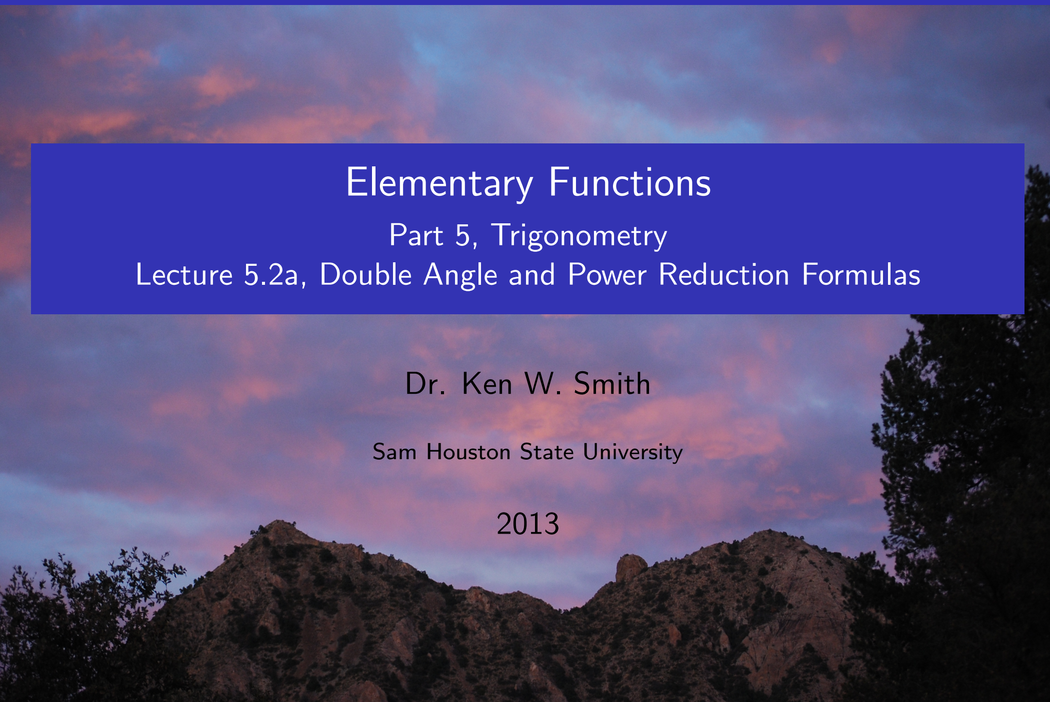

Elementary Functions

Part 5, Trigonometry

Lecture 5.2a, Double Angle and Power Reduction Formulas

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Double-angle & power-reduction identities

In the previous presentation we developed formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$

These formulas lead naturally to another set of identities involving double angles power reduction and half-angles.

Double-angle & power-reduction identities

Recall the sum-of-angle identities from an earlier presentation:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we replace α and β with the same angle, θ, these identities describe the sine and cosine of 2θ, expressing trig functions of a doubled angle in terms of the original.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{1}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{2}$$

Double-angle & power-reduction identities

The equation

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

is particularly applicable.

It can be used to reduce the power on cosine and sine.

For example, suppose we add this equation to the Pythagorean Identity

$$\cos^2\theta + \sin^2\theta = 1.$$

Double-angle & power-reduction identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$1 = \cos^2\theta + \sin^2\theta.$$

We get

$$1 + \cos 2\theta = 2\cos^2\theta.$$

After dividing by 2, we obtain an equation for $\cos^2\theta$.

$$\boxed{\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)} \qquad (3)$$

Double-angle & power-reduction identities

In a similar manner, we could *subtract* the double angle formula from the Pythagorean identity:

$$1 = \cos^2\theta + \sin^2\theta.$$
$$-(\cos 2\theta = \cos^2\theta - \sin^2\theta)$$

getting $2\sin^2\theta = 1 - \cos 2\theta$.

After dividing both sides by 2 we have a formula for $\sin^2\theta$:

$$\boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)} \qquad (4)$$

Double-angle & power-reduction identities

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

These formulas are sometimes called "power reduction formulas" because they often allow us reduce the power on one of the trig functions, if the power is an even integer.

For example, we can reduce $\cos^4 x = (\cos^2 x)^2$ by substituting for $\cos^2 x$ and then expanding the expression.

$$\cos^4 x = (\frac{1+\cos 2x}{2})^2 = \frac{1}{4}(1+2\cos 2x+\cos^2 2x) = \frac{1}{4}(1+2\cos 2x+(\frac{1+\cos}{2}$$

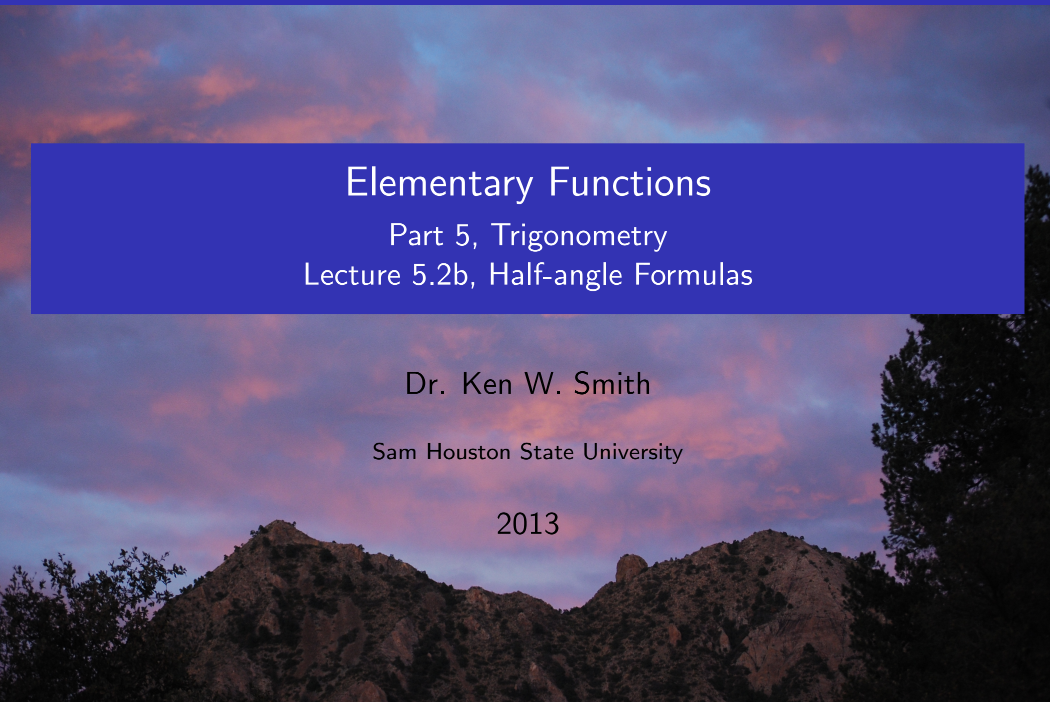
We may simplify the last expression by using the common denominator 8 and so write:

$$\cos^4 x = \frac{1}{8}(3 + 4\cos 2x + \cos 4x)).$$

Sum and Difference Formulas

In the next presentation we look at half-angle formulas.

(End)



Elementary Functions

Part 5, Trigonometry
Lecture 5.2b, Half-angle Formulas

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Half-angle formulas

The power reduction formulas can also be interpreted as half-angle formulas. Consider again the power reduction formulas

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta) \text{ and } \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta).$$

Replace 2θ by α (and replace θ by $\frac{1}{2}\alpha$.) Now our equations are

$$\cos^2(\frac{\alpha}{2}) = \frac{1}{2}(1+\cos\alpha)$$

and

$$\sin^2(\frac{\alpha}{2}) = \frac{1}{2}(1-\cos\alpha)$$

Now take square roots of both sides:

$$\cos(\frac{\alpha}{2}) = \pm\sqrt{\frac{1}{2}(1+\cos\alpha)} \tag{5}$$

$$\sin(\frac{\alpha}{2}) = \pm\sqrt{\frac{1}{2}(1-\cos\alpha)} \tag{6}$$

Half-angle formulas

$$\cos(\frac{\alpha}{2}) = \pm\sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$\sin(\frac{\alpha}{2}) = \pm\sqrt{\frac{1}{2}(1-\cos\alpha)}$$

If we want an equation for $\tan(\frac{\alpha}{2})$ we divide $\sin\frac{\alpha}{2}$ by $\cos\frac{\alpha}{2}$ to get

$$\tan(\frac{\alpha}{2}) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \tag{7}$$

Some worked problems

Use half-angle formulas to find the exact value of $\cos\frac{\pi}{12}$.

Solution. We view $\pi/12$ as half of the angle $\alpha = \pi/6$. So using the half-angle formula for cosine, with $\alpha = \pi/6$, we have

$$\cos(\tfrac{\pi}{12}) = \sqrt{\frac{1+\cos(\pi/6)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}.$$

Use half-angle formulas to find the exact value of $\sin\frac{\pi}{12}$.

Solution. Again we view $\pi/12$ as half of the angle $\alpha = \pi/6$ and here we use half-angle formula for sine.

$$\sin(\tfrac{\pi}{12}) = \sqrt{\frac{1-\cos(\pi/6)}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}.$$

Use half-angle formulas to find the exact value of $\cos\frac{\pi}{24}$.

Solution. The angle $\pi/24$ is half of the angle $\alpha = \pi/12$. Fortunately we just computed the cosine and sine of $\pi/12$ so we can use the half-angle formula again:

$$\cos(\frac{\pi}{24}) = \sqrt{\frac{1+\cos(\pi/12)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{3}}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2+\sqrt{3}}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}}.$$

Use half-angle formulas to find the exact value of $\sin\frac{\pi}{24}$.

Solution.

$$\sin(\tfrac{\pi}{24}) = \sqrt{\frac{1-\cos(\pi/12)}{2}} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{3}}}{2}}{2}} = \boxed{\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}}.$$

Is there a pattern to the formula for sine and cosine if we start at $\pi/6$ and keep cutting the angle in half?

$$\sin(\tfrac{\pi}{12}) = \tfrac{\sqrt{2-\sqrt{3}}}{2}.$$
$$\sin(\tfrac{\pi}{24}) = \tfrac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}.$$

Since the sine function helps give us a chord on the circle, we could then develop a formula for the circumference of a circle and then develop, from that, a formula for π.

Archimedes did this by (essentially) computing $\sin(\frac{\pi}{96})$!

In the next presentation we look at the Law of Sines.

(End)