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Completely bounded homomorphisms of the Fourier algebras

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Abstract

For locally compact groups G and H let $\mathrm{A}(G)$ denote the Fourier algebra of G and $\mathrm{B}(H)$ the Fourier–Stieltjes algebra of H. Any continuous piecewise affine map $\alpha : Y \subset H \to G$ (where Y is an element of the open coset ring) induces a completely bounded homomorphism $\Phi_\alpha : \mathrm{A}(G) \to \mathrm{B}(H)$ by setting $\Phi_\alpha u = u \circ \alpha$ on Y and $\Phi_\alpha u = 0$ off of Y. We show that if G is amenable then any completely bounded homomorphism $\Phi : \mathrm{A}(G) \to \mathrm{B}(H)$ is of this form; and this theorem fails if G contains a discrete nonabelian free group. Our result generalises results of Cohen (Amer. J. Math. 82 (1960) 213–226), Host (Bull. Soc. Math. France (1986) 114) and of the first author (J. Funct. Anal. (2004) 213). We also obtain a description of all the idempotents in the Fourier–Stieltjes algebras which are contractive or positive definite.

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1. Introduction

For any pair of locally compact abelian groups G and H, Cohen [10] characterised all of the bounded homomorphisms from the group algebra $\mathrm{L}^1(G)$ to the measure algebra $\mathrm{M}(H)$ in terms of piecewise affine maps between their dual groups $\widehat{H}$ and $\widehat{G}$. In doing so he made use of an equally profound discovery of his [9] characterising idempotent measures on abelian groups, a result which won him the Bôchner Memorial Prize in 1964. These results generalised results of many authors [3,27,28,30,43,44]. Cohen's work is also exposed nicely in [45]. As this characterisation is in terms of dual groups, it is more naturally formulated in terms of the algebras of Fourier and Fourier–Stieltjes transforms $\mathrm{A}(\widehat{G})$ and $\mathrm{B}(\widehat{H})$. There is a formulation of the Fourier and Fourier–Stieltjes algebras $\mathrm{A}(G)$ and $\mathrm{B}(H)$, due to Eymard [16], which can be done for any locally compact groups, and which generalises the group algebra and the measure algebra for dual groups. Moreover, these are commutative semi-simple Banach algebras, even for non-commutative groups. A longstanding question in harmonic analysis has been to determine to which extent Cohen's theorem holds in the non-abelian setting. Various intermediate results, described below, have been given over the years, and our main objective in this paper is to give a definitive solution to this problem.

The first decisive step in generalising Cohen's results is due to Host [22]. He first discovered the general form of idempotents in the Fourier–Stieltjes algebras, a significant result. He then identified the role of tensor products in obtaining the characterisation of bounded homomorphisms from $\mathrm{A}(G)$ to $\mathrm{B}(H)$. Unfortunately, by a result of Losert [34], the Banach algebra $\mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)$ (projective tensor product) is isomorphic to $\mathrm{A}(G \times G)$ only if G has an abelian subgroup of finite index. Hence it was only for such groups that Host established his result on homomorphisms.

In the intervening years the theory of operator spaces and completely bounded maps was developed by Paulsen, Blecher, Effros and Ruan amongst many others. See [15]. It was recognised in [4,14] that $\mathrm{A}(G)$ and $\mathrm{B}(H)$ can be regarded as operator spaces. The first major application of this was given by Ruan in [42] where it was shown that the operator space structure on $\mathrm{A}(G)$ gives rise to a more tractable cohomology theory than traditional Banach algebra cohomology (such as in [26]). It thus makes sense to speak about *completely bounded homomorphisms* from $\mathrm{A}(G)$ to $\mathrm{B}(H)$. Any results on such generalise the results of Cohen and Host since it was shown by Forrest and Wood [18] that any bounded linear map from $\mathrm{A}(G)$ to any operator space, is automatically completely bounded if and only if G has an abelian subgroup of finite index. The advantage of having the context of operator spaces is that it gives us the operator projective tensor product $\widehat{\otimes}$ [5,14]. By this, $\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G)$ can be naturally identified with $\mathrm{A}(G \times G)$ [13]. Using these techniques, the first author [24,25] took the next decisive step and characterised all of the completely bounded homomorphisms form $\mathrm{A}(G)$ to $\mathrm{B}(H)$, when G is an amenable discrete group.

In this article we generalise this result to what appears to be the fullest extent possible. We make note of the fact that for any pair of locally compact groups G and H, a continuous piecewise affine map $\alpha : Y \subset H \to G$ induces a completely bounded homomorphism from $\mathrm{A}(G)$ to $\mathrm{B}(H)$ (Proposition 3.1). We then show that if G is amenable then every completely bounded homomorphism from $\mathrm{A}(G)$ to $\mathrm{B}(H)$ is thus induced (Main Result: Theorem 3.7). Along the way we make significant use of a positive *bounded approximate diagonal* for $\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G)$, following the construction of Aristov, Runde and the second author [1, Lemma 3.4]—which simplifies a construction of Ruan [42]. As a complement to our main theorem we show that for any group which contains a discrete non-abelian free group the main result fails (Proposition 3.8), lending strong evidence that amenability is an indispensable assumption. This makes use of Leinert's free sets [31]. We also indicate how, for most (amenable) groups, our main result can fail for bounded homomorphisms from $\mathrm{A}(G)$ to itself which are not completely bounded (Remark 3.6).

In order to refine our main result, i.e. to characterise completely contractive and "completely positive" homomorphisms, we obtain a description of contractive and of positive definite idempotents in Fourier–Stieltjes algebras (Theorem 2.1). This result is well known for abelian groups, but does not appear to be in the literature for general groups. It is a special case of the significant theorem of Host [22], though it is not mentioned nor covered by him.

1.1. Preliminaries

If G is any locally compact group let $\mathrm{A}(G)$ denote its *Fourier algebra* and $\mathrm{B}(G)$ denote its *Fourier–Stieltjes algebra*, as defined in [16]. We recall that $\mathrm{B}(G)$ consists of all matrix coefficients of continuous unitary representations, i.e. functions of the form $s \mapsto \langle \pi(s)\xi|\eta\rangle$ where $\pi : G \to \mathcal{U}(\mathcal{H})$ is a homomorphism, continuous when the unitary group $\mathcal{U}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ is endowed with the weak operator topology. We also recall that $\mathrm{A}(G)$ is the space of all matrix coefficients of the left regular representation $\lambda_G : G \to \mathcal{U}(\mathrm{L}^2(G))$, given by left translation operators on $\mathrm{L}^2(G)$, the Hilbert space of (equivalence classes of) square-integrable functions. The norms on $\mathrm{A}(G)$ and $\mathrm{B}(G)$ are given by the dualities indicated below.

We note that $\mathrm{A}(G)$ has bounded dual space $\mathrm{A}(G)^* \cong \mathrm{VN}(G)$, where $\mathrm{VN}(G)$ is the von Neumann algebra generated by λ_G. The Fourier–Stieltjes algebra is the predual of the *enveloping von Neumann algebra* $\mathrm{W}^*(G)$ which is generated by the universal representation ϖ_G [12]. On the other hand, $\mathrm{B}(G)$ is the dual of the *enveloping C*-algebra* $\mathrm{C}^*(G)$. We note that $\mathrm{W}^*(G)$ satisfies the universal property for group von Neumann algebras: if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation and VN_π is the von Neumann algebra it generates, then there is $*$-homomorphism $\Pi : \mathrm{W}^*(G) \to \mathrm{VN}_\pi$ such that $\Pi(\varpi(s)) = \pi(s)$ for each s in G. We note that both $\mathrm{A}(G)$ and $\mathrm{B}(G)$ are semi-simple commutative Banach algebras under pointwise operations. Moreover, $\mathrm{A}(G)$ is an ideal in $\mathrm{B}(G)$. Furthermore, $\mathrm{A}(G)$ has Gel'fand spectrum G, and is *regular* on G in the sense that for any compact subset K of G, and any open set U containing K, there is an element u in $\mathrm{A}(G)$ such that $u|_K = 1$ and $\mathrm{supp}(u) \subset U$.

Our standard reference for operator spaces and completely bounded maps is [15], though we will indicate other references below. Any C*-algebra $\mathcal{A}$ is an *operator space* in the sense that for $n = 1, 2, \ldots$ the algebra of $n \times n$ matrices over $\mathcal{A}$, $\mathrm{M}_n(\mathcal{A})$ admits a unique norm which makes it into a C*-algebra. A linear map $T : \mathcal{A} \to \mathcal{B}$ between C*-algebras is called *completely bounded* if it is bounded and its amplifications $T^{(n)} : \mathrm{M}_n(\mathcal{A}) \to \mathrm{M}_n(\mathcal{B})$, given by $T^{(n)}[a_{ij}] = [Ta_{ij}]$, give a bounded family of norms

$\{\|T^{(n)}\| : n = 1, 2, \ldots\}$. In this case we write $\|T\|_{\mathrm{cb}} = \sup\{\|T^{(n)}\| : n = 1, 2, \ldots\}$. Moreover we say T is *completely contractive* if $\|T\|_{\mathrm{cb}} \leqslant 1$; that T is a *complete isometry* if each $T^{(n)}$ is an isometry; and that T is *completely positive* if each $T^{(n)}$ is a positive map (see [38]). Examples of completely bounded maps are $*$-homomorphisms, which are also completely positive and contractive, and multiplications by fixed elements in C*-algebras.

If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras with preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, we say a map $\Phi : \mathcal{M}_* \to \mathcal{N}_*$ is *completely bounded* (*contractive*), if its adjoint $\Phi^* : \mathcal{N} \to \mathcal{M}$ is such. However, it is often convenient to consider completely bounded maps on the space $\mathcal{M}_*$ by noting that it admits an operator space structure via the identifications $\mathrm{M}_n(\mathcal{M}_*) \cong \mathcal{CB}^\sigma(\mathcal{M}, \mathrm{M}_n)$, $n = 1, 2, \ldots$, where $\mathcal{CB}^\sigma(\mathcal{M}, \mathrm{M}_n)$ is the space of normal completely bounded maps form $\mathcal{M}$ to the finite-dimensional von Neumann algebra of $n \times n$ complex matrices [4,14]. We note that these spaces $\mathcal{M}_*$, with the above matricial structures, are completely isometrically isomorphic to subspaces of C*-algebras [41], which are not generally operator subalgebras. In particular, these structures are used to create the *operator projective tensor product* $\mathcal{M}_* \widehat{\otimes} \mathcal{N}_*$ [5,14]. This tensor product admits the very useful formula $(\mathcal{M}_* \widehat{\otimes} \mathcal{N}_*)^* \cong \mathcal{M} \overline{\otimes} \mathcal{N}$ [13], where $\mathcal{M} \overline{\otimes} \mathcal{N}$ is the von Neumann tensor product. We note for any locally compact group G that multiplication extends to a completely contractive linear map $\mu : \mathrm{B}(G) \widehat{\otimes} \mathrm{B}(G) \to \mathrm{B}(G)$. Indeed $\mu^* : \mathrm{W}^*(G) \to \mathrm{W}^*(G) \overline{\otimes} \mathrm{W}^*(G)$ is the $*$-homomorphism which extends $\varpi_{G \times G}(s, t) \mapsto \varpi_G(s) \otimes \varpi_G(t)$. Hence we say that $\mathrm{B}(G)$ is a *completely contractive Banach algebra*. In particular, multiplication by a fixed element $v \mapsto uv$ on $\mathrm{B}(G)$ is completely bounded with $\|v \mapsto uv\|_{\mathrm{cb}} = \|u\|_{\mathrm{B}(G)}$.

1.2. Piecewise affine maps

In this section we give a quick survey of piecewise affine maps on groups. These maps are natural generalisations of group homomorphisms and are the natural morphisms on finite collections of cosets. We will require general versions of several results from [45] concerning abelian groups. Unfortunately modification of the original proofs is required, and we give these below.

Let G be a group. A *coset* of G is any subset C of G for which there is a subgroup H of G, and an element s in G such that $C = sH$. We note that for H and s as above, we have that $Hs = ss^{-1}Hs$, which means that we need not distinguish between left and right cosets. The following result is [45, 3.7.1] in the case that G is abelian.

Proposition 1.1. *A subset C of G is a coset if and only if for every r, s and t* in *C, $rs^{-1}t \in C$ too. Moreover, $C^{-1}C$ is a subgroup for which $C = sC^{-1}C$ for any s* in *C.*

Proof. Necessity is trivial, so we will prove only sufficiency. We will show that $H = C^{-1}C$ is a subgroup and $C = sH$ for any s in C. If $s, t \in H$, then $s = s_1^{-1}s_2$ and $t = t_1^{-1}t_2$ where $s_i, t_i \in C$ for $i = 1, 2$. Then

$$st = s_1^{-1}(s_2 t_1^{-1} t_2) \in C^{-1}C \quad \text{and} \quad s^{-1} = s_2^{-1}s_1 \in C^{-1}C$$

whence $H = C^{-1}C$ is a subgroup. Now if $s \in C$ and $t \in H$ with $t = t_1^{-1}t_2$ as above, then $st = st_1^{-1}t_2 \in C$, so $sH \subset C$. Also, $C = ss^{-1}C \subset sC^{-1}C = sH$. Hence $C = sH$. $\square$

Now let H be another group. A map $\alpha : C \subset H \to G$ is called *affine* if C is a coset and for r, s, t in C

$$\alpha(rs^{-1}t) = \alpha(r)\alpha(s)^{-1}\alpha(t).$$

It is clear from Proposition 1.1 above, that the range $\alpha(C)$ of α is also a coset. Hence if $s \in C$, then

$$s^{-1}C \ni t \mapsto \alpha(s)^{-1}\alpha(st) \in \alpha(s)^{-1}\alpha(C) \tag{1.1}$$

is a homomorphism between subgroups.

We let $\Omega(H)$ denote the *coset ring* of the group H, which is the smallest ring of subsets which contains every coset. A map $\alpha : Y \subset H \to G$ is called *piecewise affine* if

$$\text{(i) there are pairwise disjoint } Y_i \in \Omega(H), \text{ for } i = 1, \dots, n$$
$$\text{such that } Y = \dot{\bigcup_{i=1}^{n}} Y_i \text{ (disjoint union)} \tag{1.2}$$

and

$$\text{(ii) each } Y_i \text{ is contained in a coset } L_i \text{ on which there is an}$$
$$\text{affine map } \alpha_i : L_i \to G \text{ such that } \alpha_i|_{Y_i} = \alpha|_{Y_i}.$$

If $\alpha : Y \subset H \to G$ is a function we define the *graph* of α to be the set

$$\Gamma_\alpha = \{(s, \alpha(s)) : s \in Y\}. \tag{1.3}$$

The following lemma is given for abelian groups in [45, 4.3.1]. Our proof is adapted from the one given there.

Lemma 1.2. *Let $\alpha : Y \subset H \to G$ be a function. Then α enjoys the following properties.*

(i) *If Γ_α is a subgroup then so too is Y and α is a homomorphism of subgroups.*
(ii) *If Γ_α is a coset then so too is Y and α is an affine map.*
(iii) *If $\Gamma_\alpha \in \Omega(H \times G)$, then α is a piecewise affine map.*

Proof. (ii) Let $r, s, t \in Y$. Then since Γ_α is a coset,

$$(r, \alpha(r))(s, \alpha(s))^{-1}(t, \alpha(t)) = (rs^{-1}t, \alpha(r)\alpha(s)^{-1}\alpha(t)) \in \Gamma_\alpha$$

which implies that Y is coset since $rs^{-1}t \in Y$. Since Γ_α is a graph, $(rs^{-1}t, \alpha(rs^{-1}t)) \in \Gamma_\alpha$ too and $\alpha(r)\alpha(s)^{-1}\alpha(t) = \alpha(rs^{-1}t)$.

(i) If Γ_α is a subgroup, it is a coset containing the identity, whence so too is Y. It follows that $\alpha(e) = e$ and α is a homomorphism.

(iii) Since $\Gamma_\alpha \in \Omega(H \times G)$, there exists a finite collection of subgroups Σ of $H \times G$ such that $\Gamma_\alpha \in \mathcal{R}(\Sigma)$, the smallest ring of subsets generated by cosets of elements of Σ. We may assume that Σ is closed under intersections. We may also assume that if one element of Σ is a subgroup of another, then the index of the first subgroup in the second is infinite.

It is then possible to write

$$\Gamma_\alpha = \dot{\bigcup_{i=1}^{n}} E_i \quad \text{where each} \quad E_i = L_i \setminus \bigcup_{j=1}^{m_i} M_{ij}$$

and each L_i and M_{ij} are cosets of elements of Σ with $M_{ij} \subset L_i$ for each i, j. Note that each E_i is itself a graph.

We claim that each L_i is a graph. If not, there are elements (s, t_1) and (s, t_2) in L_i with $t_1 \neq t_2$, so $(e, t) = (e, t_1^{-1} t_2) \in L_i^{-1} L_i$. Now if $(s, \alpha(s))$ is any element of E_i, then $(s, \alpha(s)t) \in L_i L_i^{-1} L_i = L_i$, so $(s, \alpha(s)t) \in M_{ij}$ for some j, since E_i is a graph. Hence

$$(s, \alpha(s)) = (s, \alpha(s)t)(e, t)^{-1} \in M_{ij}(e, t)^{-1}.$$

We note that $M_{ij}(e, t)^{-1}$ may not be in $\mathcal{R}(\Sigma)$, but it is a coset of a subgroup of infinite index in $L_i^{-1} L_i$. Thus

$$E_i \subset \bigcup_{j=1}^{m_i} M_{ij}(e, t)^{-1} \quad \text{whence} \quad L_i \subset \bigcup_{j=1}^{m_i} \left(M_{ij} \cup M_{ij}(e, t)^{-1} \right).$$

Hence the subgroup $L_i^{-1} L_i$ can be covered by finitely many cosets of subgroups which are of infinite index in itself, which is impossible by [36] (also see Proposition 2.2 for an analytic proof of this). Thus L_i is a graph and we may write

$$L_i = \{(s, \alpha_i(s)) : s \in K_i = p_1(L_i)\},$$

where $p_1 : H \times G \to H$ is the standard projection. Now if we let $Y_i = p_1(E_i)$ and $N_{ij} = p_1(M_{ij})$ we see that

$$Y_i = K_i \setminus \bigcup_{j=1}^{m_i} N_{ij} \tag{1.4}$$

since $p_1|_{L_i}$ has inverse $s \mapsto (s, \alpha_i(s))$. We also see from (ii) that $\alpha_i : K_i \to G$ is an affine map and that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$. $\square$

Now let us suppose that G and H are topological groups and the topology of G is locally compact and Hausdorff. If S is any subset of H we let $\overline{S}$ be the closure of S. The following result for abelian groups is given in [45, 4.2.4 & 4.5.2]. While our proof of (i) differs from the one given there, the proof of (ii) is similar and is included for convenience of the reader. We let $\Omega_o(H)$ denote the *open coset ring*, the smallest ring of subsets of H containing all open cosets.

Lemma 1.3. (i) *If $\alpha : C \subset H \to G$ is affine, and continuous on C, then it admits a continuous extension to an affine map $\bar{\alpha} : \overline{C} \to G$.*

(ii) *If $\alpha : Y \subset H \to G$ is piecewise affine and continuous on Y, and Y is open in H, then α admits a continuous extension $\bar{\alpha} : \overline{Y} \to G$. Moreover, $\overline{Y}$ is open in H, admits a decomposition $\overline{Y} = \dot{\bigcup}_{i=1}^{n} Y_i$ as in (1.2) where each $Y_i \in \Omega_o(H)$.*

Proof. (i) Let $r \in C$. Then the homomorphism $t \mapsto \alpha(r)^{-1}\alpha(rt)$ from $r^{-1}C$ to $\alpha(r)^{-1}\alpha(C)$ is continuous on $C^{-1}C = r^{-1}C$, and hence left uniformly continuous by [21, 5.40(a)]–i.e. for every (compact) neighborhood W of e_G, the unit in G, there is a neighbourhood U of e_H in H such that

$$\text{if } s^{-1}t \in U \text{ then } \alpha(rs)^{-1}\alpha(rt) = \alpha(rs)^{-1}\alpha(r)\alpha(r)^{-1}\alpha(rt) \in W. \tag{1.5}$$

If $s_0 \in \overline{C}$, then any net $(s_i)_i$ from C which converges to s_0 is left Cauchy: for any neighbourhood U of e_H, there is i_U such that $i, j \geqslant i_U$ implies that $s_i^{-1}s_j \in U$. Hence the net $(r^{-1}s_i)_i$ is left Cauchy as well, and so if U is chosen to satisfy (1.5) and $i, j \geqslant i_U$ then

$$\alpha(s_i)^{-1}\alpha(s_j) = \alpha(rr^{-1}s_i)^{-1}\alpha(rr^{-1}s_j) \in W$$

so $(\alpha(s_i))_i$ is left Cauchy in G. However, Hausdorff locally compact groups are complete, whence we obtain a unique limit $\bar{\alpha}(s_0) = \lim_i \alpha(s_i)$. That $\bar{\alpha} : \overline{C} \to G$ is affine follows from continuity of the group operations.

(ii) Since we assume that $\alpha : Y \subset H \to C$ is piecewise affine, we can decompose

$$Y = \dot{\bigcup_{i=1}^{n}} Y_i \quad \text{where each} \quad Y_i = K_i \setminus \bigcup_{j=1}^{m_i} N_{ij}$$

as in (1.4). We may reorder the indices so that $\overline{Y}_1, \ldots, \overline{Y}_{n'}$ represents the collection of closures having non-empty interiors. For $i = 1, \ldots, n'$ it follows that the coset $\overline{K}_i$ must have nonempty interior, and hence is open. We may reorder the second indices so that for each i, $N_{i1}, \ldots, N_{im_i'}$ is the collection of cosets which have nonempty interiors in $\overline{K}_i$, and hence are both closed and open. Then

$$\overline{Y}_i = \overline{K}_i \setminus \bigcup_{j=1}^{m_i'} N_{ij}$$

and hence is open. Thus

$$\overline{Y} = \dot{\bigcup_{i=1}^{n'}} \overline{Y}_i.$$

Since $\alpha|_{Y_i}$ is continuous, the affine map $\alpha_i : K_i \to G$ such that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$ is continuous on Y_i, and, by uniformity of the topology, continuous on K_i. By (i) we may extend $\alpha_i : K_i \to G$ to a continuous affine map $\bar{\alpha}_i : \overline{K}_i \to G$. We thus let $\bar{\alpha} : \overline{Y} \to G$ be determined by $\bar{\alpha}|_{\overline{Y}_i} = \bar{\alpha}_i|_{\overline{Y}_i}$ for each $i = 1, \ldots, n'$. $\square$

We note that if C is a closed coset, in the hypotheses of (i) above, then it may not be the case that $\alpha(C)$ is closed in G. Consider, for example, the map $n \mapsto e^{in}$ from the integers $\mathbb{Z}$ to the circle group $\mathbb{T}$. See Corollary 3.11, for a condition which guarantees that the range of α is closed.

Let G and H be locally compact Hausdorff groups, which are usually referred to as simply "locally compact". If $\alpha : Y \subset H \to G$, then we say α is a *continuous piecewise affine map* if

(i) α is piecewise affine, and
(ii) Y is both open and closed in H.

2. On idempotents in Fourier–Stieltjes algebras

The major result of Host [22] states that any idempotent in $\mathrm{B}(G)$, for any locally compact group G, is the indicator function of a set from $\Omega_o(G)$, the ring of sets generated by cosets of open subgroups. While the following proposition borrows from Host's methods, it cannot be directly deduced from [22].

Theorem 2.1. *Let G be a locally compact group and u be an idempotent in* $\mathrm{B}(G)$.

(i) $\|u\| = 1$ *if and only if* $u = 1_C$ *for some open coset C in G.*
(ii) *u is positive definite if and only if* $u = 1_H$ *for some open subgroup H of G.*

Proof. Sufficiency for each of (i) and (ii) above is well known. Let us review it, briefly. If H is an open subgroup of G let G/H denote the discrete space of left cosets of H and $\pi_H : G \to \mathcal{U}\left(\ell^2(G/H)\right)$ the *quasi-left regular representation*, given by $\pi_H(s)\delta_{tH} = \delta_{stH}$. Then $1_H = \left\langle \pi_H(\cdot)\delta_H | \delta_H \right\rangle$, and hence is positive definite with $\|1_H\| = 1_H(e) = 1$. If $C = sH$ for some s in G then

$$1_C = \left\langle \pi_H(\cdot)\delta_H | \delta_{sH} \right\rangle = \left\langle \pi_H(s)^* \pi_H(\cdot)\delta_H | \delta_H \right\rangle = s * 1_H$$

and hence $\|1_C\| = \|s * 1_H\| = \|1_H\| = 1$. Thus it remains to prove necessity for each (i) and (ii).

(i) By [16] there exist a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ and vectors ξ, η in $\mathcal{H}$ such that

$$u = \langle \pi(\cdot)\xi|\eta \rangle \quad \text{and} \quad \|\xi\| = \|\eta\| = \|u\| = 1.$$

Since $\pi(s)$ is a unitary for any s in G, $\|\pi(s)\xi\| = 1$, and hence, by the Cauchy–Schwarz inequality we have that

$$u(s) = \langle \pi(s)\xi|\eta \rangle = 1 \quad \text{if and only if} \quad \pi(s)\xi = \eta.$$

Since u is idempotent, we thus see that

$$u(s) = \langle \pi(s)\xi|\eta \rangle = \begin{cases} 1 & \text{if } \pi(s)\xi = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C = \mathrm{supp}(u) = \{s \in G : u(s) = 1\}$, and then we have that

$$C = \{s \in G : \pi(s)\xi = \eta\} = \{s \in G : \xi = \pi(s)^*\eta\}.$$

If $r, s, t \in C$, then we have that

$$\pi(rs^{-1}t)\xi = \pi(r)\pi(s)^*\pi(t)\xi = \pi(r)\pi(s)^*\eta = \pi(r)\xi = \eta$$

so $rs^{-1}t \in C$. Hence C is a coset by Proposition 1.1. C is open since u is continuous.

(ii) First, since $\|u\| = u(e)$, and u is idempotent, we have either that $u = 0$ or $\|u\| = 1$. The first case is trivial. In the second, we have from (i) that $u = 1_C$ for some open coset C in G. Since $e \in C$, we must have that C itself is a subgroup. $\square$

We note that if G is an abelian group, then for any nontrivial idempotent u of $\mathrm{B}(G)$ we have either that $\|u\| = 1$ or $\|u\| \geqslant \frac{1}{2}(1+\sqrt{2})$ by [47]. We do not know if a similar result holds for nonabelian groups.

The next result is well-known and a proof can be found in [36]. For abelian groups an interesting analytic proof is given in [45, 4.3.3]. We offer another analytic proof which is valid for any infinite group.

Proposition 2.2. *It is impossible to cover a group G with finitely many cosets of subgroups of infinite index.*

Proof. We will consider G to be a discrete group. Let $\mathrm{WAP}(G)$ be the C*-algebra of *weakly almost periodic functions* on G (see [8]). Then $\mathrm{WAP}(G) \supset \mathrm{B}(G)$ and $\mathrm{WAP}(G)$ admits a unique translation invariant mean m. Let Ξ denote the ring of subsets E of G such that $1_E \in \mathrm{WAP}(G)$. Then $\Xi \supset \Omega(G)$ by [22] (or by Theorem 2.1 above). Let $\tilde{m} : \Xi \to [0,1]$ be the finitely additive measure given by $\tilde{m}(E) = m(1_E)$. Then

$\tilde{m}(G) = 1$. Thus if C is a coset of a subgroup of infinite index in G, $\tilde{m}(C) = 0$. Thus for any finite collection $C_1, \ldots, C_n$ of such, $\tilde{m}(\bigcup_{i=1}^{n} C_i) \leqslant \sum_{i=1}^{n} \tilde{m}(C_i) = 0$, whence $C_1, \ldots, C_n$ cannot cover G. $\square$

3. The main result

3.1. Affine maps induce completely bounded homomorphisms

Let us begin with the converse of our main result. Note that if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras with respective preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, we say that a map $\Phi : \mathcal{M}_* \to \mathcal{N}_*$ is *completely positive* if its adjoint, $\Psi^* : \mathcal{N} \to \mathcal{M}$ is completely positive.

Proposition 3.1. *Let G and H be locally compact groups. If $\alpha : Y \subset H \to G$ is a continuous piecewise affine map, then $\Phi_\alpha : \mathrm{A}(G) \to \mathrm{B}(H)$ given by*

$$\Phi_\alpha u(h) = \begin{cases} u(\alpha(h)) & \text{if } h \in Y, \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

is a completely bounded homomorphism. Moreover, Φ_α is completely contractive if α is affine, and completely positive if α is a homomorphism on an open subgroup.

Proof. We will build the proof up in stages, beginning with homomorphisms, then moving to affine maps, and then to piecewise affine maps.

First, suppose that α *is a homomorphism and Y is an open subgroup*. Then by [16] or [2, 2.10], $\Phi_{\alpha|_Y} : \mathrm{A}(G) \to \mathrm{B}(Y)$ is an isometric homomorphism whose adjoint $\Phi^*_{\alpha|_Y} : \mathrm{W}^*(Y) \to \mathrm{VN}(G)$ is the $*$-homomorphism such that $\Phi^*_{\alpha|_Y}(\varpi_Y(y)) = \lambda_G \circ \alpha(y)$ for each y in Y. Since Y is an open subgroup of H, $\mathrm{B}(Y)$ injects contractively into $\mathrm{B}(H)$ via the map which sends v in $\mathrm{B}(Y)$ to the function $\tilde{v}$, which takes the values of v on Y and 0 otherwise. This fact is well known, and follows from [40, Proposition 1.2], for example. Note that $\{\tilde{v} : v \in \mathrm{B}(Y)\} = m_Y \mathrm{B}(H)$ where $m_Y : \mathrm{B}(H) \to \mathrm{B}(H)$ is multiplication by the idempotent 1_Y. Hence Φ_α is the composition of maps

$$\mathrm{A}(G) \xrightarrow{\Phi|_{\alpha_Y}} \mathrm{B}(Y) \xrightarrow{v \mapsto \tilde{v}} m_Y \mathrm{B}(H) \hookrightarrow \mathrm{B}(H).$$

The adjoint of the inclusion map $m_Y \mathrm{B}(H) \hookrightarrow \mathrm{B}(H)$ is m_Y^*. We have that $I = \varpi_H(e) = m_Y^* \varpi_H(e) = m_Y^* I$ and $\left\| m_Y^* \right\|_{\mathrm{cb}} = \|m_Y\|_{\mathrm{cb}} = \|1_Y\| = 1$. Hence, by [15, 5.1.2], m_Y^* is completely positive. The adjoint of the map $v \mapsto \tilde{v}$ is the $*$-isomorphism $\theta : \overline{\mathrm{span}}^{w^*}\{\varpi_H(y) : y \in Y\} \to \mathrm{W}^*(Y)$ such that $\theta(\varpi_H(y)) = \varpi_Y(y)$ for any y in Y. Hence it follows that $\Phi_\alpha^* = m_Y^* \circ \theta \circ \Phi^*_{\alpha|_Y}$ is completely positive and contractive.

Second, suppose that α *is affine and Y is an open coset*. Fix an element h in Y and let $\beta : h^{-1}Y \to \alpha(h)^{-1}\alpha(Y) \subset G$ be the homomorphism from (1.1). Then Φ_α is the composition of maps

$$\mathrm{A}(G) \xrightarrow{u \mapsto \alpha(h)^{-1} * u} \mathrm{A}(G) \xrightarrow{\Phi_\beta} \mathrm{B}(H) \xrightarrow{v \mapsto h * v} \mathrm{B}(H),$$

where for $s * u(t) = u(s^{-1}t)$ and $h * v(h') = v(h^{-1}h')$ for s, t in G and h, h' in H. Then for fixed s in G [respectively, h in H], the translation operator $u \mapsto s * u$ on $\mathrm{A}(G)$ [$v \mapsto h * v$ on $\mathrm{B}(H)$] has adjoint which is multiplication by the unitary $\lambda_G(s)^*$ on $\mathrm{VN}(G)$ [$\varpi_H(h)^*$ on $\mathrm{W}^*(H)$], which is a complete isometry. Hence it follows that Φ_α is a complete contraction.

Finally, we suppose that α *is a continuous piecewise affine map and* $Y \in \Omega_o(H)$. Then, by Lemma 1.3 (ii), we can write

$$Y = \dot{\bigcup_{i=1}^{n}} Y_i \quad \text{where each} \quad Y_i = L_i \setminus \bigcup_{j=1}^{m_i} M_{ij}$$

and each L_i and M_{ij} is an open coset. Moreover, for each i there is a continuous affine map $\alpha_i : L_i \to G$ such that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$. Let $\ell^1(n)$ be the n-dimensional ℓ^1-space with contractive summing basis $\{\delta_i : i = 1, \ldots, n\}$. Let $A : \mathrm{A}(G) \to \mathrm{B}(H)\widehat{\otimes}\ell^1(n)$ be given for u in $\mathrm{A}(G)$ by the weighted amplification

$$Au = \sum_{i=1}^{n} \Phi_{\alpha_i} u \otimes \delta_i.$$

Then A is completely bounded with $\|A\|_{\mathrm{cb}} \leqslant n$. Letting $\{\chi_i : i = 1, \ldots, n\}$ in $\ell^\infty(n)$ be the dual basis to $\{\delta_i : i = 1, \ldots, n\}$, we let the weighted diagonal multiplication map $M : \mathrm{B}(H)\widehat{\otimes}\ell^1(n) \to \mathrm{B}(H)\widehat{\otimes}\ell^1(n)$ be given by

$$M = \sum_{j=1}^{n} m_{1_{Y_i}} \otimes \langle \cdot, \chi_j \rangle \delta_j \quad \text{so} \quad M \sum_{i=1}^{n} v_i \otimes \delta_i = \sum_{i=1}^{n} 1_{Y_i} v_i \otimes \delta_i.$$

Hence M is completely bounded with $\|M\|_{\mathrm{cb}} \leqslant \sum_{i=1}^{n} \left\|1_{Y_i}\right\|$. Let tr be the linear functional on $\ell^1(n)$ implemented by $\chi_1 + \cdots + \chi_n$, so that $\mathrm{id}_{\mathrm{B}(H)} \otimes \mathrm{tr} : \mathrm{B}(H)\widehat{\otimes}\ell^1(n) \to \mathrm{B}(G)$ is given by

$$\mathrm{id}_{\mathrm{B}(H)} \otimes \mathrm{tr}\left(\sum_{i=1}^{n} v_i \otimes \delta_i\right) = v_1 + \cdots + v_n$$

and $\left\|\mathrm{id}_{\mathrm{B}(H)} \otimes \mathrm{tr}\right\|_{\mathrm{cb}} \leqslant \|\mathrm{tr}\| = 1$. Then we have that Φ_α is the composition of maps

$$\mathrm{A}(G) \xrightarrow{A} \mathrm{B}(H)\widehat{\otimes}\ell^1(n) \xrightarrow{M} \mathrm{B}(H)\widehat{\otimes}\ell^1(n) \xrightarrow{\mathrm{id}_{\mathrm{B}(H)}\otimes \mathrm{tr}} \mathrm{B}(H)$$

and is thus completely bounded. $\square$

Corollary 3.2. *If* $\alpha : Y \subset H \to G$ *is a continuous piecewise affine map then the map* $\Psi_\alpha : \mathrm{B}(G) \to \mathrm{B}(H)$ *given similarly as* Φ_α *in* (3.1) *is a completely bounded homomorphism. Moreover,* Ψ_α *is completely contractive if* α *is affine, and* Ψ_α *is completely positive if* α *is a homomorphism on an open subgroup. If* G *is amenable, then* $\|\Psi_\alpha\|_{\mathrm{cb}} = \|\Phi_\alpha\|_{\mathrm{cb}}$.

Proof. That Ψ_α has the properties claimed can be seen by making trivial modifications to the proof of the proposition above: replace λ_G by ϖ_G, and $\mathrm{VN}(G)$ by $\mathrm{W}^*(G)$.

If G is amenable, there is a norm 1 bounded approximate identity $(u_i)_i$ for $\mathrm{A}(G)$ by [33]. If $v \in \mathrm{B}(G)$ with $\|v\| = 1$, then $(vu_i)_i$ is a contractive net which converges to v in the multiplier topology, and hence uniformly on compact subsets of G. Hence $\Phi_\alpha(vu_i) \to \Psi_\alpha v$ uniformly on compact subsets of G. Since $(\Phi_\alpha(vu_i))_i$ is bounded by $\|\Phi_\alpha\|$ it follows that it converges to $\Psi_\alpha v$ in the weak* topology, and hence $\|\Psi_\alpha v\| \leqslant \|\Phi_\alpha\|$. Thus $\|\Psi_\alpha\| \leqslant \|\Phi_\alpha\|$.

We can obtain that $\|\Psi_\alpha\|_{\mathrm{cb}} \leqslant \|\Phi_\alpha\|_{\mathrm{cb}}$, similarly. Indeed, for any $n \times n$ matrix $[v_{kl}]$ over $\mathrm{B}(G)$, the amplified multiplication maps $[v_{kl}] \mapsto [v_{kl}u_i]$ are contractive. We get for any k, l that $\Phi_\alpha(v_{kl}u_i) \to \Psi_\alpha v_{kl}$ weak*, and hence $\left[\Phi_\alpha(v_{kl}u_i)\right] \to [\Psi_\alpha v_{kl}]$ weak*, from which it follows that $\left\|\Psi_\alpha^{(n)}\right\| \leqslant \left\|\Phi_\alpha^{(n)}\right\|$. Hence $\|\Psi_\alpha\|_{\mathrm{cb}} \leqslant \|\Phi_\alpha\|_{\mathrm{cb}}$. Since $\Phi_\alpha = \Psi_\alpha|_{\mathrm{A}(G)}$, we obtain the converse inequality, $\|\Phi_\alpha\|_{\mathrm{cb}} \leqslant \|\Psi_\alpha\|_{\mathrm{cb}}$. $\square$

Remark 3.3. If $G = \mathrm{F}_2$, the free group on two generators, then it may not be the case that $\|\Phi_\alpha\|_{\mathrm{cb}} = \|\Psi_\alpha\|_{\mathrm{cb}}$ for a piecewise affine map $\alpha : Y \subset \mathrm{F}_2 \to \mathrm{F}_2$ which is not itself affine. To see this let a and b be the generators for F_2. For any $n = 1, 2, \ldots$ the set $E_n = \{a^k b^k : k = 1, \ldots, n\}$ is a *free set* in the sense of [31]. Let $\alpha_n : E_n \to \mathrm{F}_2$ be the inclusion map, which is affine as E_n is finite. Then

$$\sup_n \left\|\Phi_{\alpha_n}\right\|_{\mathrm{cb}} \leqslant 2 \quad \text{while} \quad \sup_n \left\|\Psi_{\alpha_n}\right\|_{\mathrm{cb}} = \infty.$$

Indeed, first observe that for any u in $\mathrm{A}(\mathrm{F}_2)$, $\Phi_{\alpha_n} u = 1_{E_n} u$. Then, by [32, Bem. (13)], $1_{E_n} \in \mathrm{B}_2(G)$, the algebra of Herz–Schur multipliers, and is of norm no greater than 2. Then it follows [6] (or [49]) that $u \mapsto 1_{E_n} u$ is completely bounded on $\mathrm{A}(G)$. (We note that an alternative line of proof can be followed, using [7, Section 2] or [39, Section 2], by which we may see that $\sup_n \left\|\Phi_{\alpha_n}\right\|_{\mathrm{cb}} \leqslant 25$. These proofs make use of the space $\mathrm{T}_1(G)$ of Littlewood functions.) On the other hand, note that $\Psi_{\alpha_n} 1_{\mathrm{F}_2} = 1_{E_n}$. If the sequence $\{1_{E_n} : n = 1, 2, \ldots\}$ were bounded in $\mathrm{B}(G)$, then its pointwise limit 1_E, where $E = \{a^n b^n : n = 1, 2, \ldots\}$, would be in $\mathrm{B}(G)$. However, this would contradict [39, Lemma 2.7]. $\square$

3.2. Some lemmas

We will need two lemmas to proceed to our main result.

The first one gives a construction of a nice *bounded approximate diagonal* for $\mathrm{A}(G \times G)$. This construction is a simplified version of the one in the seminal paper [42] (see also [46, Section 7.4]).

Lemma 3.4. *Let G be an amenable locally compact group. Then there exists in $\mathrm{A}(G \times G)$ a net $(w_l)_l$ such that*

(i) *each w_l is positive definite and of norm* 1, *and*
(ii) *for each $(s, t) \in G \times G$*

$$\lim_l w_l(s,t) = \begin{cases} 1 & \textit{if } s=t, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First we will obtain a *positive approximate indicator* for the diagonal subgroup as in [1]. Since G is amenable, there is a positive contractive quasi-central approximate identity $(e_i)_i$ for $\mathrm{L}^1(G)$ by [35]. Moreover, by [50], $(e_i)_i$ can be chosen to have compact supports tending to the identity in G. We let $\xi_i = e_i^{1/2}$ for each i and define for (s,t) in $G \times G$

$$u_i(s,t) = \left\langle \lambda_G(s)\rho_G(t)\xi_i | \xi_i \right\rangle,$$

where $\rho_G : G \to \mathcal{U}(\mathrm{L}^2(G))$ is the right regular representation of G. Then each u_i is a positive definite function with $\|u_i\| = u(e,e) = 1$. It follows, using computations exactly as in [1, Theorom 2.4], that

$$\lim_i u_i(s,t) = \begin{cases} 1 & \textit{if } s=t, \\ 0 & \textit{otherwise.} \end{cases}$$

(We note that if G is a small invariant neighbourhood group, there is a very simple construction of a net, having the properties of $(u_i)_i$, in [48].)

Since G is amenable, $G \times G$ is amenable and hence there exists a net $(v_j)_j$ of $\mathrm{A}(G \times G)$ comprised of norm 1 positive definite functions which converges uniformly on compact sets to $1_{G\times G}$. To see this, by [23, Propositon 6.1] (also see [37, 4.21]) there exists a bounded net of elements $(v'_j)_j$ from $\mathrm{A}(G \times G)^+$ which converges uniformly on compact sets to $1_{G\times G}$, and normalise by taking $v_j = \frac{1}{v'_j(e,e)} v'_j$. (Note that by [19], this net is a bounded approximate identity for $\mathrm{A}(G \times G)$. This gives an alternative proof to [33].) Then the product net with elements

$$w_l = v_j u_i$$

is the desired bounded approximate diagonal. □

The next lemma is a straightforward technical result on complete positivity. Though it is surely well-known, we include a proof for convenience of the reader.

Lemma 3.5. *Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{N}$ be von Neumann algebras and let $\Phi : (\mathcal{M}_1)_* \to (\mathcal{M}_2)_*$ be a completely positive map. Then $\Phi \otimes \mathrm{id}_{\mathcal{N}_*} : (\mathcal{M}_1)_* \widehat{\otimes} \mathcal{N}_* \to (\mathcal{M}_2)_* \widehat{\otimes} \mathcal{N}_*$ is also completely positive.*

Proof. By definition we have that $\Phi^* : \mathcal{M}_2 \to \mathcal{M}_1$ is completely positive. Hence the normal map $\Phi^* \otimes \mathrm{id}_{\mathcal{N}} : \mathcal{M}_2 \overline{\otimes} \mathcal{N} \to \mathcal{M}_1 \overline{\otimes} \mathcal{N}$ makes sense. Moreover, it is simple to verify, using elementary tensors in $\mathcal{M}_2 \overline{\otimes} \mathcal{N}$, that $\Phi^* \otimes \mathrm{id}_N = (\Phi \otimes \mathrm{id}_{\mathcal{N}_*})^*$, where we identify $\mathcal{M}_i \overline{\otimes} \mathcal{N} \cong \left((\mathcal{M}_i)_* \widehat{\otimes} \mathcal{N}_*\right)^*$ for $i = 1, 2$. Thus we need only to see that $\Phi^* \otimes \mathrm{id}_{\mathcal{N}}$ is completely positive.

First, let $b = \Phi^*(I)^{1/2}$. By [15, 5.1.6] there is a unital completely positive map $\theta : \mathcal{M}_2 \to \mathcal{M}_1$ such that $\Phi^* = b\theta(\cdot)b$. Hence

$$\Phi^* \otimes \mathrm{id}_{\mathcal{N}} = (b \otimes I)\theta \otimes \mathrm{id}_{\mathcal{N}}(\cdot)(b \otimes I).$$

Now $\theta \otimes \mathrm{id}_{\mathcal{N}}$ is completely positive since $\theta \otimes \mathrm{id}_{\mathcal{N}}(I \otimes I) = I \otimes I$ while $\|\theta \otimes \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leqslant \|\theta\|_{\mathrm{cb}} = \|\theta(I)\| = 1$, and we employ [15, 5.1.2]. Thus $\Phi^* \otimes \mathrm{id}_{\mathcal{N}}$ is a composition of completely positive maps whence it is completely positive. $\square$

Remark 3.6. If G is any locally compact group, the map $\imath : \mathrm{A}(G) \to \mathrm{A}(G)$ given by $\imath u(s) = u(s^{-1})$ (s in G) is positive—it takes positive definite elements to positive definite elements—contractive and a homomorphism. However, by [17, Proposition 1.5], $\imath$ is completely bounded only if G has an abelian subgroup of finite index. Thus if G does not admit an abelian subgroup of finite index, then $\mathrm{A}(G)$ admits a bounded homomorphism which is not completely bounded.

3.3. The main result

Theorem 3.7. *Let G and H be locally compact groups with G amenable, and let $\Phi : \mathrm{A}(G) \to \mathrm{B}(H)$ be a completely bounded homomorphism. Then there is a continuous piecewise affine map $\alpha : Y \subset H \to G$ such that for each h* in *H*

$$\Phi u(h) = \begin{cases} u(\alpha(h)) & \text{if } h \in Y, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, α is affine if Φ is completely contractive, and α is a homomorphism defined on an open subgroup if Φ is completely positive.

Proof. First, it will be convenient for us to let $G_\infty = G\dot{\cup}\{\infty\}$ be either the one-point compactification of G if G is not compact, or the topological coproduct if G is compact. Each element u of $\mathrm{A}(G)$ extends to a continuous function on G_∞ by setting $u(\infty) = 0$. Now, as observed in [25], since G is the Gel'fand spectrum of $\mathrm{A}(G)$, for each h in H there is an $\alpha(h)$ in G_∞ such that

$$\Phi u(h) = u(\alpha(h)). \tag{3.2}$$

The map $\alpha : H \to G_\infty$ is continuous. Indeed, suppose not. Then there is an h_0 in H, a neighbourhood U of $\alpha(h_0)$ and a net $(h_i)_i$ in H such that $h_i \to h_0$ but $\alpha(h_i) \notin U$ for any i. If $\alpha(h_0) \in G$, find a u in $\mathrm{A}(G)$ such that $\mathrm{supp}(u) \subset U$ and $u(\alpha(h_0)) = 1$. Then

$$1 = u(\alpha(h_0)) = \Phi u(h_0) = \lim_i \Phi u(h_i) = \lim_i u(\alpha(h_i)) = 0$$

which is absurd. If $\alpha(h_0) = \infty$, then $K = G_\infty \setminus U$ is compact. Find u in $\mathrm{A}(G)$ such that $u|_K = 1$ and $\mathrm{supp}(u)$ is compact, and we reach a similar contradiction as above. We thus have that

$$\alpha : H \to G_\infty \text{ is continuous and } Y = \alpha^{-1}(G) \text{ is open in } H. \tag{3.3}$$

Note that

$$Y = \{h \in H : \text{ there exists a } u \text{ in } \mathrm{A}(G) \text{ such that } \Phi u(h) \neq 0\}. \tag{3.4}$$

We will now suppose that Φ is completely bounded, and address the cases that it is completely contractive or completely positive later. Then the map $\Phi \otimes \mathrm{id}_{\mathrm{A}(G)} : \mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G) \to \mathrm{B}(H)\widehat{\otimes}\mathrm{A}(G)$ is completely bounded with $\left\|\Phi \otimes \mathrm{id}_{\mathrm{A}(G)}\right\|_{\mathrm{cb}} \leqslant \|\Phi\|_{\mathrm{cb}}$. We can identify $\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G)$ completely isometrically with $\mathrm{A}(G \times G)$ via

$$u \otimes v \mapsto u \times v, \tag{3.5}$$

where $u \times v(s,t) = u(s)v(t)$ for (s,t) in $G \times G$. Indeed, the adjoint of this map is the isomorphism $\mathrm{VN}(G \times G) \cong \mathrm{VN}(G)\overline{\otimes}\mathrm{VN}(G)$, which is spatially implemented. We can also identify $\mathrm{B}(H)\widehat{\otimes}\mathrm{A}(G)$ completely isometrically as a subspace of $\mathrm{B}(H \times G)$ via a map like (3.5), where $u \times v(h,t) = u(h)v(t)$ for (h,t) in $H \times G$. Indeed, the adjoint of such a map would be the canonical homomorphism from $\mathrm{W}^*(H \times G)$ to $\mathrm{W}^*(H)\overline{\otimes}\mathrm{VN}(G)$, which extends $\varpi_{H\times G}(h,t) \mapsto \varpi_H(h) \otimes \lambda_G(t)$. This is a complete quotient map. Furthermore, the inclusion $\mathrm{B}(H \times G) \hookrightarrow \mathrm{B}(H_d \times G_d)$ is a completely positive isometry. Thus the composition of maps

$$\mathrm{A}(G \times G) \cong \mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G) \xrightarrow{\Phi\otimes\mathrm{id}_{\mathrm{A}(G)}} \mathrm{B}(H)\widehat{\otimes}\mathrm{A}(G) \hookrightarrow \mathrm{B}(H \times G) \hookrightarrow \mathrm{B}(H_d \times G_d)$$

forms a map J_Φ, such that $\|J_\Phi\| \leqslant \|J_\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|_{\mathrm{cb}}$. On any elementary product of functions $u \times v$ in $\mathrm{A}(G \times G)$ and for any (h,t) in $H \times G$ we have

$$J_\Phi u \times v(h,t) = (\Phi u) \times v(h,t) = u(\alpha(h))v(t),$$

where we recall that $u(\infty) = 0$. We can extend this to linear combinations of elementary products and hence, by continuity, to any element w in $\mathrm{A}(G \times G)$, so we obtain for any (h,t) in $H \times G$

$$J_\Phi w(h,t) = \begin{cases} w(\alpha(h),t) & \text{if } h \in Y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Now, let $(w_l)_l$ be the net from Lemma 3.4, and consider the net $(J_\Phi w_l)_l$ in $\mathrm{B}(H_d \times G_d)$. We have for each (h,t) in $H \times G$ that

$$\lim_l J_\Phi w_l(h,t) = \begin{cases} \lim\limits_l w_l(\alpha(h),t) & \text{if } h \in Y, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} 1 & \text{if } h \in Y \text{ and } \alpha(h) = t, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $(J_\Phi w_l)_l$ converges pointwise to 1_{Γ_α}, where Γ_α is the graph of α, as in (1.3). Since $(J_\Phi w_l)_l$ is bounded, we conclude that $1_{\Gamma_\alpha} \in \mathrm{B}(H_d \times G_d)$, which implies by [22] that $\Gamma_\alpha \in \Omega(H \times G)$. This implies, by Lemma 1.2 (iii), that α is piecewise affine. Since (3.3) holds, we can apply Lemma 1.3 to see that α extends to a continuous piecewise affine map $\bar{\alpha} : \overline{Y} \to G$. However, it follows from this that Y is closed. Indeed if $h_0 \in \overline{Y}$, there is an element u of $\mathrm{A}(G)$ such that $u(\bar{\alpha}(h_0)) \neq 0$. Then for any net $(h_i)_i$ in Y, converging to h_0 we have

$$\Phi u(h_0) = \lim_i \Phi u(h_i) = \lim_i u(\alpha(h_i)) = u(\bar{\alpha}(h_0)) \neq 0$$

and it follows from (3.4) that $h_0 \in Y$. Hence $\bar{\alpha} = \alpha$ and α is itself a continuous piecewise affine map.

Now suppose that Φ is completely contractive. It then follows that the net $(J_\Phi w_l)_l$ is completely contractive, and hence $1_{\Gamma_\alpha} \in \mathrm{B}(H_d \times G_d)$ with $\|1_{\Gamma_\alpha}\| \leqslant 1$. By Theorem 2.1 we conclude that Γ_α is a coset, which in turn forces α to be an affine map by Lemma 1.2(ii).

Finally, if Φ is completely positive, then $\Phi \otimes \mathrm{id}_{\mathrm{A}(G)}$ is a completely positive map by Lemma 3.5, and hence J_Φ is completely positive. Thus the net $(J_\Phi w_l)_l$ is a net of norm 1 positive definite functions. It follows then that 1_{Γ_α} is positive definite, which, by Theorem 2.1 forces Γ_α to be a group. Hence α is a homomorphism on some open subgroup by Lemma 1.2(i). □

The theorem stated above appears to be the best result possible. We have already seen in Remark 3.6 that there exist bounded homomorphisms of Fourier algebras which are not completely bounded. Moreover, we can show for a large class of nonamenable groups, namely those which contain a closed noncommutative free group, that Theorem 3.7 fails. We note that a *nonamenable* group G contains a discrete copy of a noncommutative free group if G is almost connected (see [37, 3.8]), or if G is linear with the discrete topology (see [51] or, for some special cases, see [11]). We are indebted to B.E. Forrest for indicating the following example to us.

Proposition 3.8. *If G is a nonamenable group which contains a discrete copy of the free group F_2, then there exists a completely bounded homomorphism $\Phi : \mathrm{A}(G) \to \mathrm{A}(\mathrm{F}_2)$ which is not of the form Φ_α as in (3.1) for a piecewise affine map α.*

Proof. The restriction map $u \mapsto u|_{F_2}$ from $A(G)$ to $A(F_2)$ is a contractive quotient map by [20]. Moreover, the adjoint of this restriction map is the $*$-homomorphism from $VN(F_2)$ to $VN(G)$ which extends $\lambda_{F_2}(s) \mapsto \lambda_G(s)$, so it is completely positive. Now let a and b be the generators for F_2 and $E = \{a^n b^n : n = 1, 2, \ldots\}$, so E is a *free set*. As in Remark 3.3, we see that the map $v \mapsto 1_E v$ is a completely bounded map on $A(F_2)$. We let Φ be the composition of maps

$$A(G) \xrightarrow{u \mapsto u|_{F_2}} A(F_2) \xrightarrow{v \mapsto 1_E v} A(F_2).$$

Then $\Phi = \Phi_\alpha$ where $\alpha : E \subset F_2 \to G$ is the inclusion map. However $1_E \notin B(F_2)$ by [39, Lemma 2.7]. Hence, by [22], $E \notin \Omega(F_2)$, so α is not piecewise affine. $\square$

3.4. Some consequences

In what follows we will always let G and H be locally compact groups with G amenable.

The first one is a direct application of Theorem 3.7 and Corollary 3.2.

Corollary 3.9. *Any completely bounded homomorphism* $\Phi : A(G) \to B(H)$ *extends to a completely bounded homomorphism* $\Psi : B(G) \to B(H)$ *with* $\|\Psi\|_{cb} = \|\Phi\|_{cb}$.

The next corollary follows from the fact that a connected group admits no continuous piecewise affine maps which are not themselves affine. See Lemma 1.3.

Corollary 3.10. *If H is connected, then any completely bounded homomorphism* $\Phi : A(G) \to B(H)$ *is of form* (3.1) *for an affine map* α. *In particular*, Φ *is completely contractive*.

Now we will consider homomorphisms between $A(G)$ and $A(H)$. If Y and X are locally compact spaces, we say that a map $\alpha : Y \to X$ is *proper* if $\alpha^{-1}(K)$ is compact in Y for every compact subset K of X. For abelian groups the result below can be found in [10].

Corollary 3.11. *A map* $\Phi : A(G) \to A(H)$ *is a completely bounded homomorphism if and only if* $\Phi = \Phi_\alpha$ *as in* (3.1) *where* α *is a continuous piecewise affine map which is proper*. *In particular*, α *is a closed map*.

Proof. As in the proof of Theorem 3.7, let us employ the convention that $\alpha : H \to G_\infty$.

Since $\Phi : A(G) \to A(H) \subset B(H)$, the existence of a piecewise affine map α such that $\Phi = \Phi_\alpha$ follows from our main result. Now if there were a compact subset K of G such that $\alpha^{-1}(K)$ is not compact, then for any $u \in A(G)$ such that $u|_K = 1$, we would have that $\Phi u = u \circ \alpha$ would not vanish at infinity, and hence would not be in $A(H)$. Hence necessity is proven.

To obtain the converse, let $A_c(G)$ denote the subspace of $B(G)$ consisting of functions of compact support. Then $A_c(G)$ is a dense subspace of $A(G)$. Similarly define $A_c(H)$.

If $\Phi = \Phi_\alpha$ for a continuous piecewise affine map α, then $\Phi : \mathrm{A}(G) \to \mathrm{B}(H)$ is a completely bounded homomorphism by Proposition 3.1. Since for any u in $\mathrm{A}_c(G)$, $\mathrm{supp}(u \circ \alpha) = \alpha^{-1}(\mathrm{supp}(u))$, we have that $\Phi(\mathrm{A}_c(G)) \subset \mathrm{A}_c(H)$, and hence $\Phi(\mathrm{A}(G)) \subset \mathrm{A}(H)$.

Now we shall see that α is closed. Let E be a nonempty closed subset of H, $s_0 \in \overline{\alpha(E)}$ and $\mathcal{U}$ be a neighbourhood basis at s_0 consisting of relatively compact sets. Then $\{\alpha^{-1}(\overline{U}) \cap E\}_{U \in \mathcal{U}}$ is a family of compact subsets of H having the finite intersection property, whence $L = \bigcap_{U \in \mathcal{U}} \alpha^{-1}(\overline{U}) \cap E \neq \emptyset$. If $h_0 \in L$, then $\alpha(h_0) = s_0$, since $\alpha(h_0)$ is contained in $\overline{U}$ for each U in $\mathcal{U}$. □

We can thus obtain an analogue of Walter's Theorem [52].

Corollary 3.12. *Let $\Phi : \mathrm{A}(G) \to \mathrm{A}(H)$ be an completely contractive isomorphism. Then there exists an element s_0 in G and a topological group isomorphism $\beta : H \to G$ such that*

$$\Phi u(h) = u(s_0 \beta(h))$$

for each h in H. Moreover, if Φ is completely positive then $s_0 = e$.

Proof. Let $\alpha : Y \subset H \to G$ be the affine map whose existence is guaranteed by Corollary 3.11. Since Φ is surjective and $\mathrm{A}(H)$ is a point-separating regular algebra on H, $Y = H$ and α is injective. Since Φ is injective, $\alpha(H)$ is dense in G. Hence, as α is closed, we obtain that $\alpha(H) = G$. Thus α is bijective, and hence open, so it is a homeomorphism. Let $s_0 = \alpha(e_H)$ and $\beta = s_0^{-1}\alpha(\cdot)$, so β is a homomorphism as in (1.1).

If Φ is completely positive then α is a homomorphism and hence $\beta = \alpha$. □

It is known, to us through Z.-J. Ruan, that the result above obtains in the case that Φ is a complete isometry, without assuming that G is amenable. The proof follows the one given by Walter.

Let us close with a characterisation of the range of a completely bounded homomorphism of Fourier algebras, which is due to Kepert [29] in the case that G and H are abelian, and due to the first author [25] in the case that either G is discrete and amenable or G is abelian, with general H. Below, as with all results in this section, we assume the G is amenable and H is a general locally compact group.

Theorem 3.13. *If $\Phi : \mathrm{A}(G) \to \mathrm{A}(H)$ is a completely bounded homomorphism with adjoint map $\Phi^* : \mathrm{VN}(H) \to \mathrm{VN}(G)$ then*

$$\Phi(\mathrm{A}(G)) = \left\{ u \in \mathrm{A}(H) : \begin{array}{c} u(h) = 0 \ \textit{if} \ \Phi^*(\lambda_H(h)) = 0, \ \textit{and} \\ u(h) = u(h') \ \textit{if} \ \Phi^*(\lambda_H(h)) = \Phi^*(\lambda_H(h')) \end{array} \right\}.$$

That $\Phi(\mathrm{A}(G))$ is contained in the set of the latter description is clear. Note that, in fact, $\Phi^*(\lambda_H(h)) = \lambda_G(\alpha(h))$ if $\Phi^*(\lambda_H(h)) \neq 0$, where α is the proper completely affine map promised by Corollary 3.11. The converse inclusion is not trivial, and its proof is in [25, Section 5]. To adapt that proof to be sufficiently general we need only note Herz's extension theorem [20]—if F is a closed subgroup of G then for each u in $\mathrm{A}(F)$ there is an element $\tilde{u}$ in $\mathrm{A}(G)$ such that $\tilde{u}|_F = u$ and $\|\tilde{u}\| = \|u\|$; and we must use Corollary 3.11 in place of the main theorem of [25].

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