Equation of energy conversion

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Abstract. We have obtained the equation of energy conversion, an alternative equation to the equations of motion, from the total time derivative of energy. Using this equation, we solved some familiar cases, including gravitational rotation, coupled pendula and Lorenz force. In these familiar cases, we obtained the same results as the Euler-Lagrange equations, except for the Lorenz force in the time-dependent situation. We also obtained Jefimenko equations for electric and magnetic field in time-dependent case and we showed that these can not be obtained from Euler-Lagrange equations. Jefimonko equations describe the electromagnetic interaction in the time-dependent situation and the general case for electromagnetic interaction can be obtained by using these equations. So obtaining them in a correct way is important for the generalization and this work can be a step towards the generalization.

1. Introduction

The equations of motion are required to analyze a physical system, and can be obtained three different ways. The first method involves writing the equations directly from Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$. The second way is to derive them from the Lagrangian. The third way is to obtain them from the Hamiltonian.

To obtain the equations of motion from the Lagrangian, we need the Euler-Lagrange equations. We can derive these equations in two different ways. In the first method, we start with Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$ and use D'Alembert's principle to write an equation corresponding to the derivatives of kinetic energy T and potential energy U. To obtain the Euler-Lagrange equations, we need the definition of the Lagrangian, $L = T - U$. At this point, there is a problem concerning potential energy which should be kept in mind. If potential energy depends on velocity, then we cannot obtain the Euler-Lagrange equations from this Lagrangian in general. We only know how to get them from Lagrangian in the time-independent case for the electromagnetic fields. Therefore, in velocity-dependent systems, the Lagrangian method can be problematic and in the following sections we will see that it fails if we consider linear velocity-dependent potential in the time-dependent situation. The second method of deriving the Euler-Lagrange equations utilizes the defined Lagrangian to obtain an action integral. Then we take the variation of this action integral. Finally, we obtain the Euler-Lagrange equations using Hamilton's principle [1].

The third method for obtaining the equations of motion requires the Hamiltonian and also uses Hamilton's equations. Firstly, we define momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and use Legendre transformations to obtain the Hamiltonian. Then we write the differentials for the Hamiltonian and Lagrangian. Lastly, we can obtain Hamilton's equations by comparing these two differentials. Therefore, Hamilton's equations are equivalent to the Euler-Lagrange equations in momentum space[1].

In this work, we will use a different approach to obtain the equations of motion. This method employs the result of the total time derivative of energy. Taking the total time derivative of the energy is common, but it is not used to obtain the equations of the motion. In this work, we will use it as such. The energy is one of the most basic quantity in physics and to obtain a relation derived from its variation can give us a framework that can be consistent with the conservation of energy, the most basic physical law. Firstly, we will derive an equation from the total time derivative of energy, and from this equation, we will then derive the equations of motion. We will also use this equation to analyze some well-known cases.

As mentioned previously, an issue arises in the velocity-dependent potential energy case. This situation is problematic in Lagrangian formalism because this case is excluded in the derivation of the Euler-Lagrange equations from D'Alembert's principle [1, 6, 7, 8]. However it is still used for velocity-dependent potentials, such as electromagnetism. In this work, we will also analyze electromagnetism using our technique.

2. Equation of energy conversion

Now we will consider energy as a function of position, velocity and time $E(q_i, \dot{q}_i, t)$. Then we can write the total differential of energy as

$$dE = \frac{\partial E}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial E}{\partial q_i} dq_i + \frac{\partial E}{\partial t} dt. \tag{1}$$

We use Einstein's summation convention for this equation as well as equations that follow, i.e. repeated indices are summed up. Hereafter, we will consider that position and velocity are functions of time only. Then we can write the total time derivative of energy as

$$\frac{dE}{dt} = \frac{\partial E}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial E}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial E}{\partial t}. \tag{2}$$

For energy, it can be shown that $\frac{dE}{dt} = \frac{\partial E}{\partial t}$ [2]. This relation can also be obtained using Poision brackets. Then from Eq.(2), we can write

$$\frac{\partial E}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial E}{\partial q_i}\frac{dq_i}{dt} = 0. \tag{3}$$

This equation describes how energy changes its form within a system. So hereafter, it is called the equation of energy conversion. In other words, Eq.(3) tells us that energy can be converted in a system from one form to another, and that any change in one form of energy can result in a change to the other forms of energy. So this equation is consistent with energy conservation and tells us that energy does not appear or disappear; it only changes form. The responsible agent for this conversion of energy is force, which is given by the equation of motion.

In the following sections, we will see that the equation of energy conversion utilizes information from the equations of motion. With the help of some examples, we will analyze how the equation of energy conversion defines the motion of particles, or in general, the evolution of a system.

3. Comparison of the equation of energy conversion with the Euler-Lagrange equations and examples

3.1. Comparison with the Euler-Lagrange equations: Velocity independent potential in the time-independent case

The Lagrangian is defined as

$$L = T - U \tag{4}$$

where T is the kinetic energy and U is the potential energy. The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{5}$$

With this form of the equations, it is hard to show the equivalence of the Euler-Lagrange equations to the ones that are obtained from the equation of energy conversion. There are some differences between them and we will consider these in the following sections. For the sake of simplicity, we will consider the case where $T(\dot{\vec{q}}) = \frac{1}{2}m\dot{q}_i^2$ and $U(\vec{q})$; this is the most common case in classical mechanics. Here, the Euler-Lagrange equations are

$$m\ddot{q}_i + \frac{\partial U}{\partial q_i} = 0. \tag{6}$$

Now let us obtain equation of energy conversion in this case. From energy $E = T + U$ and using Eq.(3), we can obtain the equation of energy conversion as

$$\left(m\ddot{q}_i + \frac{\partial U}{\partial q_i}\right)\dot{q}_i = 0. \tag{7}$$

In this equation, the term in the parenthesis is equal to zero if this multiplication is linearly-independent. Hereafter we will consider this linearly-independent case. This is the same result as the Euler-Lagrange equations, i.e. Eq.(6). Hence, they are equivalent in the considered case.

3.2. Example 1: Gravitational rotation

For gravitational rotation, we can write energy as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r}. \tag{8}$$

This equation is independent of θ. Then the corresponding conserved momentum can be written as $p_\theta = \frac{\partial E}{\partial \dot{\theta}} = mr^2\dot{\theta}$. This is similar to $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ and they are equivalent in this example. After stating conserved quantity, we can rewrite Eq.(8) as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}. \tag{9}$$

Now, we have obtained a 1-dimensional equation, since the only variables are r and $\dot{r}$. So we can write the equation of energy conversion as

$$m\dot{r}\ddot{r} - \frac{p_\theta^2}{mr^3}\dot{r} + \frac{GMm}{r^2}\dot{r} = 0. \tag{10}$$

Then we have $m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{GMm}{r^2}$. This is the same equation as the one obtained from the Euler-Lagrange equations[1].

The solution of the gravitational rotation from the equation of energy conversion is simpler than obtaining the solution from Lagrangian formalism. Since we used conservation of energy and equation of energy conversion, two statement, instead of using Lagrangian, Euler-Lagrange equations and conservation of energy, three statement.

3.3. Example 2: Coupled pendula

The previous example was reduced to 1-dimensional form using the conserved quantity. Now we will solve a 2-dimensional example: coupled pendula. Consider two simple twin pendula of mass m and length l which are coupled to each other with a massless spring and have spring constant k. If we consider small oscillations, we can write energy as

$$E = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + \frac{mgl}{2}(\theta_1^2 + \theta_2^2) + \frac{kl^2}{2}(\theta_1 - \theta_2)^2. \tag{11}$$

Using the relations $\beta_1 = \frac{\theta_1+\theta_2}{2}$ and $\beta_2 = \frac{\theta_1-\theta_2}{2}$, we can rewrite Eq.(11) as

$$E = \frac{1}{2}ml^2(\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{mgl}{2}(\beta_1^2 + \beta_2^2) + kl^2\beta_2^2. \tag{12}$$

In this case, we can write the equation of energy conversion as

$$\dot{\beta}_1(ml^2\ddot{\beta}_1 + mgl\beta_1) + \dot{\beta}_2(ml^2\ddot{\beta}_2 + mgl\beta_2 + 2kl^2\beta_2) = 0. \tag{13}$$

The terms in the parenthesis are equal to zero for linearly-independent velocities. So we obtain

$$\begin{aligned} \ddot{\beta}_1 &= -\tfrac{g}{l}\beta_1, \\ \ddot{\beta}_2 &= -\left(\tfrac{g}{l} + \tfrac{2k}{m}\right)\beta_1. \end{aligned} \tag{14}$$

These are the same equations as the ones obtained from the Euler-Lagrange equations [3].

3.4. Comparison with the Euler-Lagrange equations: Linear velocity-dependent potential in the time-independent case

Now we will consider linear velocity dependence in potentials for the time-independent case. We will separate potential in two parts as $U(\vec{q}, \dot{\vec{q}}) = U_1(\vec{q}) + U_2(\vec{q}, \dot{\vec{q}})$. Here we will consider the linear velocity dependence in a special potential similar to electromagnetic interactions and then we can write $U_2(\vec{q}, \dot{\vec{q}}) = \dot{q}_i f_i(\vec{q})$. Since a potential should be a scalar quantity, $f_i(\vec{q})$ is considered as velocity-independent vectorial components of the velocity-dependent part of the potential. Now we can write the Lagrangian as $L = \frac{1}{2}m\dot{q}_i^2 - U_1(\vec{q}) - q_i f_i(\vec{q})$. Then we can obtain equations of motion as

$$m\ddot{q}_i + \dot{q}_j\left(\frac{\partial f_j(\vec{q})}{\partial q_i} - \frac{\partial f_i(\vec{q})}{\partial q_j}\right) + \frac{\partial U_1}{\partial q_i} = 0. \tag{15}$$

Now for comparison we will calculate equation of energy conversion and in this case the energy is $E = \frac{1}{2}m\dot{q}_i^2 + U_1(\vec{q}) + q_i f_i(\vec{q})$ and we have the equation of energy conversion as

$$m\dot{q}_i\ddot{q}_i + \ddot{q}_i f_i(\vec{q}) + \dot{q}_i\dot{q}_j\frac{\partial f_j(\vec{q})}{\partial q_i} + \dot{q}_i\frac{\partial U_1}{\partial q_i} = 0. \tag{16}$$

If the velocity-dependent part of the potential energy is conserved we have $\frac{d}{dt}(\dot{q}_i f_i(\vec{q})) = 0$. Then after taking this total time derivative we can write $\ddot{q}_i f_i(\vec{q}) = -\dot{q}_i\dot{q}_j\frac{\partial f_i(\vec{q})}{\partial q_j}$. Finally we obtain

$$\dot{q}_i\left[m\ddot{q}_i + \dot{q}_j\left(\frac{\partial f_j(\vec{q})}{\partial q_i} - \frac{\partial f_i(\vec{q})}{\partial q_j}\right) + \frac{\partial U_1}{\partial q_i}\right] = 0. \tag{17}$$

While getting this we have used that changing dummy indices does not effect any equation. The term in the square parenthesis is same with the equations of motion. If we consider linearly independent velocities we obtain the equations of motion. So we can say that if the velocity dependent potential is conserved then Euler-Lagrange equations and equation of the energy conversion gives same result in the considered case.

3.5. Example 3: Lorentz force from potentials in the time-independent case

First we will consider time-independent potentials for electromagnetic fields. For a particle with charge q and mass m, we can write the total energy as

$$E = \frac{1}{2} m\dot{x}_i^2 - q\dot{x}_i A_i + q\phi \tag{18}$$

where ϕ is the electric potential and $\vec{A}$ is the vector potential. In the time-independent case, they are

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x'})}{|\vec{x} - \vec{x'}|} d^3x', \tag{19}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x'})}{|\vec{x} - \vec{x'}|} d^3x'. \tag{20}$$

In this case the equation of energy conversion is

$$m\ddot{x}_i\dot{x}_i - q\ddot{x}_i A_i - q\dot{x}_i\dot{x}_j \frac{\partial A_i}{\partial x_j} + q\dot{x}_i \frac{\partial \phi}{\partial x_i} = 0. \tag{21}$$

Eq.(21) has a unfamiliar term $q\ddot{x}_i A_i$. Before proceeding further with this term, it is better to first consider the total time derivative of magnetic energy,

$$\frac{d(q\dot{x}_i A_i)}{dt} = q\ddot{x}_i A_i + q\dot{x}_i\dot{x}_j \frac{\partial A_i}{\partial x_j}. \tag{22}$$

If we consider the case that the total time derivative of magnetic energy is equal to zero, we can write $q\ddot{x}_i A_i = -q\dot{x}_i\dot{x}_j \frac{\partial A_i}{\partial x_j}$. Then

$$m\ddot{x}_i\dot{x}_i + q\dot{x}_i\dot{x}_j \frac{\partial (A_i)}{\partial x_j} - q\dot{x}_i\dot{x}_j \frac{\partial (A_i)}{\partial x_j} + q\dot{x}_i \frac{\partial \phi}{\partial x_i} = 0. \tag{23}$$

In Eq.(23), the second and third terms cancel each other out. These two terms are related with the magnetic energy and the change in the forms of the energy is described by Eq.(23). This means that there is not any change in the form of the magnetic energy due to these two magnetic interaction terms, which is the expected result and this is known as that the magnetic energy does no work. Now, in the third term, let us replace the indices i and j with one another to obtain the Lorenz force. Since they are dummy indices, this does not change equivalence. From Eq.(23) we then obtain

$$\dot{x}_j \left[m\ddot{x}_j + q\dot{x}_i \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) + q \frac{\partial \phi}{\partial x_j} \right] = 0. \tag{24}$$

By eliminating $\dot{x}_j$ and using the double cross product property, we can write the equation

$$m\ddot{\vec{x}} = -q\nabla\phi + q\dot{\vec{x}} \times (\nabla \times \vec{A}). \tag{25}$$

If we look at the definitions of the electric and the magnetic field in terms of the potentials $\vec{E} = -\nabla\phi$ and $\vec{B} = \nabla \times \vec{A}$, we see that the terms on the right hand side are equal to Lorentz force, i.e., $\vec{F} = q\vec{E} + q\dot{\vec{x}} \times \vec{B}$.

3.6. Comparison with the Euler-Lagrange equations: Linear velocity-dependent potential in the time-dependent case

Now linear velocity-dependent potential in the time-dependent case will be considered. We will follow nearly same steps with the Section 3.4. A particle interacts with a potential via carriers of a interaction. Then we will consider retarded time since it is required by the carriers of the potential interactions to travel from the source of the fields to the particle. In this case the potential can be separated in two parts as $U(\vec{q}, \dot{\vec{q}}, t_r) = U_1(\vec{q}, t_r) + U_2(\vec{q}, \dot{\vec{q}}, t_r)$ and we can write $U_2(\vec{q}, \dot{\vec{q}}, t_r) = \dot{q}_i f_i(\vec{q}, t_r)$. So the Lagrangian is $L = \frac{1}{2}m\dot{q}_i^2 - U_1(\vec{q}, t_r) - \dot{q}_i f_i(\vec{q}, t_r)$ and then we can obtain equations of motion as

$$m\ddot{q}_i + \dot{q}_j\left(\frac{\partial f_j(\vec{q}, t_r)}{\partial q_i} - \frac{\partial f_i(\vec{q}, t_r)}{\partial q_j}\right) - \dot{q}_j\frac{\partial t_r}{\partial x_j}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r} - \frac{\partial t_r}{\partial t}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r} + \frac{\partial U_1}{\partial q_i} = 0. \quad (26)$$

Now we will obtain the equation of energy conversion. In this case the energy is $E = \frac{1}{2}m\dot{q}_i^2 + U_1(\vec{q}, t_r) + q_i f_i(\vec{q}, t_r)$. This case is different than the previous examples and due to presence of t_r, we have an extra term in the equation of energy conversion. We write it as

$$\frac{\partial E}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial E}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial E}{\partial t_r}\frac{\partial t_r}{\partial q_j}\frac{dq_j}{dt} = 0. \quad (27)$$

While obtaining Eq.(27) we used $\frac{\partial E}{\partial t_r}\frac{\partial t_r}{\partial t} = \frac{\partial E}{\partial t}$ and $\frac{dE}{dt} = \frac{\partial E}{\partial t}$. Then the equation of energy conversion can be obtained as

$$m\dot{q}_i\ddot{q}_i + \ddot{q}_i f_i(\vec{q}, t_r) + \dot{q}_i\dot{q}_j\frac{\partial f_j(\vec{q}, t_r)}{\partial q_i} + \dot{q}_i\dot{q}_j\frac{\partial t_r}{\partial q_j}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r} + \dot{q}_i\frac{\partial U_1(\vec{q}, t_r)}{\partial q_i} + \dot{q}_i\frac{\partial t_r}{\partial q_i}\frac{\partial U_1(\vec{q}, t_r)}{\partial t_r} = 0. \quad (28)$$

If the velocity dependent part of the potential energy is conserved we can write $\frac{d}{dt}(\dot{q}_i f_i(\vec{q}, t_r)) = 0$. So we have $\ddot{q}_i f_i(\vec{q}, t_r) = -\dot{q}_i\dot{q}_j\frac{\partial f_i(\vec{q}, t_r)}{\partial q_j} - \dot{q}_i\dot{q}_j\frac{\partial t_r}{\partial q_j}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r} - \dot{q}_i\frac{\partial t_r}{\partial t}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r}$. Using this in Eq.(28) we obtain

$$\begin{aligned} \dot{q}_i \quad \Bigg[\quad & m\ddot{q}_i + \dot{q}_j\left(\frac{\partial f_j(\vec{q}, t_r)}{\partial q_i} - \frac{\partial f_i(\vec{q}, t_r)}{\partial q_j}\right) + \dot{q}_j\left(\frac{\partial t_r}{\partial q_i}\frac{\partial f_j(\vec{q}, t_r)}{\partial t_r} - \frac{\partial t_r}{\partial q_j}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r}\right) - \frac{\partial t_r}{\partial t}\frac{\partial f_i(\vec{q}, t_r)}{\partial t_r} \\ + \quad & \frac{\partial U_1(\vec{q}, t_r)}{\partial q_i} + \frac{\partial t_r}{\partial q_i}\frac{\partial U_1(\vec{q}, t_r)}{\partial t_r}\Bigg] = 0. \end{aligned} \quad (29)$$

If we compare Eq.(26) with the terms inside the square parenthesis of Eq.(29) we see that there are two more terms in the equation of motion which is obtained from equation of energy conversion. So it is obvious that Euler-Lagrange equations and equation of the energy conversion do not give same result for the linear velocity-dependent potential in the time-dependent case.

3.7. Example 4: Lorentz force from potentials in the time-dependent case

Now we will consider electromagnetic interaction in the time-dependent case. Eq.(18) is still valid for energy in the time-dependent case. However, this time the definitions of scalar and vector potential are different and can be written as [4]

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|}d^3x', \quad (30)$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r'}, t_r)}{|\vec{r} - \vec{r'}|}d^3x' \quad (31)$$

where

$$t_r = t - \frac{|\vec{r} - \vec{r'}|}{c} \tag{32}$$

and t_r refers to retarded time. Retarded time is present here because electromagnetic fields need time t_r to reach the interaction point from the source. One can obtain detailed information from books, e.g., [4]. Due to the position dependence of t_r we will use Eq.(27) and we have

$$\ddot{x}_i(m\dot{x}_i - qA_i) - q\dot{x}_j\frac{\partial(\dot{x}_iA_i)}{\partial x_j} + q\dot{x}_j\frac{x_j - x'_j}{|\vec{r} - \vec{r'}|c}\frac{\partial(\dot{x}_iA_i)}{\partial t_r} + q\dot{x}_i\frac{\partial\phi}{\partial x_i} - q\dot{x}_i\frac{x_i - x'_i}{|\vec{r} - \vec{r'}|c}\frac{\partial\phi}{\partial t_r} = 0. \tag{33}$$

Then after taking the derivatives we obtain

$$\begin{aligned} m\ddot{x}_i\dot{x}_i - q\ddot{x}_iA_i + q\dot{x}_i\dot{x}_j\tfrac{\mu_0}{4\pi}\int\tfrac{J_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_j - x'_j)d^3x' + q\dot{x}_i\dot{x}_j\tfrac{\mu_0}{4\pi}\int\tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_j - x'_j)d^3x' \\ -q\dot{x}_i\tfrac{1}{4\pi\epsilon_0}\int\tfrac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_i - x'_i)d^3x' - q\dot{x}_i\tfrac{1}{4\pi\epsilon_0}\int\tfrac{\dot{\rho}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_i - x'_i)d^3x' = 0. \end{aligned} \tag{34}$$

Again we will consider the relation

$$\begin{aligned} \tfrac{d(q\dot{x}_iA_i)}{dt} &= q\ddot{x}_jA_j + q\dot{x}_i\tfrac{\partial A_i}{\partial x_j}\tfrac{\partial x_j}{\partial t} + q\dot{x}_i\tfrac{\partial A_i}{\partial t_r}\tfrac{\partial t_r}{\partial x_j}\tfrac{\partial x_j}{\partial t} + q\dot{x}_i\tfrac{\partial A_i}{\partial t_r}\tfrac{\partial t_r}{\partial t} \\ &= q\ddot{x}_jA_j - q\dot{x}_i\dot{x}_j\tfrac{\mu_0}{4\pi}\int\left[\tfrac{J_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_j - x'_j) + \tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_j - x'_j)\right]d^3x' \\ &\quad +q\dot{x}_i\tfrac{\mu_0}{4\pi}\int\tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d^3x'. \end{aligned} \tag{35}$$

If we consider the case $\frac{d(q\dot{x}_iA_i)}{dt} = 0$ we have

$$\begin{aligned} q\ddot{x}_iA_i &= q\dot{x}_i\dot{x}_j\tfrac{\mu_0}{4\pi}\int\left[\tfrac{J_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_j - x'_j) + \tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_j - x'_j)\right]d^3x' \\ &\quad -q\dot{x}_i\tfrac{\mu_0}{4\pi}\int\tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|}d^3x'. \end{aligned} \tag{36}$$

Using this relation in the Eq.(33), we obtain

$$\begin{aligned} \dot{x}_i\Big\{m\ddot{x}_i &- \tfrac{q\dot{x}_j\mu_0}{4\pi}\int\Big[\tfrac{J_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_j - x'_j) - \tfrac{J_j(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_i - x'_i) \\ &\qquad + \tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_j - x'_j) - \tfrac{\dot{J}_j(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_i - x'_i)\Big]d^3x' \\ &- \tfrac{q}{4\pi\epsilon_0}\int\left[\tfrac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(x_i - x'_i) + \tfrac{\dot{\rho}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(x_i - x'_i) - \tfrac{\dot{J}_i(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|c^2}\right]d^3x'\Big\} = 0. \end{aligned} \tag{37}$$

Similar arguments in the Section 3.5 on dummy indices are still valid in this case and using $\left(\dot{\vec{r}} \times \vec{J} \times (\vec{r} - \vec{r'})\right)_i = \dot{x}_jJ_i(x_j - x'_j) - \dot{x}_jJ_j(x_i - x'_i)$, we can write this equation in vectorial form as

$$\begin{aligned} m\ddot{\vec{r}} &= \tfrac{q\mu_0}{4\pi}\dot{\vec{r}} \times \int\left[\tfrac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3} \times (\vec{r} - \vec{r'}) + \tfrac{\dot{\vec{J}}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c} \times (\vec{r} - \vec{r'})\right]d^3x' \\ &+ \tfrac{q}{4\pi\epsilon_0}\int\left[\tfrac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(\vec{r} - \vec{r'}) + \tfrac{\dot{\rho}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(\vec{r} - \vec{r'}) - \tfrac{\dot{\vec{J}}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|c^2}\right]d^3x'. \end{aligned} \tag{38}$$

The terms in the parenthesis corresponds to $\vec{E}$ and $\vec{B}$, respectively and they are

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3} + \frac{\dot{\vec{J}}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}\right] \times (\vec{r} - \vec{r'})d^3x', \tag{39}$$

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^3}(\vec{r} - \vec{r'}) + \frac{\dot{\rho}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|^2c}(\vec{r} - \vec{r'}) - \frac{\dot{\vec{J}}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|c^2}\right]d^3x'. \tag{40}$$

These are known as Jefimenko equations. These equations describe electric field $\vec{E}(\vec{r},t)$ and magnetic field $\vec{B}(\vec{r},t)$ in the time-dependent scalar potential $\phi(\vec{r},t)$ and vector potential $\vec{A}(\vec{r},t)$. Then in terms of these time-dependent electric and magnetic field, from Eq.(38) we have the Lorentz force $\vec{F} = q\vec{E}(\vec{r},t) + q\vec{\dot{r}} \times \vec{B}(\vec{r},t)$.

If the sources of the electromagnetic field, charge density ρ and current density $\vec{J}$, depend on time their density and position can change with time. These variations affect on the strength of the electromagnetic field and the distance between source and interaction point. These electromagnetic fields are carried by photons and photon intensity and travel distance change with time and these changes are represented by $\rho(\vec{r'},t_r)$ and $\vec{J}(\vec{r'},t_r)$. The differentiations of the sources with respect to retarded time t_r, $\frac{\partial\rho(\vec{r'},t_r)}{\partial t_r}$ and $\frac{\partial\vec{J}(\vec{r'},t_r)}{\partial t_r}$, represent the variations on the sources and they are the reasons of the difference between our derivation and the derivations from Lagrangian formalism. In the derivation from Lagrangian formalism $\frac{\partial\rho(\vec{r'},t_r)}{\partial t_r}$ and one of the terms related with $\frac{\partial\vec{J}(\vec{r'},t_r)}{\partial t_r}$ are missing and one can not achieve Jefimenko equations. On the other hand these are present in derivations from our formalism and we obtained Jefimenko equations correctly. These means that effects on the force due to the mentioned variations on the density and distance can not be reached from Lagrangian formalism.

4. Conclusion and discussion

In this work, we used a new technique to analyze changes in systems. By taking the total time derivative of energy and using the relation $\frac{dE}{dt} = \frac{\partial E}{\partial t}$, we obtained an equation in units of power, i.e., the equation of energy conversion. Except for the linear velocity-independent potential in the time-dependent case, we reached the same results as other formalisms. From the equation of energy conversion, we obtained solutions for two well-known examples, gravitational rotation and coupled pendula, and Lorentz force for time-independent and time-dependent potentials. Using these derivations, we will compare our technique to others in the following paragraphs.

Firstly, we will describe our problem-solving technique for a physical system having energy E. The first relation that we should consider is $\frac{dE}{dt} = \frac{\partial E}{\partial t}$. If $\frac{\partial E}{\partial t}$ is not equal to zero, then we can say that energy is changing with time and this change is explicitly given by it. If it is zero, we can say that energy is conserved and we can apply the equation of energy conversion. As a second step, we should calculate conserved momenta if there are symmetric coordinates and we should eliminate corresponding velocities from the energy equation. In the third step, we can write the equation of energy conversion using the derivatives of energy with respect to positions and corresponding velocities. Here, by eliminating common velocity multiplier from all terms we can obtain equations of motion for the linearly-independent velocities. In general, however, we do not need the equation of motion; we can use equation of energy conversion directly. This is the technique that we used to analyze evolutions of a system. Importantly, it applies nearly the same steps as other formalisms.

There are differences between our formalism and the others. The first difference is related to specific definitions. The Lagrangian does not have any physical definitions, since it is not a physical quantity. Also, from the historical point of view, it was developed to derive the equation of motion. In books, it is defined only as a function and there is no other definition[1, 6, 7, 8]. The Hamiltonian is another function which is derived from the Lagrangian, and in some cases it is the same as the energy. But in general, it is not equal to the energy. Hence, we can say that, in general, both the Lagrangian and the Hamiltonian are not physical quantities. On the other hand, our paradigm uses energy, which is a measurable physical quantity.

The second difference is in our understanding way of physical situations. Typically, we understand physical situations with the equations of motion. On one side of the equation of motion, there is a force or a similar quantity and on the other side, relevant acceleration. This equation of motion tells us that if there is a force, it results in acceleration. This is the basis of the Lagrangian formalism; the force together with the acceleration is the basic physical quantity. However, the subtraction of kinetic and potential energy from one another in the definition of the Lagrangian means that the Lagrangian is itself not a physical quantity. So in the Lagrangian formalism we obtain physical quantities from a unphysical quantity, Lagrangian. In our case, we employ the equation of energy conversion. This equation describes how energy changes its form within a system and the agent of this change is the force. Also, knowledge of the equations of motion is involved in our technique and we obtain relevant force from it. As it is clear, we derive all these information from energy. The basic quantity for our formalism is energy and it forms the foundation upon which we have constructed our formalism. So in our technique force and change in energy are obtained from a physical quantity, energy.

The third difference between our formalism and others is its relative simplicity. If one compares our derivations and calculations with the other techniques, the algebraic simplicity is apparent. This difference also extends to logical simplicity. In our case, we have energy and the equation of energy conversion from the differentiation of energy. In other cases, we have the Lagrangian or Hamiltonian and the Euler-Lagrange or Hamilton's equations. However, these formalisms need energy relation and its conservation to be able to solve a problem in most cases, whereas these energy-related terms are naturally present in our case. Thus, we have two statements, while the other formalisms require three and this is expressed in the solution of gravitational rotation. This means that our formalism is logically simpler.

There is another important issue that we should point out. In the velocity-independent potential cases, we obtained the same results as the other formalisms, and shown their equivalence. However, there are some problems related with linear velocity-dependent potential only in the time-dependent case and we observed these in the derivation of Lorentz force. This type of potential is problematic for the Lagrangian formalism, since Lagrangian is defined only by considering velocity-independent potentials[1, 6, 7, 8]. Despite this definition, some books use the Lagrangian for a velocity-dependent potential case to calculate Lorentz force [1, 3]. They even use time-dependent $\vec{A}$ and time-independent ϕ, which is an erroneous mixture of two cases. More importantly, they take the total time derivative of $\vec{A}$ in the wrong manner. Their derivations are incorrect because they ignore a crucial property of electromagnetism. That is, the electromagnetic fields need some time to travel from the source to interaction point. If one considers a time-dependent situation due to the abovementioned property of electromagnetic fields, one has to use retarded potentials and take the derivative in a similar manner applied in the Section 3.7. We considered velocity-dependent potentials in the Sections 3.5 and 3.7 and reached force correctly. We obtained Lorentz force for the time-independent case, and the Lorentz force together with the Jefimenko equations for the time-dependent case.

To sum up, we used the energy relation and from its variation we obtained the equation of energy conversion. The equation of energy conversion is successful in obtaining equations of motion in a correct way even for time-dependent linear velocity-dependent potential case. These show that it can be a promising framework for some other situations.

References

[1] Goldstein, H.: Classical Mechanics, 2nd ed. Narosa Publishing House, New. Delhi (1998)

[2] Landau, L. D., Lifshitz, E. M.: Mechanics, 3rd ed. Butterworth-Heinemann, Oxford (1976)
[3] McCauley,J. L.: Classical Mechanics: Transformations, Flows, Integrable and Chaotic Dynamics. Cambridge University Press (1997)
[4] Griffiths, D. J.: Introduction to Electrodynamics, 3rd ed. Benjamin Cummings, Upper Saddle River, NJ. (1999)
[5] Jackson, J. D.: Classical Electrodynamics, 3rd ed. Wiley (1999)
[6] Iro, H.: A Modern Approach to Classical Mechanics. World Scientific (2002)
[7] Greenwood, D. T.: Advanced Dynamics. Cambridge University Press (2003)
[8] Chow, T. L.: Classical Mechanics. John Wiley and Sons (1995)