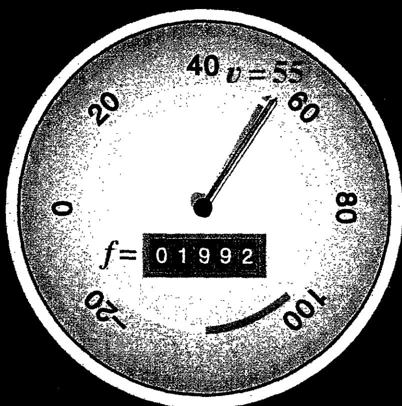
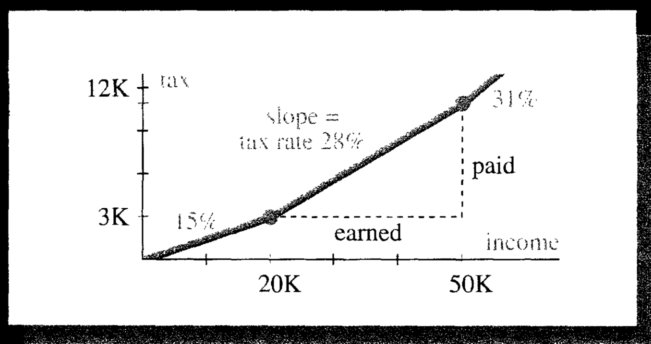
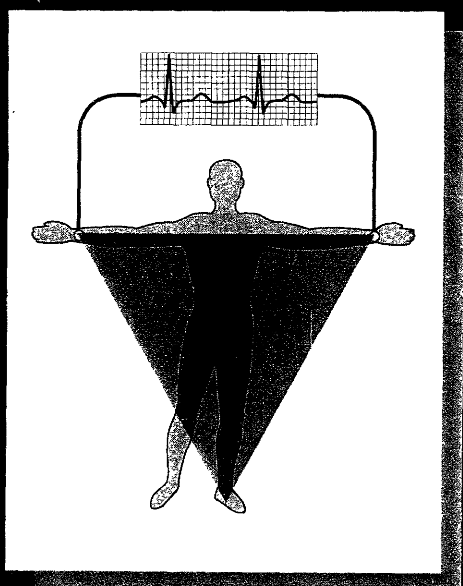
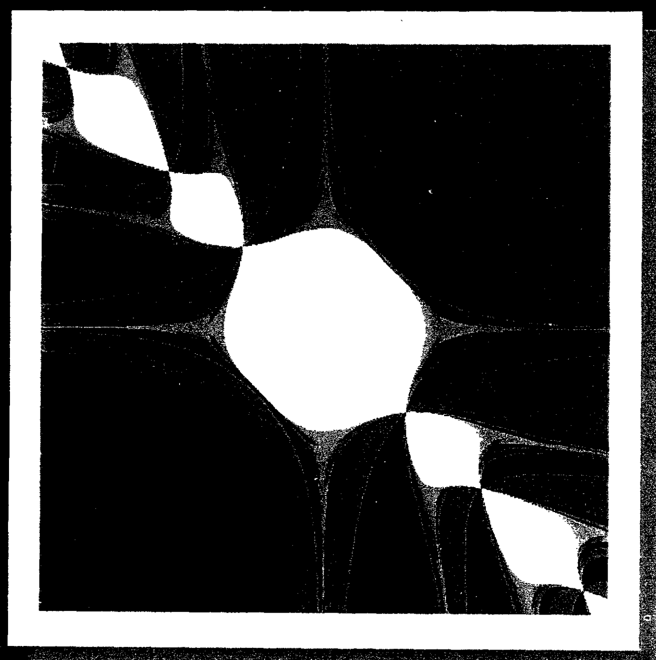
CALCULUS

Gilbert Strang









Derivatives

Sum: $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

Power: $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

Chain: $\frac{d}{dx}z(y(x)) = \frac{dz}{dy}\frac{dy}{dx}$

Inverse: $\frac{dx}{dy} = \frac{1}{dy/dx}$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}e^{cx} = ce^{cx}$

$\frac{d}{dx}b^x = b^x \ln b$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

Limits and Continuity

$\frac{\sin x}{x} \to 1 \quad \frac{1-\cos x}{x} \to 0 \quad \frac{1-\cos x}{x^2} \to \frac{1}{2}$

$a_n \to 0 : |a_n| < \epsilon$ for all $n > N$

$a_n \to L : |a_n - L| < \epsilon$ for all $n > N$

$f(x) \to L : |f(x) - L| < \epsilon$ for $0 < |x-a| < \delta$

$f(x) \to f(a)$: Continuous at a if $L = f(a)$

$\frac{f(x)-f(a)}{x-a} \to f'(a)$: Derivative at a

$\frac{f(x)-f(a)}{x-a} = f'(c)$: Mean Value Theorem

$\frac{f(x+\Delta x)-f(x)}{\Delta x} \to f'(x)$: Derivative at x

$\frac{f(x+\Delta x)-f(x-\Delta x)}{2\,\Delta x} \to f'(x)$: Centered

$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ l'Hôpital's Rule for $\frac{0}{0}$

Maximum and Minimum

Critical: $f'(x) = 0$ or no f' or endpoint

Minimum $\quad f'(x) = 0$ and $f''(x) > 0$

Maximum $\quad f'(x) = 0$ and $f''(x) < 0$

Inflection point $\quad f''(x) = 0$

Newton's Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Iteration $x_{n+1} = F(x_n)$ attracted to fixed point $x^* = F(x^*)$ if $|F'(x^*)| < 1$

Stationary in 2D: $\partial f/\partial x = 0$, $\partial f/\partial y = 0$

Minimum $\quad f_{xx} > 0 \quad f_{xx}f_{yy} > f_{xy}^2$

Maximum $\quad f_{xx} < 0 \quad f_{xx}f_{yy} > f_{xy}^2$

Saddle point $\quad f_{xx}f_{yy} < f_{xy}^2$

Newton in 2D $\begin{cases} g + g_x\,\Delta x + g_y\,\Delta y = 0 \\ h + h_x\Delta x + h_y\Delta y = 0 \end{cases}$

Algebra

$\frac{x/a}{y/b} = \frac{bx}{ay} \qquad x^{-n} = \frac{1}{x^n} \qquad \sqrt[n]{x} = x^{1/n}$

$(x^2)(x^3) = x^5 \quad (x^2)^3 = x^6 \quad x^2/x^3 = x^{-1}$

$ax^2 + bx + c = 0$ has roots $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x^2 + 2Bx + C = 0$ has roots $x = -B \pm \sqrt{B^2 - C}$

Completing square $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$

Partial fractions $\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

Mistakes $\quad \frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c} \qquad \sqrt{x^2+a^2} \neq x + a$

Fundamental Theorem of Calculus

$\frac{d}{dx}\int_a^x v(t)dt = v(x) \qquad \int_a^b \frac{df}{dx}dx = f(b) - f(a)$

$\frac{d}{dx}\int_{a(x)}^{b(x)} v(t)dt = v(b(x))\frac{db}{dx} - v(a(x))\frac{da}{dx}$

$\int_0^b y(x)dx = \lim_{\Delta x \to 0} \Delta x[y(\Delta x) + y(2\Delta x) + \cdots + y(b)]$

Circle, Line, and Plane

$x = r\cos\omega t, y = r\sin\omega t,$ speed ωr

$y = mx + b$ or $y - y_0 = m(x - x_0)$

Plane $ax + by + cz = d$ or

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Normal vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Distance to (0,0,0): $|d|/\sqrt{a^2+b^2+c^2}$

Line $(x,y,z) = (x_0,y_0,z_0) + t(v_1,v_2,v_3)$

No parameter: $\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$

Projection: $\mathbf{p} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}, |\mathbf{p}| = |\mathbf{b}|\cos\theta$

Trigonometric Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$ (divide by $\cos^2 x$)

$1 + \cot^2 x = \csc^2 x$ (divide by $\sin^2 x$)

$\sin 2x = 2 \sin x \cos x$ (double angle)

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin(s \pm t) = \sin s \cos t \pm \cos s \sin t$ (Addition

$\cos(s \pm t) = \cos s \cos t \mp \sin s \sin t$ formulas)

$\tan(s+t) = (\tan s + \tan t)/(1 - \tan s\ \tan t)$

$c^2 = a^2 + b^2 - 2ab\cos\theta$ (Law of cosines)

$a/\sin A = b/\sin B = c/\sin C$ (Law of sines)

$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \tan^{-1}\frac{b}{a})$

$\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$\sin(\frac{\pi}{2} \pm x) = \cos x$ and $\cos(\frac{\pi}{2} \pm x) = \mp \sin x$

$\sin(\pi \pm x) = \mp \sin x$ and $\cos(\pi \pm x) = -\cos x$

Trigonometric Integrals

$\int \sin^2 x\, dx = \frac{x - \sin\ x\cos\ x}{2} = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4}$

$\int \cos^2 x\, dx = \frac{x + \sin\ x\cos\ x}{2} = \int \frac{1+\cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4}$

$\int \tan^2 x\, dx = \tan x - x$

$\int \cot^2 x\, dx = -\cot x - x$

$\int \sin^n x\, dx = -\frac{\sin^{n-1} x\ \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

$\int \cos^n x\, dx = +\frac{\cos^{n-1} x\ \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$

$\int \tan\ x\, dx = -\ln|\cos x|$

$\int \cot\ x\, dx = \ln|\sin x|$

$\int \sec x\, dx = \ln|\sec x + \tan x|$

$\int \csc x\, dx = \ln|\csc x - \cot x| = -\ln|\csc x + \cot x|$

$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$

$\int \sin px \sin qx\, dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$

$\int \cos px \cos qx\, dx = \frac{\sin(p-q)x}{2(p-q)} + \frac{\sin(p+q)x}{2(p+q)}$

$\int \sin px \cos qx\, dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)}{2(p+q)}$

Additional integrals follow the index

Integration by Parts

$\int \ln x\, dx = x\ln x - x$

$\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}$

$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

$\int e^{cx}\sin kx\, dx = \frac{e^{cx}}{c^2+k^2}(c\sin kx - k\cos kx)$

$\int e^{cx}\cos kx\, dx = \frac{e^{cx}}{c^2+k^2}(c\cos kx + k\sin kx)$

$\int x\sin x\, dx = \sin x - x\cos x$

$\int x\cos x\, dx = \cos x + x\ \sin x$

$\int x^n \sin x\, dx = -x^n\cos x + n\int x^{n-1}\cos x\, dx$

$\int x^n \cos x\, dx = +x^n\sin x - n\int x^{n-1}\sin x\, dx$

$\int \sin^{-1} x\, dx = x\ \sin^{-1} x + \sqrt{1-x^2}$

$\int \tan^{-1} x\, dx = x\ \tan^{-1} x - \frac{1}{2}\ln(1+x^2)$

Integrals with x^2 and a^2 and $D = b^2 - 4ac$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| = \frac{1}{a}\tanh^{-1}\frac{x}{a}$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}|$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x + \sqrt{x^2+a^2}|$

$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}$

$\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{\sqrt{x^2+a^2}-a}{x}\right|$

$$\int \frac{dx}{ax^2+bx+c} \begin{aligned}[t] &= \frac{1}{\sqrt{D}}\ln\left|\frac{2ax+b-\sqrt{D}}{2ax+b+\sqrt{D}}\right|,\ D>0 \\ &= \frac{2}{\sqrt{-D}}\tan^{-1}\frac{2ax+b}{\sqrt{-D}},\ D<0 \\ &= \frac{-2}{2ax+b},\ D=0 \end{aligned}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} \begin{aligned}[t] &= \frac{1}{\sqrt{a}}\ln|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}| \\ &= \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-2ax-b}{\sqrt{D}}, a<0 \end{aligned}$$

Definite Integrals

$\int_0^\infty x^n e^{-x}dx = n! = \Gamma(n+1)$

$\int_0^\infty e^{-a^2x^2}dx = \sqrt{\pi}/2a$

$\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$

$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2}\cos^n x\, dx =$

$\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}\left(\frac{\pi}{2}\right)$ (n even) or $\frac{2}{3}\frac{4}{5}\cdots\frac{n-1}{n}$ (n odd > 1)

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Gilbert Strang
Massachusetts Institute of Technology

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The figures on the cover are from Sections 1.1, 1.2, 11.2, and 13.3. Electrocardiogram by Dr. Frank Netter © CIBA-GEIGY. The back cover is explained in Section 1.6.

Other texts from Wellesley-Cambridge Press

Introduction to Applied Mathematics, Gilbert Strang, ISBN 0-9614088-0-4.

An Analysis of the Finite Element Method, Gilbert Strang and George Fix, ISBN 0-13-032946-0.

Introduction to Linear Algebra, Gilbert Strang, ISBN 0-9614088-5-5 (to be published in 1993).

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Preface

Calculus is a course with a purpose. It describes growth and decay and change. Population has a growth rate—which changes. The value of money has a decay rate—which may grow. Whatever happens in life, the one sure thing is change. To understand and model those changes—that is where calculus is needed and used.

The central ideas are firmly established. The author's responsibility is to explain them clearly, with examples that go directly to the point—*and light it up*. By far the greater part of the effort behind this book was devoted to clear and lively expression. It must be on this basis first, that a new text is judged. The words should attract the reader—and help the ideas to take hold.

This preface mentions some changes in emphasis. It is in no way surprising that changes should come (carefully). But you will understand that ***this is a basic text for calculus****. It is for all institutions, and it is written directly to the student.* The important points are highlighted, not only visually but verbally. This is a textbook and not a test bank. I hope you approve of the presentation.

I also hope that readers will see the spirit behind this book. The best part of mathematics is in doing it, and we jump right in. With teaching and writing, the start sets the tone—and we want the class to be active. For all students there should be something new. Instead of ending each section with a summary (reader passive), we ask the student to contribute the key words. That reinforces confidence—which is essential to learning.

INTRODUCTION TO CALCULUS

The first chapter previews calculus—*with a purpose*. That purpose is to see functions in action. Especially we show ***pairs of functions***, the distance $f(t)$ and the velocity $v(t)$. In my experience, the intuition from driving a car is a free gift to calculus teachers. The equation $f = vt$ is understood (but not always in the language of algebra). The speedometer represents the derivative, the odometer shows the integral. By learning specific functions $f(t)$ and $v(t)$, calculus begins.

This book emphasizes, more than in the past, the visual part of learning mathematics. The figures are not in the margins, where they are missed. Formulas are connected to graphs. The best thing a computer can do is to display the function. We see it grow to a maximum, or cross zero, or oscillate. Graphs are a key to mathematics outside the classroom.

We also emphasize, earlier than usual, the meaning of differential equations. Those are models of life. They express how change is determined—by the present position and the outside force. I believe it is essential to meet some real equations in a calculus course. We cannot depend on a later course to make clear what this one is about.

I hope you will find that the text concentrates on the important points. It is useless to include far more than anyone can read. And "grinding it out" is not mathematics at its best. We are asking students for a real effort, ***which we must not waste***. It is easy to bury the purpose of calculus under a million equations. It is harder, but right, to stay with the ideas that matter the most.

On this principle that less is more, I will summarize:

1. Practice with functions and graphs is all-important.
2. The models of calculus are differential equations.
3. Examples need not be artificially complicated. Mathematics is hard enough.
4. It is not required to cover every topic.
5. There is no point in preparing for real problems and then never seeing them.

CAREFUL CHANGES

This is a mainstream textbook, but not a copy of all the others. I greatly respect what they have taught, while still believing that a more active course teaches more permanently. The book is not different by being "easier" or "harder" or "more applied"—but by its effort to speak frankly and directly. I absolutely do not accept that calculus is too difficult for our students, or too remote from their lives. On the contrary, we can teach what students need—which is not constant.

One change in direction (from outside) is toward computing. That is a good thing, and getting better. The book aims to support this movement, *without forcing it*. Computer-based graphics is a fantastic way to appreciate functions. We include examples (and programs) because this activity is valuable. But it takes time—computing is no easy cure-all. The problems of learning calculus don't disappear in the presence of a mouse.

This book is equally intended for courses taught in the traditional way. The text reinforces what is explained in class, with a fresh approach. I am convinced that seeing an idea *often* and *clearly* is the way to master it.

A second change (from inside) is directly mathematical. Most new functions are created from a chain like $f(g(x))$. This fact is buried deep inside too many textbooks. The chain rule is not just another formula, and $f(g(x)) = x$ doesn't happen by accident. Those topics are much too basic to miss.

The special case of *iteration*, when f and g are the same and the chain keeps going, leads to convergence or divergence or chaos. Derivatives are in control—a perfect application of calculus. Iterations begin with Newton's method (the best way to solve equations in practice). When we teach that, we are teaching the real thing. Then Chapter 3 attempts to solve $x^2 + 1 = 0$, and Newton's method ends in chaos.

This is a time of transition. The book has to encourage new ideas, while preserving the structure on which calculus is built. *Certainly the structure can be less heavy*. So can the book. We do not need to be crushed by this subject, to appreciate its power. The book could be lighter still* by eliminating topics, but I don't think the author should decide the syllabus. This is a free country, with free choices.

*We envision a future book for short calculus courses (not to be called *Calculus Lite*).

May I repeat that the changes are careful. A revolution is very likely to end at 2π. The teacher's confidence is fully as important as the student's, and order is essential—this subject was not created by random choice. The emphasis can change, while the pattern remains completely familiar. *My goal is to support a conventional course with fresh ideas.* The book also supports a "lean and lively" course, and the *full range* in between.

APPLICATIONS

I hope the electrocardiogram is an attractive example. It comes first in Chapter 3, and again in Chapter 11, to illustrate periodic motion and vectors and projection. Biology uses calculus—so do management and economics. I have been astonished (now that I notice) how often derivatives are in the newspapers. One pleasure of learning a language is to hear it spoken and see it written.

The book aims to connect mathematics to other learning. That comes from the examples we choose and the way we present them. Science and engineering are excellent sources. Rates of change appear everywhere in sports (and business). *Applications cannot be forced*—a long trip through a distant topic is not welcomed or followed. It is the author's job to make the steps few and clear. I will mention six applications, all of them optional, that can be in the course or for reference:

1. Probability (including mean and variance).
2. Complex numbers (in differential equations).
3. Models in life sciences: logistic and mass action and "MM" equations.
4. Mathematics of finance: compound interest, loans, continuous *vs.* discrete growth. "Business calculus" should be calculus.
5. Numerical methods: integration, maximum and minimum, convergence of iterations. *Calculus in use.*
6. Economics: the supply-demand equation (perfect competition) is different from profit maximization by a monopoly.

Engineering and physics appear throughout—we prepare for them very carefully. The treatment of linear algebra, and later the optimization of $f(x, y)$, reflect their enormous importance in practice. The book ends with a discussion of which courses to take next.

ORGANIZATION

An experienced teacher will see two things—the order of topics is close to traditional, while each topic is slightly new. When an idea is thought through, some insight always comes. An extremely small example ends Section 1.1, about the graph of a function. It asks why **X** cannot be a graph, and which letters can. (*Not many. There is room for debate.*) For inverse functions, °F to °C is good because of $\frac{5}{9}$ and $\frac{9}{5}$. For a piecewise function we turn to income tax. Even the cardboard box of maximum volume is finally given a decent top.

There are also serious decisions, of which I hope you approve. The sine and cosine are introduced early. Calculus relies heavily on a very few functions (amazingly few). ***They are understood by being used.*** We explain and review them—never expecting a perfect memory, or requiring everything at once. A student can compare $2j$ and j^2 and 2^j before exponentials are fully defined. When 2^x does come, its derivative is found in two steps: first as $c2^x$—emphasizing what it means to have dy/dx proportional to y—and then we discover the constant c (by means of e).

Effort on e^x is well spent. It is the most important function created by calculus. The logarithm fills a gap among the integrals, but for applications nothing can touch e^x.

Part of calculus is about these particular functions. The other part is more general—about $y(x)$ and dy/dx. Their relation is deep, but we can reach it. This book emphasizes key ideas rather than complicated examples. The work is there to do, but it need not be drudgery. Why spoil such a beautiful subject? My effort is to make the book lighter, spiritually as well as materially.

I am convinced that a book and a course can be cheerful and human without losing sight of their goal—***to teach real mathematics***. Limits are at the heart of calculus, not to be missed (or overdone). We pay attention to the logic behind "necessary" and "sufficient" and "if and only if." When a proof is needed we give it. *The mathematics is all there*—the great problem of teaching is to find the right time and the right expression.

SPECIAL ACKNOWLEDGEMENTS

A lot of generous people have helped this book. I believe they know how much it is appreciated. For class testing, Dan Drucker at Wayne State was perfect—by insisting that the best traditions of calculus be preserved. For critical reading, Bob Lynch at Purdue is a master. A thousand small things make the difference, when a book is in use day by day. Only one other source—totally unexpected—pointed out so many sentences to rewrite. Darien Lauten opened my eyes to what a good teacher in a regular New Hampshire school can do. For one thing, the calculus classes at Oyster River High School accept nothing but clarity. This book is intended to be taken personally, which they did. Their comments were anonymous, and mostly friendly, and they changed the book.

One person knows how often these pages were typed. I thank Sophia Koulouras for constant support—she gave encouragement from start to finish.

I also want to express, most clearly of all, the happiness of these three years. That was Jill's contribution, unique and unequaled. It is felt on every page. This book is dedicated to her as it is offered to you—with thanks for such good fortune.

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