

## MATH 578: SOLUTION MANUAL FOR ASSIGNMENT 5

1. Exercise 3.5: Suppose that an  $\nu$ -stage ERK method of order  $\nu$  is applied to the linear scalar equation  $y' = \lambda y$ . Prove that

$$y_n = \left[ \sum_{k=0}^{\nu} \frac{1}{k!} (h\lambda)^k \right]^n y_0, \quad n = 0, 1, \dots.$$

*Proof.* Assume the exact solution of

$$\begin{cases} y' = \lambda y \\ y(t_n) = y_n \end{cases}$$

is  $\tilde{y} = \tilde{y}(t)$ . Then, we have

$$\begin{aligned} \tilde{y}(t_{n+1}) &= \tilde{y}(t_n + h) = \tilde{y}(t_n) + h\tilde{y}'(t_n) + \frac{h^2}{2!}\tilde{y}''(t_n) + \dots + \frac{h^\nu}{\nu!}\tilde{y}^\nu(t_n) + O(h^{\nu+1}) \\ &= \tilde{y}(t_n) + h[\lambda\tilde{y}(t_n)] + \frac{h^2}{2!}[\lambda^2\tilde{y}(t_n)] + \dots + \frac{h^\nu}{\nu!}[\lambda^\nu\tilde{y}(t_n)] + O(h^{\nu+1}) \\ &= \left[ \sum_{k=0}^{\nu} \frac{1}{k!} (h\lambda)^k \right] \tilde{y}(t_n) + O(h^{\nu+1}). \end{aligned}$$

To make local truncation error to be  $O(h^{\nu+1})$ , we need the scheme to be

$$y_{n+1} = \left[ \sum_{k=0}^{\nu} \frac{1}{k!} (h\lambda)^k \right] y_n.$$

Hence, we can obtain

$$y_n = \left[ \sum_{k=0}^{\nu} \frac{1}{k!} (h\lambda)^k \right]^n y_0, \quad n = 0, 1, \dots.$$

□

2. Exercise 3.6: Determine all choices of  $b$ ,  $c$  and  $A$  such that the two-stage IRK method

$$\begin{array}{c|cc} c & A \\ \hline & b^T \end{array}$$

is of order  $p \geq 3$ .

**Solution:** Consider equation

$$y' = f(t, y).$$

Suppose

$$\xi_1 = y_n + h[a_{11}K_1 + a_{12}K_2], \quad \xi_2 = y_n + h[a_{21}K_1 + a_{22}K_2], \quad (1)$$

$$y_{n+1} = y_n + h(b_1K_1 + b_2K_2), \quad (2)$$

where  $K_1 = f(t_n + c_1 h, \xi_1)$ ,  $K_2 = f(t_n + c_2 h, \xi_2)$ .

Doing Taylor expansion of  $K_1$  and  $K_2$  at  $(t_n, y_n)$ , and substituting the result into equations (1), we have

$$\begin{aligned} K_1 &= f + h[f_t c_1 + f_y c_1 f] + h^2[f_y(a_{11}c_1 + a_{12}c_2)(f_t + f_y f) + \frac{f_{tt} + f_{yy}f^2 + 2f_{ty}f}{2}c_1^2] + O(h^3) \\ K_2 &= f + h[f_t c_2 + f_y c_2 f] + h^2[f_y(a_{21}c_1 + a_{22}c_2)(f_t + f_y f) + \frac{f_{tt} + f_{yy}f^2 + 2f_{ty}f}{2}c_2^2] + O(h^3) \end{aligned}$$

Substituting the above  $K_1$  and  $K_2$  equations into equation (2) and comparing the coefficients with

$$y(t_{n+1}) = y(t_n) + hf + \frac{h^2}{2}[f_t + f_y f] + \frac{h^3}{6}[f_{tt} + f_y f_t + (f_y)^2 f + 2ff_{yt} + f_{yy}(f)^2] + O(h^4),$$

we have the following equations:

$$\left\{ \begin{array}{l} a_{11} + a_{12} = c_1 \\ a_{21} + a_{22} = c_2 \\ b_1 + b_2 = 1 \\ b_1 c_1 + b_2 c_2 = 1/2 \\ b_1 c_1^2 + b_2 c_2^2 = 1/3 \\ b_1(a_{11}c_1 + a_{12}c_2) + b_2(a_{21}c_1 + a_{22}c_2) = 1/6 \end{array} \right.$$

3. Exercise 3.7: Write the theta method (1.13) as a Runge-Kutta method.

**Solution:** The theta method is

$$y_{n+1} = y_n + h \cdot [\theta f(t_n, y_n) + (1 - \theta)f(t_{n+1}, y_{n+1})].$$

The Runge-Kutta method is

$$y_{n+1} = y_n + h \cdot [b_1 f(t_n + c_1 h, \xi_1) + b_2 f(t_n + c_2 h, \xi_2)].$$

Comparing the two schemes, we have

$$b_1 = \theta, \quad b_2 = 1 - \theta, \quad c_1 = 0, \quad c_2 = 1, \quad \xi_1 = y_n, \quad \xi_2 = y_{n+1}.$$

And since

$$\xi_1 = y_n + h[a_{11}f(t_n + c_1 h, \xi_1) + a_{12}f(t_n + c_2 h, \xi_2)]$$

and

$$\xi_2 = y_n + h[a_{21}f(t_n + c_1 h, \xi_1) + a_{22}f(t_n + c_2 h, \xi_2)],$$

substituting  $\xi_1 = y_n$ ,  $\xi_2 = y_{n+1}$ ,  $c_1 = 0$ ,  $c_2 = 1$  into the above two equations, we have

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{21} = \theta, \quad a_{22} = 1 - \theta.$$

Therefore, the RK table is

	0	0
	$\theta$	$1 - \theta$
	$\theta$	$1 - \theta$

4. Exercise 3.8: Derive the three-stage Runge-Kutta method that corresponds to the collocation points  $c_1 = \frac{1}{4}$ ,  $c_2 = \frac{1}{2}$ ,  $c_3 = \frac{3}{4}$  and determine its order.

**Solution:** From the lemma 3.5, we have

$$a_{j,i} = \int_0^{c_j} l_i(\tau) d\tau,$$

$$b_j = \int_0^1 l_j(\tau) d\tau,$$

where  $l_i(t) = \prod_{k=1, k \neq i}^3 \left( \frac{t - c_k}{c_i - c_k} \right)$ . Hence, the RK table is

$\frac{1}{4}$	$\frac{23}{48}$	$-\frac{1}{3}$	$\frac{5}{48}$
$\frac{1}{2}$	$\frac{7}{12}$	$-\frac{1}{6}$	$\frac{1}{12}$
$\frac{3}{4}$	$\frac{9}{16}$	0	$\frac{3}{16}$
	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$

To determine its order, define

$$g(t) = (t - \frac{1}{4})(t - \frac{1}{2})(t - \frac{3}{4}),$$

then it is easy to check that  $g$  is orthogonal to  $\Pi_0$ . In fact,

$$\int_0^1 g(\tau) d\tau = 0,$$

$$\int_0^1 g(\tau) \tau d\tau = \frac{7}{960} \neq 0.$$

Hence, from Theorem 3.7, the collocation method is of order

$$\nu + m = 3 + 1 = 4.$$

5. (computer project) Consider the ODE

$$\begin{cases} y' = \frac{1}{t^2} - \frac{y}{t} - y^2 \\ y(1) = -1 \end{cases}$$

Solve the IVP of the ODE for  $t$  in  $[1, 5]$ , using the two-stage, fourth-order Gauss-Legendre IRK method (p. 47 of the textbook) and Newton's method for solving the nonlinear equation. Could you verify numerically that the order of the method is four?

**Solution:** The two-stage, fourth-order Gauss-Legendre method is

$$y_{n+1} = y_n + h \cdot [b_1 f(t_n + c_1 h, \xi_1) + b_2 f(t_n + c_2 h, \xi_2)]$$

with

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

To get the value of  $\xi_1, \xi_2$ , we use Newton's method to solve equations

$$\begin{cases} \xi_1 = y_n + h[a_{11}f(t_n + c_1h, \xi_1) + a_{12}f(t_n + c_2h, \xi_2)] \\ \xi_2 = y_n + h[a_{21}f(t_n + c_1h, \xi_1) + a_{22}f(t_n + c_2h, \xi_2)], \end{cases}$$

We can define

$$f(\xi_1, \xi_2) := \begin{pmatrix} \xi_1 - y_n - h[a_{11}f(t_n + c_1h, \xi_1) + a_{12}f(t_n + c_2h, \xi_2)] \\ \xi_2 - y_n - h[a_{21}f(t_n + c_1h, \xi_1) + a_{22}f(t_n + c_2h, \xi_2)] \end{pmatrix}.$$

Then the Jacobian matrix becomes

$$J = I - h \cdot \begin{pmatrix} a_{11}f_y(t_n + c_1h, \xi_1) & a_{12}f_y(t_n + c_2h, \xi_2) \\ a_{21}f_y(t_n + c_1h, \xi_1) & a_{22}f_y(t_n + c_2h, \xi_2) \end{pmatrix}$$

where  $f_y = -\frac{1}{t} - 2y$ .

To verify the order of this method, we use formula

$$p = \frac{\log(e1/e2)}{\log(h1/h2)}.$$

Hence, computing by Matlab, we get the order of this method is

$$3.99317916 \approx 4.$$

## APPENDIXES

```
% Solving initial value problem
% dy/dt= 1/t^2 - y/t - y^2           y(1)=-1;
% two-stage , fourth-order Gauss-Legendre IRK method
```

```
% RK table :
A=[1/4 1/4-sqrt(3)/6;1/4+sqrt(3)/6 1/4];
b=[1/2 ; 1/2];
c=[1/2-sqrt(3)/6 ; 1/2+sqrt(3)/6];

f=@(t,y) 1/t^2-y/t-y^2;                      %right hand side of the ODE
fy=@(t,y) -1/t-2*y;

h=[0.1 0.01 0.001];                           %step size
```

```

for j=1:3

tt=1:h(j):5;
yy=zeros(size(tt));
yy(1)=-1; %initial value

% parameters in newton's method
M=100;
Tol=10^(-11);

for i=1:length(tt)-1
    % Newton's method for finding the roots of nonlinear function
    xi_1=yy(i); xi_2=yy(i); %initial guess
    FF=[1,1];
    for k=1:M
        if norm(FF)>Tol
            F= [ xi_1 ; xi_2 ] -[yy(i); yy(i)] -...
                  h(j).*A*[ f( tt(i)+c(1)*h(j), xi_1 ); f( tt(i)+c(2)*h(j), xi_2 ) ];
            J=diag(ones(2,1)) - h(j).*A*[ fy( tt(i)+c(1)*h(j), xi_1 ) 0;
                                              0 fy( tt(i)+c(2)*h(j), xi_2 ) ];
            delta_xi=-J\F;
            xi_1 = xi_1 + delta_xi(1);
            xi_2 = xi_2 + delta_xi(2);
            FF=[xi_1 ; xi_2 ] -[yy(i); yy(i)] - ...
                  h(j).*A*[ f( tt(i)+c(1)*h(j), xi_1 ); f( tt(i)+c(2)*h(j), xi_2 ) ];
        else
            yy(i+1)=yy(i)+h(j)*(b(1)*f(tt(i)+...
                c(1)*h(j), xi_1)+b(2)*f(tt(i)+c(2)*h(j), xi_2));
            break
        end
    end

end

l(j)=yy(length(tt));
end

```

```
plot(tt,yy,'b')
xlabel('t');
ylabel('y')
title(['Solution when step size is h= ', num2str(h(j))]);
```

*%compute order of the method*

```
error1=abs(l(1)-l(3));
error2=abs(l(2)-l(3));
p= log(error1/error2)/log(h(1)/h(2));
fprintf('The order of Gauss-Legendre is p=%12.8f\n', p)
```