Chapter 9. Adequacy

In this chapter, we discuss the connection between the intuitive notions of logical validity and logical truth on the one hand and truth-functional consequence and truth-functional validity on the other. Since the material is entirely philosophical, it may be omitted by someone only interested in technical details.

Our aim is to show that the definitions of truth-functional consequence and validity are <u>adequate</u>. Two senses of "adequate" might be intended. The definitions might be adequate in the <u>intensional</u> sense; they provide or at least correspond to the correct analysis of the informal notions. Or the definitions might be adequate in the <u>extensional</u> sense; they pick out the same collection of argument pair schemas or formulas as the informal notions. So in the case of logical truth, adequacy amounts to the claim that, for any formula **A** of the language $\mathcal{L}$, **A** is truth-functionally valid iff it is logically true. In the case of logical validity it amounts to the claim that **A** is a truth-functional consequence of Γ iff the argument-schema Γ ∴ **A** is logically valid.

It is important to appreciate that, in these extensional claims of adequacy, an intuitive notion (viz., logical truth or validity) appears on one side and a technical counterpart (viz., truth-functional validity or consequence) on the other side. In this respect, the claims differ from the completeness result, in which two technical notions are linked by an equivalence. Thus although the completeness result is susceptible of mathematical proof, the most that can be hoped for the adequacy claims is some sort of philosophical justification.

We shall devote ourselves entirely to the question of extensional adequacy and ignore the much more difficult question of intensional adequacy. Moreover, we shall concentrate on the case of logical truth and only consider the case of logical validity towards the end of the chapter. We therefore have to assess two conditionals:

(1) Any logically true formula is truth-functionally valid;

(2) Any truth-functionally valid formula is logically true.

The first of these conditionals is readily established. For suppose the formula **A** is not truth-functionally valid. Then $\alpha \nvDash \mathbf{A}$ for some valuation α. Let **S** be a true sentence; and let $\mathbf{A}^*$ be the result of substituting **S** for those sentence-letters of **A** true in α and **it is not the case that** S for those sentence letters of **A** not true in α. Since the truth-conditions respect the informal semantics, $\mathbf{A}^*$ is false. But any logically true sentence is true. Therefore $\mathbf{A}^*$ is not logically true and hence neither is the formula **A**.

The other conditional is much harder to establish. Suppose that the formula **A** is not logically true. Let us assume

(*) If a formula is not logically true then some concrete instance of it is false.

Then some concrete instance **S** of **A** is false. Let α be the valuation that makes a sentence-letter **P** true iff **S** is obtained from **A** by substituting a true sentence for **P**. Since the truth-conditions respect the informal semantics, $\alpha \vDash \mathbf{A}$ iff **S** is true. But **S** is false. So $\alpha \nvDash \mathbf{A}$, and **A** is not valid.

The weak point in the above justification is assumption (*). Some philosophers have claimed that a sentence is logically true, by definition, if any sentence of the same logical form is true. But it seems perfectly conceivable, either because of some poverty in the language or for some other reason, that each sentence of a given logical form could be true without any one of them being true in virtue of that form. It would therefore be preferable to avoid the dependence on (*).

One way of doing so is to take advantage of the completeness theorem of chapter 6. By this result, every valid formula is a theorem. Hence it suffices, to establish the hard direction of the adequacy claim, to show that every theorem is logically true. But this can be done by verifying that each axiom is logically true and that the rule of inference, modus ponens, preserves logical truth. Thus a case by case check replaces the general assumption of (*).

Let us now turn to the question of argument pair validity. In analogy with (1) and (2) above, it must be shown, for $\Delta=\{\mathbf{A}_1,...,\mathbf{A}_n\}$ a set of formulas with n>0, that:

(1') If the argument-schema $\Delta \therefore \mathbf{B}$ is logically valid then **B** is a truth-functional consequence of Δ;

(2') If **B** is a truth-functional consequence of Δ then the argument-schema $\Delta \therefore \mathbf{B}$ is logically valid.

The first of these claims may be established in the same straightforward way as its counterpart for logical truth. For if **B** is not a truth functional consequence of Δ, then there is a valuation α under which Δ is true and **B** is false; and this valuation may be used to construct a concrete instance of the argument pair in which the premisses are actually true and the conclusion actually false.

The second claim is even more problematic than its counterpart (2) for logical truth. Let us assume:

(**) If $\mathbf{A}_1 \supset ... \supset \mathbf{A}_{n+1}$ is logically true, then the argument $\Delta \therefore \mathbf{A}_{n+1}$ (for $\Delta=\{\mathbf{A}_1,...,\mathbf{A}_n\}$) is logically valid.

Then (2') can be reduced to (2). For suppose **B** is a truth functional consequence of **Δ**. Then by the corollary to theorem 4.3, $\mathbf{A_1 \supset ... \supset A_{n+1}}$ is truth functionally valid. By (2), $\mathbf{A_1 \supset ... \supset A_{n+1}}$ is logically true; and so by (**), the argument schema **Δ** ∴ **B** is logically valid.

Unfortunately, assumption (**) is not universally accepted. There are those who do not believe that the notions of logical truth and argument validity are connected in the simple way that the assumption seems to require. An especially problematic case is provided by the argument from **p∧¬p** to **q**. (We might take **p** to be **Snow is white** and **q** to be **2+2=4** in a concrete instance.) It is granted that the formula **(p∧¬p)⊃q** is logically true. But it is denied that the argument **(p∧¬p)** ∴ **q** is valid on the grounds that an appropriate relevant connection between premiss and conclusion is lacking.

This issue will not be taken up in this book. But the interested reader might like to consult [Anderson and Belnap] and [Anderson and Belnap].

<u>Drills, exercises and problems</u>

2**[e]**. a. Give an argument to justify the claim that **(p∨¬p)** is a logically true form
b. Give an argument to justify the claim that **(p∨there are fewer than twelve planets)** is not a logically true form.

3**[e]**. How might one argue for the logical truth of the axioms A1-A6? How might this differ from arguing just for the truth of A1-A6?

4**[e]**. Using the results of exercise 3 above, write out the details of the justification of 2 without appeal to * that was sketched in the text. Hint: The idea is to argue by induction on the length of derivations that the argument from the assumptions of a line in a derivation to the formula on that line is always correct. If the derivation is of length one it must consist of an axiom (based on no assumptions) or a formula (based only on itself as an assumption). In the first case you can use the results of exercise 3. If the derivation is of length n, for n>1, then the last line is either an axiom or a formula that follows from previous lines by modus ponens. The derivation of these previous lines is of length less than n, so we can assume that the claim holds for them.

5**[e]**. Suppose that informal account of the semantics of $\mathscr{L}$ were changed so that **S∨T** was understood as **either it is necessarily the case that** S **or it is necessarily the case that T.** Explain why SL would not be adequate.

6**[p]**. (Post completeness) a. Let **L** be any system obtained from **SL** by adding some new axiom-schemas. Prove Post's theorem, which says that **L** has the same theorems as **SL** or has all formulas as theorems.

b. Use this result to give an alternative justification of the adequacy claim.