Graphical solution of inequalities

Introduction

Graphs can be used to solve inequalities. This leaflet illustrates how.

1. Solving inequalities

We start with a very simple example which could be solved very easily using an algebraic method.

Example

Solve the inequality \( x + 3 > 0 \).

Solution

We seek values of \( x \) which make \( x + 3 \) positive. There are many such values, e.g. try \( x = 7 \) or \( x = -2 \). To find all values first let \( y = x + 3 \). Then the graph of \( y = x + 3 \) is sketched as shown below. From the graph we see that the \( y \) coordinate of any point on the line is positive whenever \( x \) has a value greater than \(-3\). That is, \( y > 0 \) when \( x > -3 \). But \( y = x + 3 \), so we can conclude that \( x + 3 \) will be positive when \( x > -3 \). We have used the graph to solve the inequality.

Example

Solve the inequality \( x^2 - 2x - 3 > 0 \).

Solution

We seek values of \( x \) which make \( x^2 - 2x - 3 \) positive. We can find these by sketching a graph of \( y = x^2 - 2x - 3 \). To help with the sketch, note that by factorising we can write \( y \) as \((x+1)(x-3)\). The graph will cross the horizontal axis when \( x = -1 \) and when \( x = 3 \). The graph is shown above on the right. From the graph note that the \( y \) coordinate of a point on the graph is positive.
when either \( x \) is greater than 3 or when \( x \) is less than \(-1 \). That is, \( y > 0 \) when \( x > 3 \) or \( x < -1 \) and so:

\[
x^2 - 2x - 3 > 0 \quad \text{when} \quad x > 3 \quad \text{or} \quad x < -1
\]

**Example**

Solve the inequality \((x - 1)(x - 2)(x - 3) > 0\).

**Solution**

We consider the graph of \( y = (x - 1)(x - 2)(x - 3) \) which is shown below. It is evident from the graph that \( y \) is positive when \( x \) lies between 1 and 2 and also when \( x \) is greater than 3. The solution of the inequality is therefore \( 1 < x < 2 \) and \( x > 3 \).

**Example**

For what values of \( x \) is \( \frac{x+3}{x-7} \) positive?

**Solution**

The graph of \( y = \frac{x+3}{x-7} \) is shown below. We can see that the \( y \) coordinate of a point on the graph is positive when \( x < -3 \) or when \( x > 7 \).

\[
\frac{x+3}{x-7} > 0 \quad \text{when} \quad x < -3 \quad \text{or when} \quad x > 7
\]

For drawing graphs like this one a graphical calculator is useful.