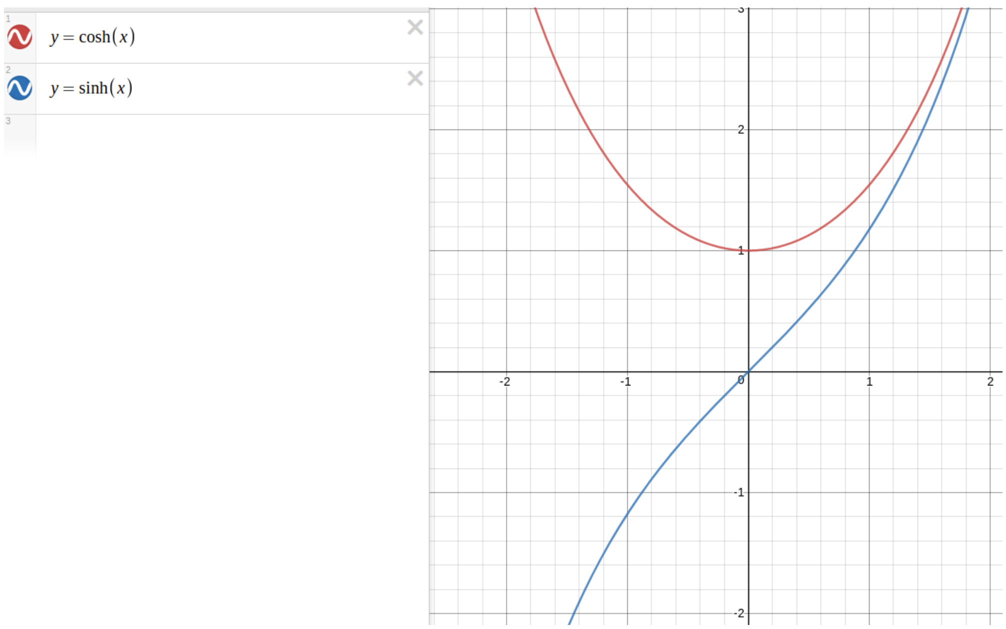
NOTHING YOU NEED TO KNOW ABOUT HYPERBOLIC (AND REGULAR) TRIG FUNCTIONS

1. Hyperbolic trig functions

The hyperbolic trig functions are defined by

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$

(They usually rhyme with 'pinch' and 'posh'.) As you can see, sinh is an odd function, and cosh is an even function. Moreover, cosh is always positive, and in fact always greater than or equal to 1. Unlike the ordinary ("circular") trig functions, the hyperbolic trig functions don't oscillate. Rather, both grow like $e^t/2$ as $t \to \infty$, and $\pm e^{-t}/2$ as $t \to -\infty$.



The derivatives of the hyperbolic trig functions are

$$\frac{d}{dt}\sinh(t) = \cosh(t), \quad \frac{d}{dt}\cosh(t) = \sinh(t).$$

Their integrals are just as easy.

$$\int \sinh(t)\,dt = \cosh(t) + C, \quad \int \cosh(t)\,dt = \sinh(t) + C.$$

We see that these two functions are solutions to the second-order differential equation

$$y'' = y.$$

In fact, they're a fundamental system of solutions – can you see why?

Since their derivatives oscillate like this, we can easily compute their Taylor series around $t = 0$.

$$\cosh(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots$$
$$\sinh(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$$

These power series converge everywhere. Note that

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \cosh(t) + \sinh(t),$$

which we knew from the definitions anyway.

One can prove addition formulas:

$$\cosh(s+t) = \cosh(s)\cosh(t) + \sinh(s)\sinh(t),$$
$$\sinh(s+t) = \sinh(s)\cosh(t) + \cosh(s)\sinh(t).$$

From this we get double-angle formulas:

$$\cosh(2t) = \cosh(t)^2 + \sinh(t)^2,$$
$$\sinh(2t) = 2\sinh(t)\cosh(t),$$

and half-angle formulas:

$$\cosh(t/2) = \sqrt{\frac{\cosh(t)+1}{2}},$$
$$\sinh(t/2) = \sqrt{\frac{\cosh(t)-1}{2}}.$$

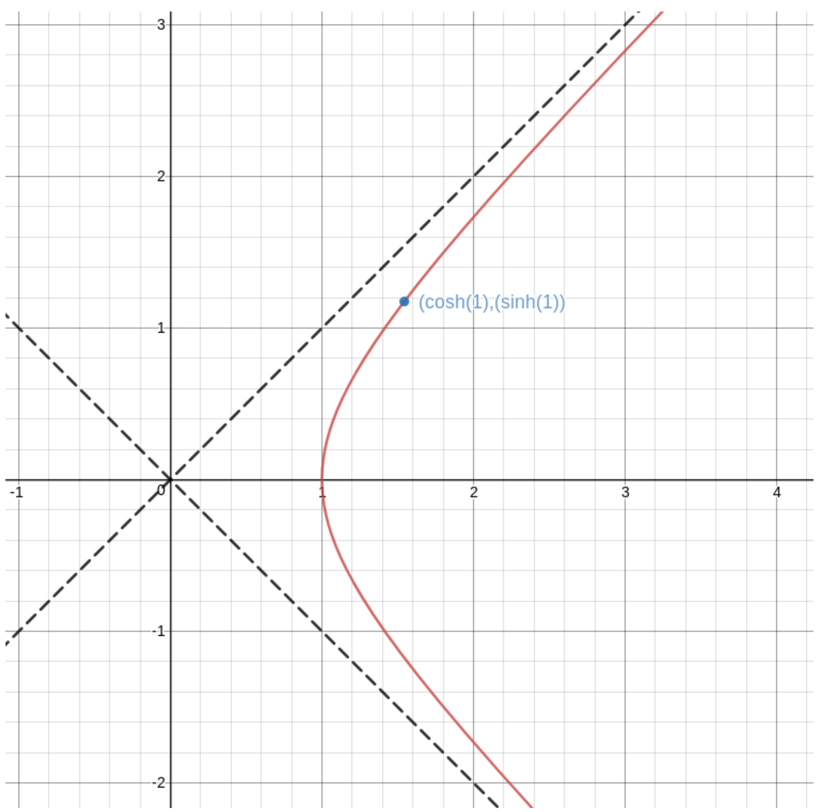
(To be precise, you have to use the fundamental identity in the next paragraph to prove these last ones.

The fundamental identity linking the two hyperbolic trig functions is

$$\cosh(t)^2 - \sinh(t)^2 = 1.$$

In other words, as t varies from $-\infty$ to ∞, $(\cosh(t), \sinh(t))$ traces out the graph of the hyperbola

$$x^2 - y^2 = 1, \quad x > 0.$$



This parametrization is particularly useful in special relativity, for reasons I'll only briefly explain. Let's say that the x-axis represents time, as measured by some clock (i.e. 'reference frame') that we take to be at rest; and the y-axis represents spatial motion in one fixed direction. If you start at $(0,0)$ and stay at rest, relative to the clock, for one second, you'll reach the spacetime point $(1,0)$. If you move in the y-direction, you'll reach other points on the graph. Since you can't move faster than the speed of light – which we'll say is 1 – you have to stay inside the cone $|y| < x$. Since you can't move backward in time, you have to stay in $x \geq 0$.

One of the basic weirdnesses of relativity is that, if you move very fast relative to the clock, you will experience time slower than the clock does. So one can ask: what is the set of spacetime points (x, y) that you can reach in one second *as you measure it*? The answer is: the hyperbola

$$x^2 - y^2 = 1, \quad x > 0!$$

Points far from the origin on this hyperbola correspond to speeds close to the speed of light, which is why the points are close to the forbidden lines $|y| = \pm x$. At these speeds, time dilates for you significantly: while you measure your motion as taking one second, the clock measures it as taking x seconds, where x is however big.

If we used *another* clock to measure time – maybe one moving at a high speed relative to the first clock – its coordinate system for spacetime would look rather different. However, the concept of 'spacetime points you can reach in one second as you measure it' doesn't have anything to do with any external clock. This means that the hyperbola $x^2 - y^2 = 1$ is *preserved* under the kinds of changes of coordinates (technical term: 'Lorentz transformations') which are meaningful in special relativity. One should compare this to a more familiar idea: circles around the origin are preserved under rotations around the origin. In other words, hyperbolas are relativistic circles. And just like rotations around the origin can be described using matrices of trig functions,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

these relativistic coordinate changes look like

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}.$$

Finally, there are a whole gamut of functions that can be defined in terms of these two: $\tanh(t)$, $\operatorname{sech}(t)$, and so on. I leave it to you to define them and discover their properties.

2. Circular trig functions

Since sinh and cosh were defined in terms of the exponential function that we know and love, proving all the properties and identities above was no big deal. On the other hand, you spent a pretty big piece of your mathematical career, maybe even a whole year of trig, studying the sine and cosine function. But if you know a little about complex numbers, the circular trig functions become just as simple as the hyperbolic ones. We'll see this by going through the same ideas we just did in the same order.

The basic idea, discussed in class already, is that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

We took this as a definition of complex exponentiation. However, we could reverse things by assuming that we already understand complex exponentiation, and deducing the properties of the trig functions from it. So, we *define*, for a real number θ,

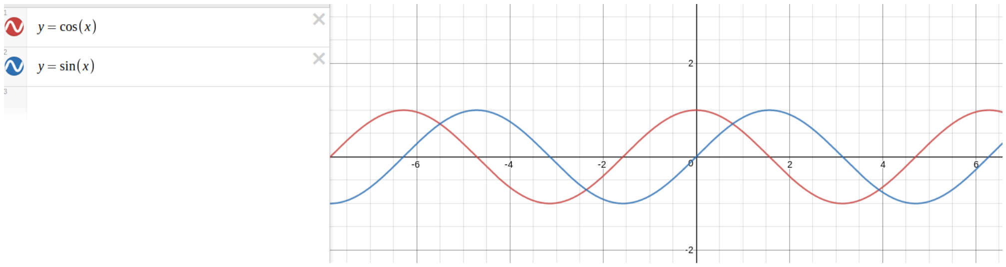
$$\sin(\theta) = \operatorname{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos(\theta) = \operatorname{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

We see from these formulas that sin is even, while cos is odd.

Note that

$$\sin(\theta) = \sinh(i\theta)/i, \quad \cos(\theta) = \cosh(i\theta).$$

So most of what follows could be deduced from the previous one, rather than directly from the complex exponential function.



How about a differential equation that sin and cos satisfy? Well, $y(\theta) = e^{i\theta}$ satisfies the complex-coefficient differential equation

$$y' = iy,$$

and differentiating again gives the real-coefficient differential equation

$$y'' = -y.$$

As we saw in class, if a complex-valued function satisfies a linear homogeneous differential equation with real coefficients, so do its real and imaginary parts. So sine and cosine are also solutions to $y'' = -y$. In fact, they are a fundamental system of solutions.

As for Taylor series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Substituting $i\theta$ for x, we get

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{theta^4}{4!} + \cdots .$$

Taking the real and imaginary parts of both sides gives

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots .$$

Finally,

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}.$$

In other words,

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos(\alpha) + i\sin(\alpha))\,(\cos(\beta) + i\sin(\beta))\,.$$

Taking the real and imaginary parts of both sides gives the expected addition formulas,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

From these we can get the double-angle formulas

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha),$$
$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha),$$

and (eventually) the half-angle formulas

$$\cos(\alpha/2) = \sqrt{\frac{1+\cos(\alpha)}{2}}$$
$$\sin(\alpha/2) = \sqrt{\frac{1-\cos(\alpha)}{2}}.$$

(Again, we have to use the fundamental identity below to get the half-angle formulas.)

We got all this from basic properties of the function $e^{i\theta}$, i. e. the fact that it behaves like an exponential function. At this point we have to bring in a non-basic property: Euler's identity

$$e^{2\pi i} = 1.$$

This can't really be deduced from anything else we've said, and has to do with the specific numbers e and π – and ordinarily, it would be proved based on the definition of $e^{i\theta}$ from trig functions, whereas we're trying to do things backwards. So let's assume we also know this, for some reason. What does it tell us about the trig functions? Well, it implies that

$$e^{i(\theta+2\pi)} = e^{i\theta},$$

which implies that sin and cos are 2π-periodic. This is a major difference from sinh and cosh.

Additionally, it implies that

$$|e^{i\theta}| = 1$$

for all θ. One way to see this is to notice that if we raise $|e^{i\theta}|$ to the power $2\pi/\theta$, which is a nonzero real number (for $\theta \neq 0$), you get $|e^{2\pi i}| = 1$. But $|e^{i\theta}|$ is a positive real number, so it has to be 1 as well. This, in turn, proves the fundamental identity

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

And this then implies that

$$|\cos(\theta)| \leq 1, \quad |\sin(\theta)| \leq 1.$$

Again, this is a major difference from the hyperbolic trig functions.

Since $|e^{i\theta}| = 1$ for all θ, the function $e^{i\theta}$ traces out the unit circle in the complex plane. So, $(\cos(\theta), \sin(\theta))$ are the coordinates of a point on that circle. We can define a system of angle measure by saying that the ray going through $(\cos(\theta), \sin(\theta))$ and the ray going through $(1, 0)$ are separated by an angle θ. This forces 2π to equal a full circle or $360°$. Of course, this is how radians are defined.

Once we know that the trig functions describe points on a circle, we can use basic geometry to recover their 'classical' definition in terms of ratios between sides of right triangles, and the remaining 'trigonal' parts of trigonometry. And again, we can define tan, sec, and all the others. This, too, is left to you.