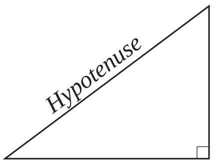
3 Pythagoras' Theorem

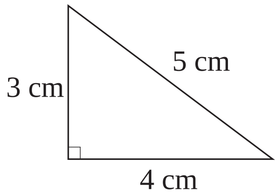
3.1 Pythagoras' Theorem

Pythagoras' Theorem relates the length of the *hypotenuse* of a right-angled triangle to the lengths of the other two sides.



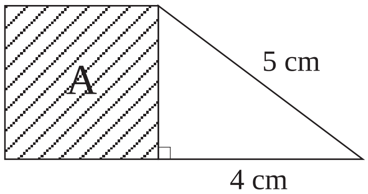
The hypotenuse is always the longest side: it is always the side opposite the right angle.

The diagram opposite shows a right-angled triangle. The length of the hypotenuse is 5 cm and the other two sides have lengths 3 cm and 4 cm.



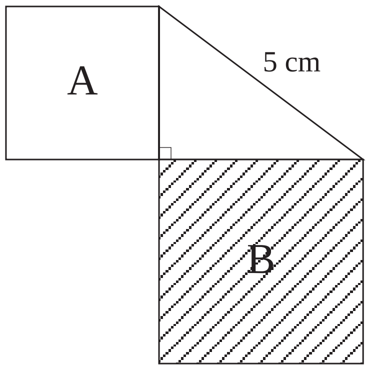
In this diagram, a square, A, has been drawn on the 3 cm side.

$$\text{Area of square A} = 3 \times 3$$
$$= 9 \text{ cm}^2$$



In this diagram, a second square, B, has been drawn on the 4 cm side.

$$\text{Area of square B} = 4 \times 4$$
$$= 16 \text{ cm}^2$$

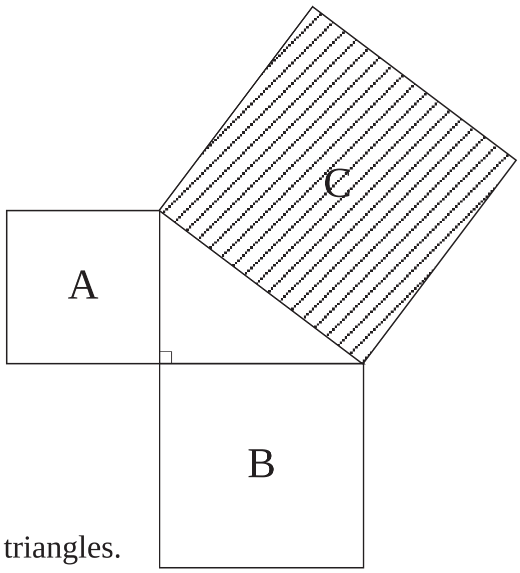


Squares A and B together have total area:

$$\text{Area A} + \text{Area B} = 9 + 16$$
$$= 25 \text{ cm}^2$$

Finally, a third square, C, has been drawn on the 5 cm side.

$$\text{Area of square C} = 5 \times 5$$
$$= 25 \text{ cm}^2$$

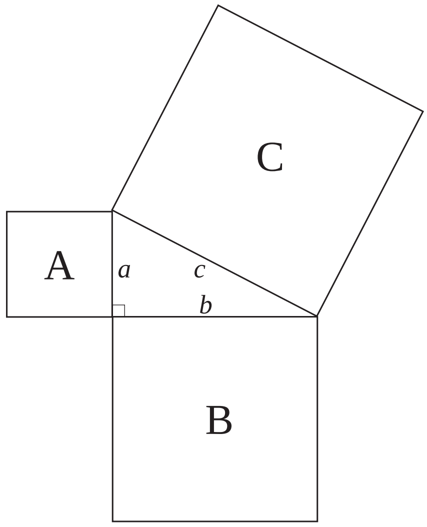


We can see that

$$\text{Area A} + \text{Area B} = \text{Area C.}$$

This formula is *always* true for right-angled triangles.

We now look at a right-angled triangle with sides a, b and c, as shown opposite.



$$\begin{aligned} \text{Area A} &= a \times a \\ &= a^2 \\ \text{Area B} &= b \times b \\ &= b^2 \\ \text{Area C} &= c \times c \\ &= c^2 \end{aligned}$$

So,

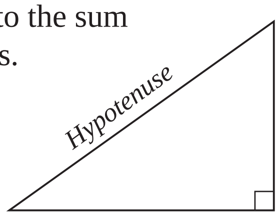
$$\text{Area A} + \text{Area B} = \text{Area C}$$

gives us the formula

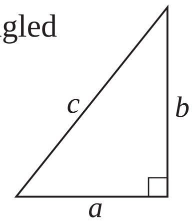
$$\boxed{a^2 + b^2 = c^2}$$

for all right-angled triangles.

Pythagoras' Theorem states that, for any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two shorter sides.



If we use the letters a, b and c for the sides of a right-angled triangle, then Pythagoras' Theorem states that

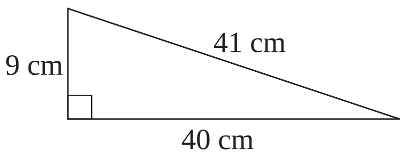


$$\mathrm{a}^2 + b^2 = c^2$$

where c is the length of the hypotenuse.

Example 1

Verify Pythagoras' Theorem for the right-angled triangle opposite:



Solution

Here $a = 9$ cm, $b = 40$ cm, $c = 41$ cm.

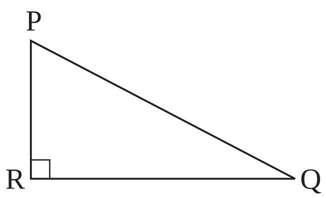
$$
\begin{array}{rcrcrcr}
a^2 & = & 9^2 & = & 9 \times 9 & = & 81 \\
b^2 & = & 40^2 & = & 40 \times 40 & = & 1600 \\
\hline
a^2 + b^2 & & & & & = & 1681 \\
 & & & & & & \\
c^2 & = & 41^2 & = & 41 \times 41 & = & 1681
\end{array}
$$

So $a^2 + b^2 = c^2$ for this triangle.

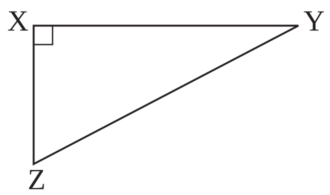
Exercises

1. Which side is the *hypotenuse* in each of the following right angled triangles:

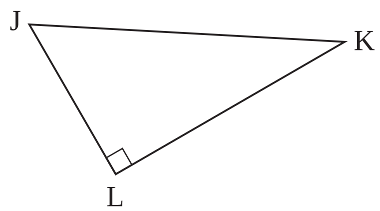
(a)



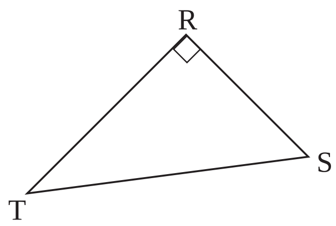
(b)



(c)



(d)



2. For each of the three diagrams at the top of the next page:

(i) calculate the area of square A,

(ii) calculate the area of square B,

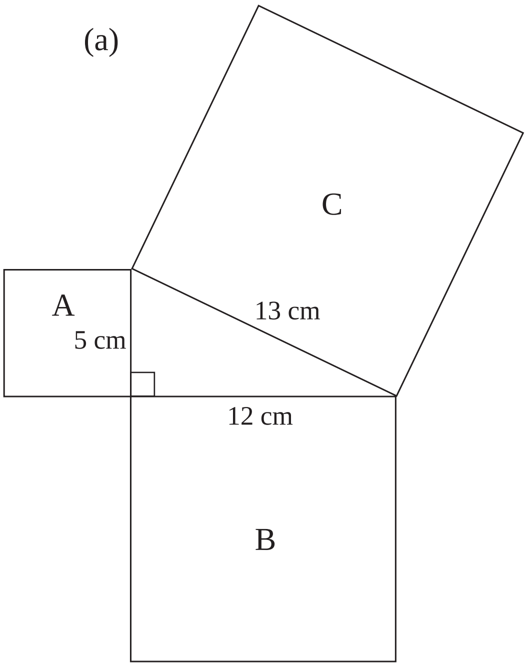
(iii) calculate the sum of area A and area B,

(iv) calculate the area of square C,

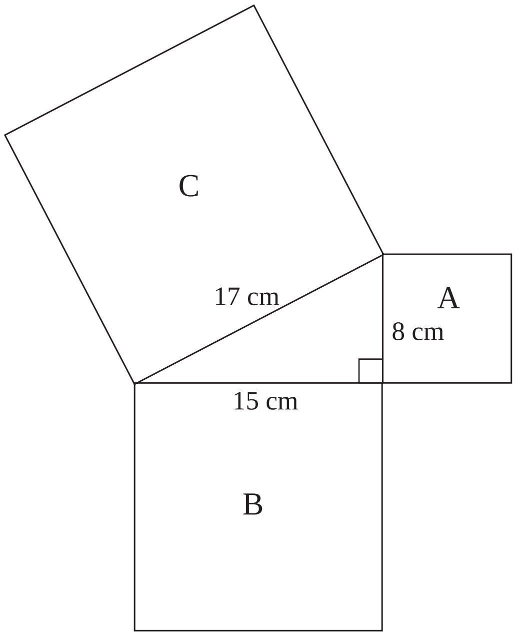
(v) check that :

$$\text{area A} + \text{area B} = \text{area C}$$

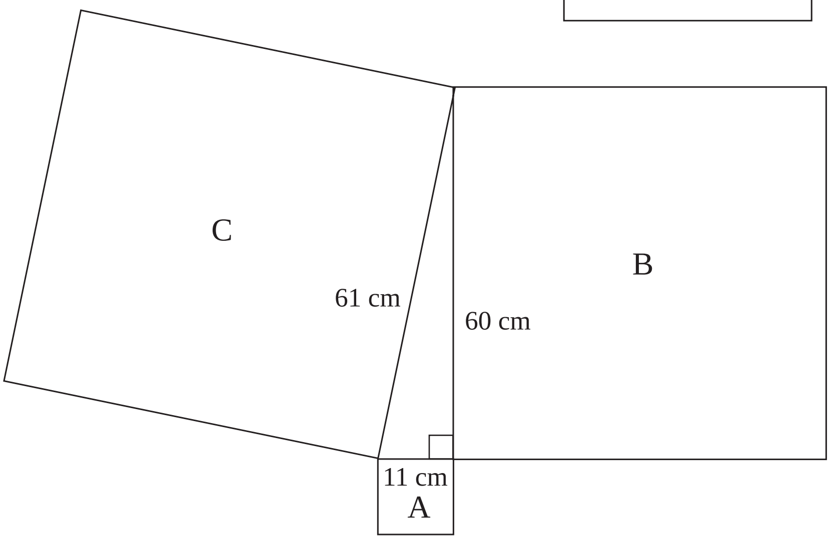
(a)



(b)

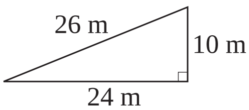


(c)

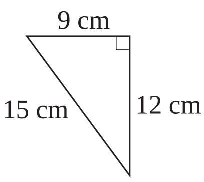


3. Using the method shown in Example 1, verify Pythagoras' Theorem for the right-angled triangles below:

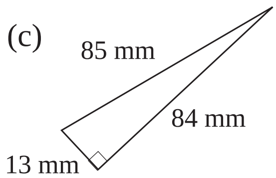
(a)



(b)



(c)



4. The whole numbers 3, 4, 5 are called a *Pythagorean triple* because $3^2 + 4^2 = 5^2$. A triangle with sides of lengths 3 cm, 4 cm and 5 cm is right-angled.

Use Pythagoras' Theorem to determine which of the sets of numbers below are Pythagorean triples:

(a) 15, 20, 25

(b) 10, 24, 26

(c) 11, 22, 30

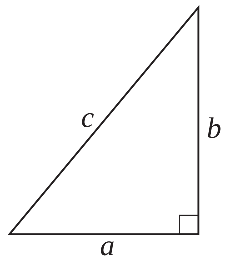
(d) 6, 8, 9

3.2 Calculating the Length of the Hypotenuse

Pythagoras' Theorem states that, for a right-angled triangle,

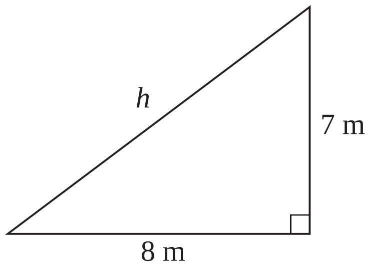
$$c^2 = a^2 + b^2$$

With this result it is very easy to calculate the length of the hypotenuse of a right-angled triangle.



Example 1

Calculate the length of the hypotenuse of a triangle in which the other two sides are of lengths 7 m and 8 m.



Solution

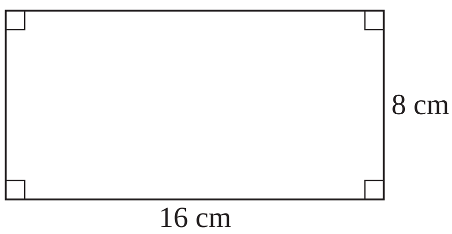
Let h be the length of the hypotenuse.

By Pythagoras' Theorem,

$$\begin{aligned} h^2 &= 8^2 + 7^2 \\ h^2 &= 64 + 49 \\ h^2 &= 113 \\ h &= \sqrt{113} \\ h &= 10.63014581 \text{ m} \\ h &= 10.6 \text{ m, correct to 1 decimal place} \end{aligned}$$

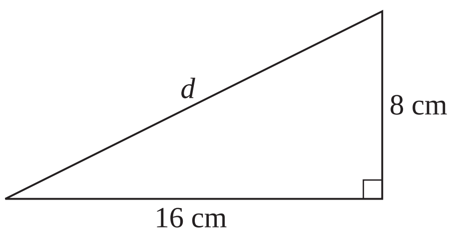
Example 2

Calculate the length of the diagonals of the rectangle opposite:



Solution

The diagram shows the right-angled triangle that you need to use to find the length of the diagonal. The hypotenuse is the diagonal of the rectangle and this is labelled d on the diagram.



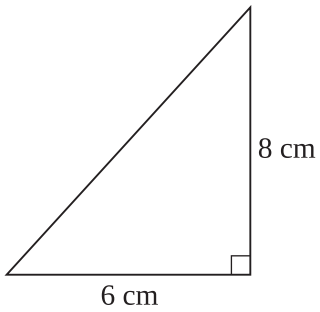
By Pythagoras' Theorem,

$$\begin{aligned} d^2 &= 16^2 + 8^2 \\ &= 256 + 64 \\ &= 320 \\ d &= \sqrt{320} \\ d &= 17.88854382 \text{ cm} \\ d &= 17.9 \text{ cm, correct to 1 decimal place} \end{aligned}$$

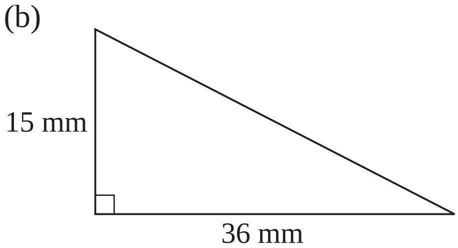
Exercises

1. Calculate the length of the hypotenuse of each of these triangles:

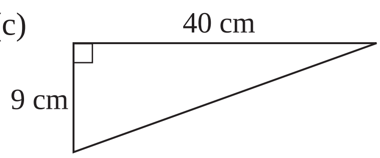
(a)



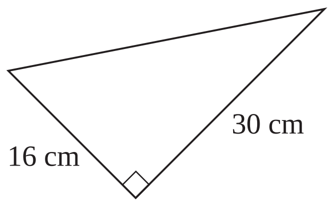
(b)



(c)

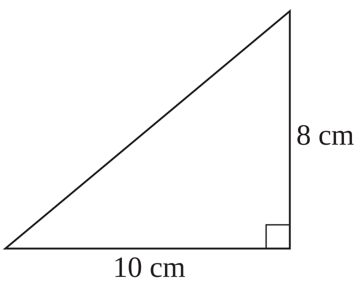


(d)

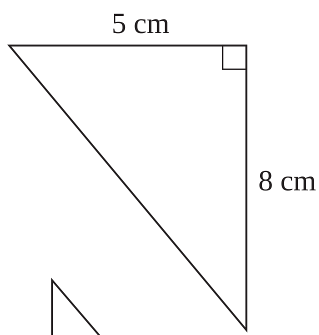


2. Calculate the length of the hypotenuse of each of the following triangles, giving your answers correct to 1 decimal place.

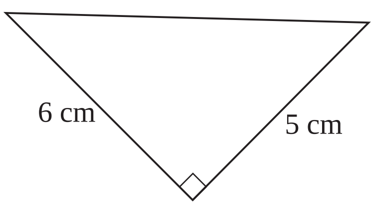
(a)



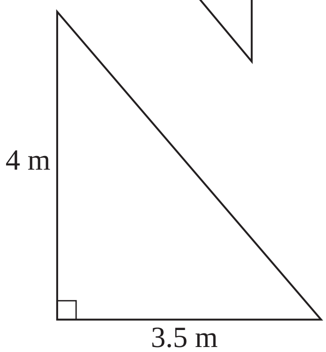
(b)



(c)



(d)



3. A rectangle has sides of lengths 5 cm and 10 cm.

   How long is the diagonal of the rectangle?

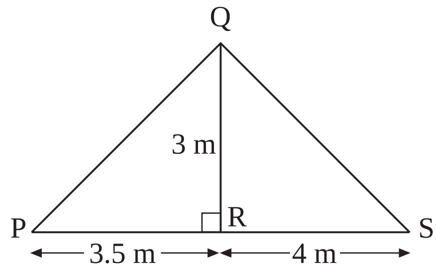
4. Calculate the length of the diagonal of a square with sides of length 6 cm.

5. The diagram shows a wooden frame that is to be part of the roof of a house:

   (a) Use Pythagoras' Theorem in triangle PQR to find the length PQ.

   (b) Calculate the length QS.

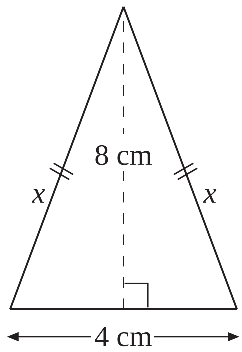
   (c) Calculate the total length of wood needed to make the frame.



6. An isosceles triangle has a base of length 4 cm and perpendicular height 8 cm. Giving your answers correct to 1 decimal place, calculate:

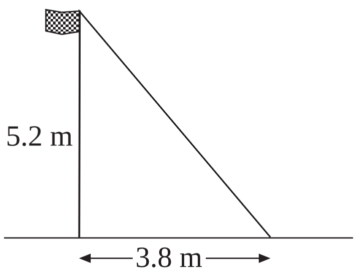
   (a) the length, x cm, of one of the equal sides,

   (b) the perimeter of the triangle.



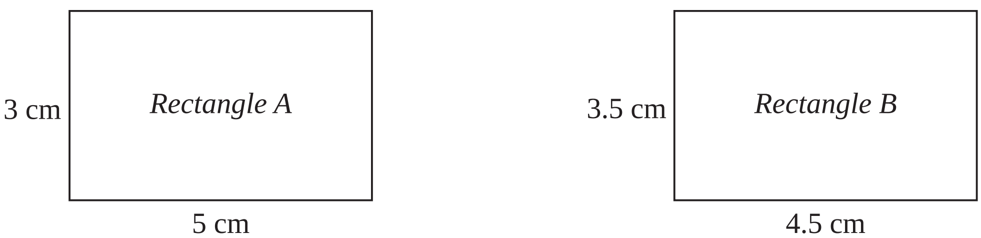
7. One end of a rope is tied to the top of a vertical flagpole of height 5.2 m. When the rope is pulled tight, the other end is on the ground 3.8 m from the base of the flagpole.

   Calculate the length of the rope, giving your answer correct to 1 decimal place.

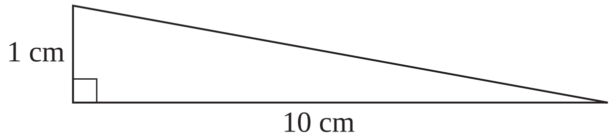


8. A rectangular lawn is 12.5 m long and 8 m wide. Matthew walks diagonally across the lawn from one corner to the other. He returns to the first corner by walking round the edge of the lawn. How much further does he walk on his return journey?

9. Which of the rectangles below has the longer diagonal?

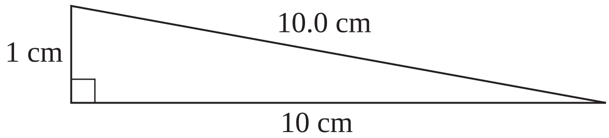


10.



(a) Use Pythagoras' Theorem to show that the length of the hypotenuse of this triangle is 10.0 cm correct to 1 decimal place.

(b) Maxine says that this triangle is isosceles because there are two sides of the same length.

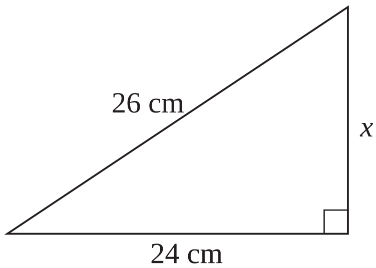


Is Maxine correct?

3.3 Calculating the Lengths of Other Sides

Example 1

Calculate the length of the side marked x in the following triangle:



Solution

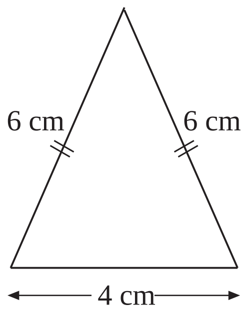
By Pythagoras' Theorem:

$$
\begin{aligned}
x^2 + 24^2 &= 26^2 \\
x^2 + 576 &= 676 \\
x^2 &= 676 - 576 \\
x^2 &= 100 \\
x &= \sqrt{100} \\
x &= 10
\end{aligned}
$$

The length of the side x is 10 cm.

Example 2

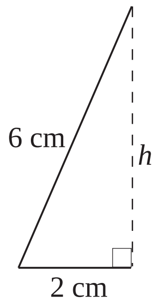
Calculate the perpendicular height of the isosceles triangle shown opposite:



Solution

The height can be calculated by using half of the original isosceles triangle, as shown:

The height has been labelled h on the diagram.



By Pythagoras' Theorem:

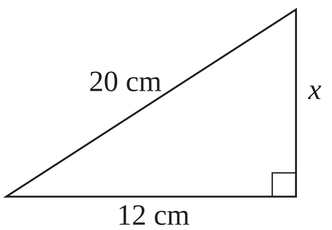
$$
\begin{aligned}
h^2 + 2^2 &= 6^2 \\
h^2 + 4 &= 36 \\
h^2 &= 36 - 4 \\
h^2 &= 32 \\
h &= \sqrt{32} \\
h &= 5.656854249
\end{aligned}
$$

The perpendicular height of the triangle is 5.7 cm to 1 decimal place.

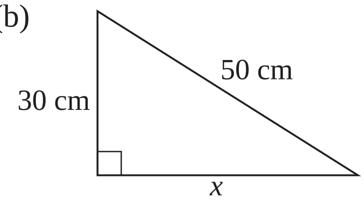
Exercises

1. Calculate the length of the side marked x in each of the following triangles:

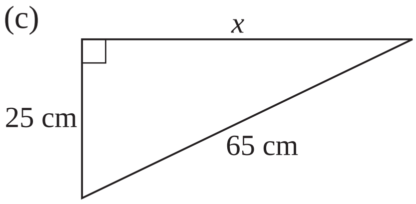
(a)



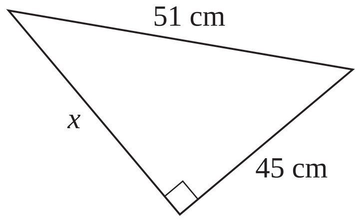
(b)



(c)

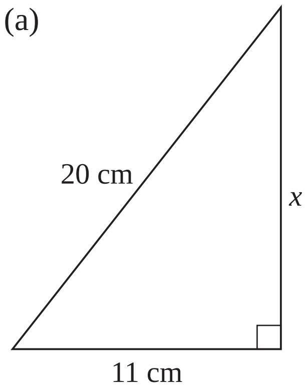


(d)

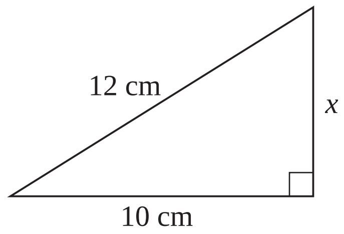


2. Calculate the length of the side marked x in each of the following triangles, giving your answer correct to 1 decimal place:

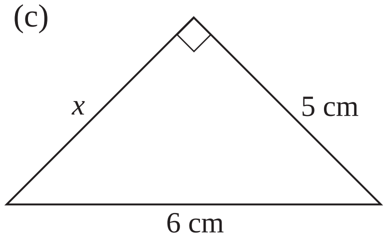
(a)



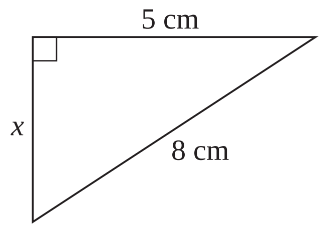
(b)



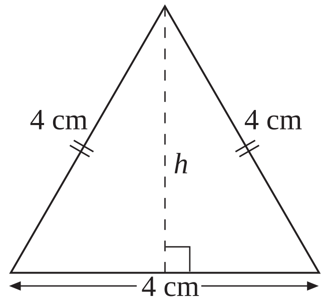
(c)



(d)

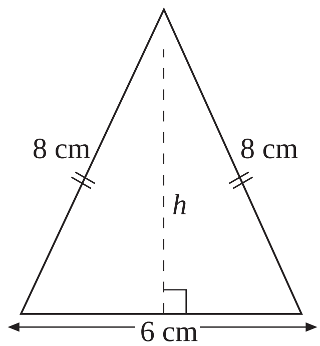


3. Calculate the perpendicular height of this equilateral triangle, giving your answer correct to 1 decimal place.



4. Calculate the perpendicular height of an equilateral triangle with sides of length 5 cm, giving your answer correct to 1 decimal place.

5. Calculate the perpendicular height of the isosceles triangle shown opposite, giving your answer correct to 1 decimal place.

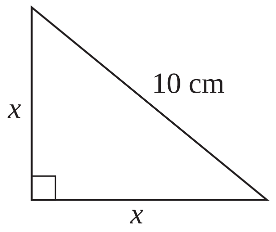


6. The width of a rectangle is 5 cm and the length of its diagonal is 13 cm.

   (a) How long is the other side of the rectangle?

   (b) What is the area of the rectangle?

7. The isosceles triangle at the top of the next page has 2 sides of length x cm. Copy and complete the calculation to find the value of x correct to 1 decimal place.



By Pythagoras' Theorem,

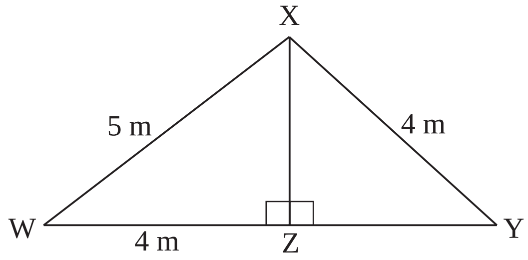
$$\begin{aligned} x^2 + x^2 &= 10^2 \\ 2x^2 &= 100 \\ x^2 &= \\ x &= \sqrt{} \\ x &= \\ x &= \quad \text{to 1 decimal place.} \end{aligned}$$

8. The length of the diagonal of a square is 8 cm. How long are the sides of the square?

9. The diagram shows part of the framework of a roof.

   (a) Calculate the length XZ.

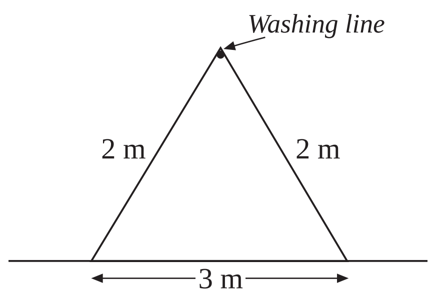
   (b) Calculate the length of YZ, correct to 1 decimal place.



10. A sheet is stretched over a washing line to make a tent, as shown in the diagram.

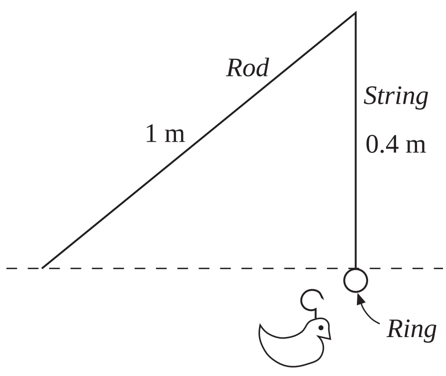
    (a) How high is the washing line above the ground? Give your answer to 1 decimal place.

    (b) If the same sheet was used and the washing line was now at a height of 1.25 m above the ground, what would be the width of the base of the tent? Give your answer correct to 1 decimal place.



11. A fishing rod is used to catch plastic ducks in a fairground game. The rod is 1 m long. A string with a ring is tied to the end of the rod. The length of the string is 0.4 m.

    When the ring is level with the lower end of the rod, as shown in the diagram, how far is the ring from that end of the fishing rod?



3.4 Problems in Context

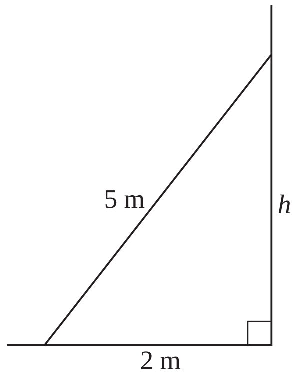
When we use Pythagoras' Theorem to solve problems in context the first key step is to draw a right-angled triangle.

Example 1

A ladder is 5 m long. The bottom of the ladder is 2 m from the foot of a wall and the top leans against the wall. How high is the top of the ladder above the ground?

Solution

The first step is to draw a triangle to represent the situation. The height to the top of the ladder has been labelled *h*. (We assume that the ground is horizontal and the wall is vertical.)



Now use Pythagoras' Theorem:

$$h^2 + 2^2 = 5^2$$

$$h^2 + 4 = 25$$

$$h^2 = 25 - 4 = 21$$

$$h = \sqrt{21}$$

$$h = 4.582575695$$

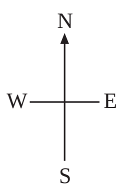
The top of the ladder is 4.58 m above the ground (to the nearest cm).

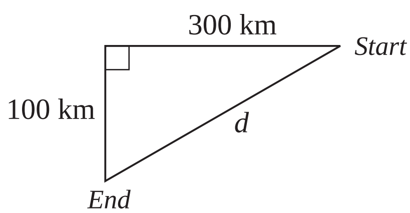
Example 2

A ship sails 300 km due west and then 100 km due south. At the end of this journey, how far is the ship from its starting position?

Solution

The first step is to draw a diagram showing the ship's journey. The distance from the starting point has been labelled *d*.





Now use Pythagoras Theorem:

$$d^2 = 300^2 + 100^2$$

$$d^2 = 90\,000 + 10\,000$$

$$d^2 = 100\,000$$

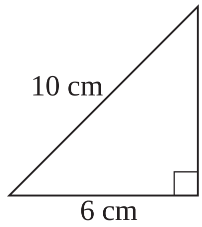
$$d = \sqrt{100\,000}$$

$$d = 316.227766$$

The distance from the starting point is 316 km to the nearest km.

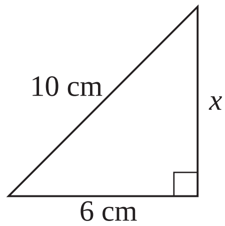
Example 3

Calculate the area of the triangle shown opposite:



Solution

The length of the unknown side has been marked *x*.



Using Pythagoras' Theorem,

$$x^2 + 6^2 = 10^2$$

$$x^2 + 36 = 100$$

$$x^2 = 100 - 36$$

$$x^2 = 64$$

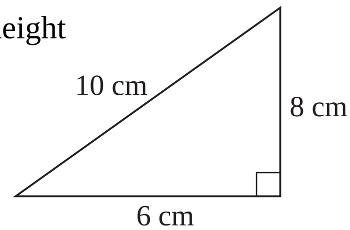
$$x = \sqrt{64}$$

$$x = 8 \text{ cm}$$

$$\text{Area of the triangle} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

$$= \frac{1}{2} \times 6 \times 8$$

$$= 24 \text{ cm}^2$$



Exercises

1. A hiker walks 300 m due north and then 400 m due east. How far is the hiker now from her starting position?

2. A ladder of length 4 m leans against a wall so that the top of the ladder is 3 m above ground level. How far is the bottom of the ladder from the wall?

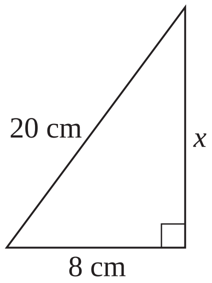
3. Two remote-controlled cars set off from the same position. After a short time one has travelled 20 m due north and the other 15 m due east. How far apart are the two cars?

4. A room should have a rectangular floor, with sides of lengths 4 m and 5 m. A builder wants to check that the room is a perfect rectangle and measures the two diagonals of the room, which should be the same length. To the nearest cm, how long should each diagonal be?

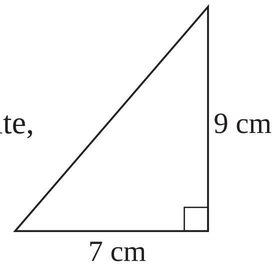
5. For the triangle shown opposite,

   (a) calculate the length x,

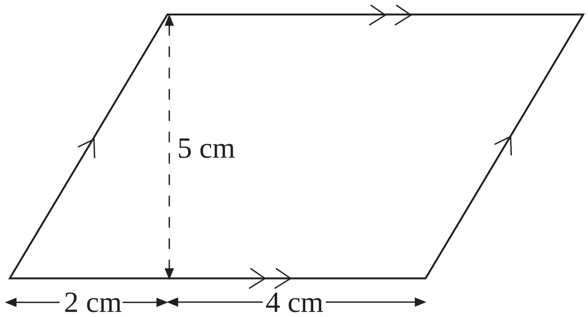
   (b) calculate the area of the triangle.



6. Calculate the perimeter of the triangle shown opposite, giving your answer correct to 1 decimal place.



7. Calculate the perimeter of the parallelogram below, giving your answer to the nearest millimetre.



8. One end of a rope of length 10 m is tied to the top of a vertical flag pole. When the rope is tight it can touch the ground at a distance of 4 m from the base of the pole. How tall is the flagpole? Give your answer correct to the nearest cm.

9. A guy rope on a tent is 1.5 m long. One end is fixed to the top of a vertical pole and the other is pegged to the ground. If the pole is 1.2 m high, how far is the pegged end of the rope from the base of the flagpole?

10. Ron's dad says that Ron must not walk on the lawn. The lawn is a rectangle with sides of lengths 10 m and 16 m. When his dad is looking, Ron walks from his house to the gate by walking along two edges of the lawn. When his dad is not looking, Ron walks diagonally across the lawn.

    How much further does Ron have to walk to get from the house to the gate when his dad is looking? Give your answer to a suitable level of accuracy.

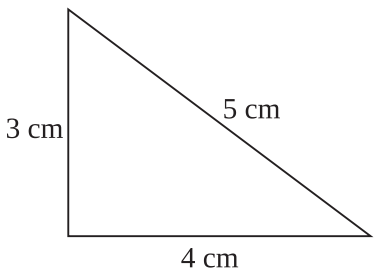
3.5 Constructions and Angles

The formula for Pythagoras' Theorem can be used to decide if a triangle is right-angled.

> In any triangle,
>
> the *longest side* faces the *largest angle*
>
> the *shortest side* faces the *smallest angle.*
>
> In a triangle with longest side *c*, and other two sides *a* and *b*,
>
> if $c^2 = a^2 + b^2$, then the angle opposite $c = 90°$;
>
> if $c^2 < a^2 + b^2$, then the angle opposite $c < 90°$ (so all three angles are acute);
>
> if $c^2 > a^2 + b^2$, then the angle opposite $c > 90°$ (i.e. the triangle has an obtuse angle).

Example 1

(a) Use a ruler and a pair of compasses to construct this triangle:



(b) Use a protractor to check that the triangle has a right angle.

(c) Confirm that Pythagoras' Theorem is true for this triangle.

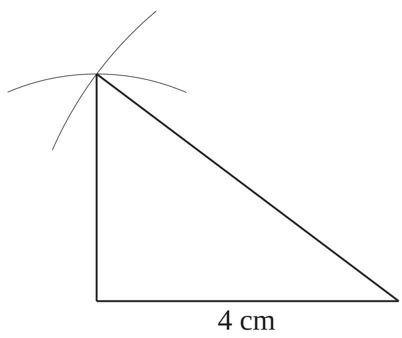
Solution

(a) First draw a line with length 4 cm.

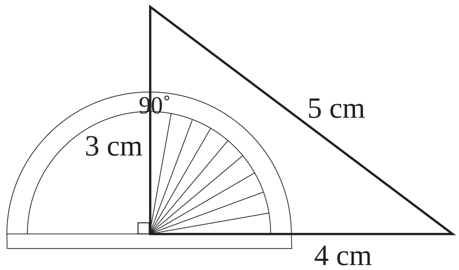
Then draw an arc of radius 3 cm with centre on the left-hand end of the line.

Next draw an arc of radius 5 cm with centre on the right-hand end of the line.

The point where the two arcs cross is the third corner of the triangle.



(b) The angle at the bottom left-hand corner measures $90°$, so the triangle has a right angle.



(c) Here $a = 4$ cm, $b = 3$ cm and $c = 5$ cm.

$$\begin{aligned} a^2 + b^2 &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

$$\begin{aligned} c^2 &= 5^2 \\ &= 25 \end{aligned}$$

Therefore

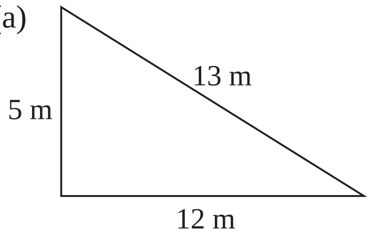
$$a^2 + b^2 = c^2$$

So Pythagoras' Theorem is true in this case, confirming that this is a right-angled triangle.

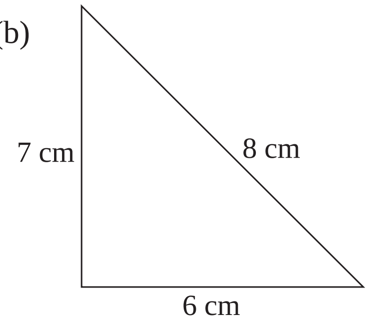
Example 2

Which of these triangles contains a right angle?

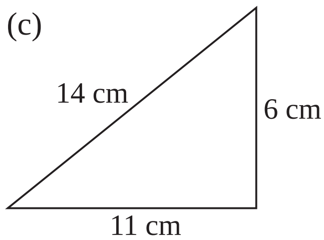
(a)



(b)



(c)



Solution

We use Pythagoras' Theorem to find out if a triangle is right-angled, using c for the longest side.

(a) In this triangle, $a = 5$, $b = 12$ and $c = 13$.

$$\begin{aligned} a^2 + b^2 &= 5^2 + 12^2 \\ &= 25 + 144 \\ &= 169 \end{aligned} \qquad \begin{aligned} c^2 &= 13^2 \\ &= 169 \end{aligned}$$

Here $a^2 + b^2 = c^2$, so this triangle does contain a right angle.

(b) In this triangle, $a = 6$, $b = 7$ and $c = 8$.

$$a^2 + b^2 = 6^2 + 7^2 \qquad c^2 = 8^2$$
$$= 36 + 49 \qquad = 64$$
$$= 85$$

Here $c^2 \neq a^2 + b^2$, so the triangle does not contain a right angle. As $c^2 < a^2 + b^2$, the angle opposite c is less than $90°$, so all the angles in this triangle are acute.

(c) Here $a = 6$, $b = 11$ and $c = 14$.

$$a^2 + b^2 = 6^2 + 11^2 \qquad c^2 = 14^2$$
$$= 36 + 121 \qquad = 196$$
$$= 157$$

Here $c^2 \neq a^2 + b^2$, so the triangle does not contain a right angle. As $c^2 > a^2 + b^2$ the angle opposite c is greater than $90°$, so the triangle contains one obtuse angle.

Exercises

1. (a) Using a ruler and a pair of compasses, construct a triangle with sides of lengths 6 cm, 8 cm and 10 cm.

   (b) Use a protractor to measure the angles of your triangle.

   (c) Is the triangle right-angled?

   (d) Use Pythagoras' Theorem to decide whether the triangle is right-angled.

   (e) Was your answer to part (c) correct?

2. Repeat question 2 for a triangle with sides of lengths 7 cm, 8 cm and 11 cm.

3. Decide which of the triangles described below:

   (a) is right-angled,

   (b) contains an obtuse angle,

   (c) contains all acute angles.

   In each case, show how you reached your conclusion.

   (i) *a triangle with sides of lengths 10 cm, 11 cm and 14 cm*

   (ii) *a triangle with sides of lengths 10 cm, 12 cm and 16 cm*

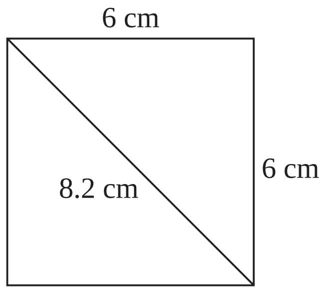
   (iii) *a triangle with sides of lengths 9 cm, 12 cm and 15 cm*

4. (a) Use an accurate construction to find out if a triangle with sides of lengths 6 cm, 7 cm and 12 cm contains a right angle.

   (b) Use Pythagoras' Theorem to check your answer to part (a).

5. Ahmed draws a square with sides of length 6 cm. He then measures a diagonal as 8.2 cm.

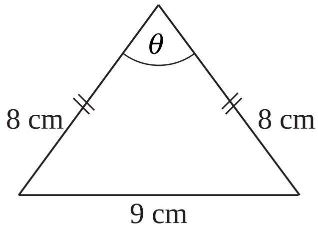
   Use Pythagoras' Theorem to decide if Ahmed has drawn the square accurately.



6. An isosceles triangle has 2 sides of length 8 cm. The length of the base is 9 cm.

   Decide, by calculation, whether the angle θ is a right angle, an acute angle or an obtuse angle.

   Show clearly how you reached your conclusion.



7. Measure the lengths of the sides and diagonal of your textbook. Use your measurements to decide whether the corners of your book are right-angled.

8. A triangle has sides of lengths 21 cm, 28 cm and x cm.

   (a) Show that the triangle has a right angle if $x = 35$.

   (b) For what values of x will the triangle contain an *obtuse* angle?

9. An isosceles triangle is known to have one side of length 18 cm and one side of length 28 cm.

   (a) Explain why the triangle cannot contain a right angle,

   (b) Show, by calculation, that it is possible for the triangle to contain three acute angles. Draw a sketch of the triangle in this case.

10. A right-angled triangle has two sides of lengths 24 cm and 32 cm. Use Pythagoras' Theorem to calculate the length of the other side.
    [Note: there are 2 possible answers.]