Complex Analysis revisited

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Introduction

What is a complex number?

All complex numbers form a field that is an extension of the real number field.

Definition

A complex number is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$. Components defined as $x = \Re(z)$, $y = \Im(z)$, $i^2 = -1$
Thus, we identify the bijection from $\mathbb{R}^2$ to $\mathbb{C}$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto z = x + iy$$

The complex field $\mathbb{C}$ is the set of pairs (x, y) with addition and multiplication defined by

$$z + w = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$z * w = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

The following laws also holds

1. $z + w = w + z$ and $zw = wz$ (commutative)
2. $(z + w) + p = z + (w + p)$ and $(zw)p = z(wp)$(associative)
3. $z(w + p) = zw + zp$ (distributive)

The **complex conjugate** of a complex number $z = x + iy$ is defined to be $\bar{z} = x - iy$.

$$\Re(z) = \frac{(z+\bar{z})}{2} \qquad \Im(z) = \frac{(z-\bar{z})}{2i}$$

Complex plane

The set of complex numbers forms the **complex plane** $\mathbb{C}$. To each complex number $z = x + iy$ we associate the point (x, y) in the Cartesian plane. Also a complex number can be represented by a vector (r, θ) in polar coordinates.

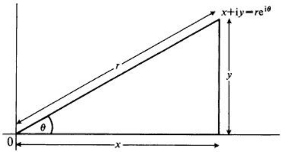
A modulus of z is

$$r = \sqrt{x^2 + y^2} = |z|.$$

From $x = r\cos\theta$ and $y = r\sin\theta$ it follows

$$z = r(\cos\theta + i\sin\theta),$$

where θ is called an argument of z.



Introduction

Definition

A function $F : \mathbb{C} \mapsto \mathbb{C}$ is called a complex function of a complex variable.

$F(z) = F(x + iy) = \Re(F(z)) + i\Im(F(z)) = F_1(x, y) + iF_2(x, y)$, where $f_1(x, y), f_2(x, y)$ are two real functions of two real variables x and y.

Also can be represent in the following way

$$F : \mathbb{R}^2 \mapsto \mathbb{R}^2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} \Re(F(x + iy)) \\ \Im(F(x + iy)) \end{bmatrix}$$

Differentiability

Definition

A complex-valued function $F(z)$ is called a differentiable in a point z_0 if exist

$$\lim_{z \to z_0} \frac{F(z)-F(z_0)}{z-z_0}$$

Thus, the complex derivative of $F(z)$ at z_0 is

$$\frac{dF}{dz}(z_0) = F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{F(z_0 + \Delta z) - F(z_0)}{\Delta z}$$

The point $z_0 + \Delta z$ may approach the point z_0 along an arbitrary curve ending at z_0. The limit is the same regardless of the path along which z_0 is approached.

Cauchy-Riemann equations

It follows limit should exist and is the same for z approaching z_0 through the paths parallel to the coordinate axes.
First, let $z = x + iy_0$ and $x \to x_0$. Then

$$F'(z_0) = \partial_x F_1 + i\partial_y F_1$$

For $z = x_0 + iy$ and $y \to y_0$ we will have

$$F'(z_0) = \partial_x F_2 - i\partial_y F_2$$

Comparing the real and the imaginary parts of two equations, we get Cauchy-Riemann equations

$$\partial_x F_1 = \partial_y F_2 \qquad \partial_y F_1 = -\partial_x F_2 \tag{1}$$

Analytic function

Equations (1) can also be rewritten as

$$\partial_x F = -i \partial_y F$$

Satisfying these equations is a necessary condition for $F(z)$ to be differentiable at point $z = z_0$, but not a sufficient condition.

Definition

We say that the complex function F is analytic at the point z_0, provided there is some $\epsilon > 0$ such that $F'(z)$ exist for all $z \in D_\epsilon(z_0)$. In other words, F must be differentiable not only at z_0, but also at all points in some ϵ neighborhood of z_0.
If F is analytic at each point in the region D, then we say that F is analytic on D.

Analytic function

Theorem

The necessary and sufficient condition for a function $F(z) = F_1 + iF_2$ to be analytic on a region D is that F_1 and F_2 have first order continuous partial derivatives on D and satisfy C-R equations(1)

If $F(z)$ is analytic in a region D, then the derivative of $F(z)$ is also an analytic function on D. Hence, the second order partial derivatives of F_1 and F_2 are also continuous. Using the C-R equations, we get the Laplace equations

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} = 0 \qquad \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} = 0$$

Analytic function

Thus, the real part and the imaginary part of an analytic function $F = F_1 + iF_2$ are harmonic functions.
We have that

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = -\frac{1}{2}i(z - \bar{z}) \tag{2}$$

By the rules of derivative, we have

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right), \qquad \frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

This implies that, a function is analytic if and only if $\partial F / \partial \bar{z} = 0$.

Laplace and Euler operators

Multiplying the last two relations we can easily derive the Laplace operator of function F

$$\Delta F = 4\frac{\partial F}{\partial z}\frac{\partial F}{\partial \bar{z}}$$

The following properties of Laplace operator holds

$$\Delta\Re = \Re\Delta = 4\Re\partial_z\partial_{\bar{z}} \qquad \Delta\Im = \Im\Delta = 4\Im\partial_z\partial_{\bar{z}}$$

The complex representation of the Euler operator is the following

$$\mathbf{x}\cdot\nabla F(\mathbf{x}) = z\frac{\partial F}{\partial z} + \bar{z}\frac{\partial F}{\partial \bar{z}}$$

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Anti-analytic function

Definition

An anti-analytic function is a function F satisfying the condition

$$\frac{\partial F}{\partial z} = 0$$

Using the result

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial F}{\partial y}$$

gives the anti-analytic version of the Cauchy-Riemann equations as

$$\frac{\partial F_1}{\partial x} = -\frac{\partial F_2}{\partial y} \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Harmonic conjugate

Definition

If the function $F_1(x, y)$ is harmonic in a domain D, we can associate with it another function $F_2(x, y)$ by means of C-R equations. The function $F_2(x, y)$ defined by this equations is harmonic in D and is called **harmonic conjugate** of $F_1(x, y)$.

It is clear that harmonic conjugate is unique up to the constant. If on a simply connected domain G, with $0 \in G$, a harmonic function $F_1(x) \in \mathbb{R}$ is given, a harmonic conjugate is constructed by

$$F_2(x) = \int_0^x (-\partial_y F_1(x(s))\dot{x} + \partial_x F_1(x(s))\dot{y})ds$$

The result doesn't depend on the path of integration.

Harmonic conjugate

$\mathbb{L}_2(\mathbb{S}^1)$ is a Hilbert space on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Let $\tilde{f}_1 \in \mathbb{L}_2(\mathbb{S}^1)$ and its Fourier expansion is

$$\tilde{f}_1(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) - b_n \sin(n\theta))$$

Extend $\tilde{f}_1$ to a harmonic function f_1 on the unit disc $D \subset \mathbb{C}$ by solving the Dirichlet problem. Let f_2, the harmonic conjugate of f_1, be fixed by taking $f_2(0) = 0$. Let $\tilde{f}_2$ denote the limit to the boundary $\mathbb{S}^1$ of D. Then

$$\tilde{f}_2(\theta) = \sum_{n=1}^{\infty} (b_n \cos(n\theta) + a_n \sin(n\theta))$$

Harmonic conjugate

$\mathbb{L}_2(\mathbb{S}^1; \mathbb{R}; \perp 1)$ is the linear subspace of all $\tilde{g} \in \mathbb{L}_2(\mathbb{S}^1)$ with $\int_0^{2\pi} \tilde{g}(\theta) d\theta = 0$
The operator

$$J : \mathbb{L}_2(\mathbb{S}^1; \mathbb{R}; \perp 1) : \tilde{f}_1 \mapsto J\tilde{f}_1 = \tilde{f}_2,$$

is orthogonal and skew-symmetric

$$J^* = -J = J^{-1}, J^2 = -I.$$

Note that $J\{\Re(a_n + ib_n)e^{in\theta}\} = \Re\{-i(a_n + ib_n)e^{in\theta}\}$.

Harmonic conjugate

The operator $N : \mathbb{L}_2(\mathbb{S}^1; \mathbb{R}; \perp 1) \to \mathbb{L}_2(\mathbb{S}^1; \mathbb{R}; \perp 1)$ is defined by

$$Nf_1 = \sum_{n=1}^{\infty} n\{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$

We have $N^* = N, J\partial_\theta = \partial_\theta J = N$ and therefore $\partial_\theta = -NJ$. For analytic functions $f(z)$ on the unit disc D we will consider a splitting in real series on $\mathbb{S}$. We put

$$f(e^{i\theta}) = \sum_{n=1}^{\infty}(a_n + ib_n)e^{in\theta} = f_1(e^{i\theta}) + if_2(e^{i\theta}) = f_1(e^{i\theta}) + iJf_1(e^{i\theta}).$$

Lemma

Lemma

On a simply connected domain $G \subset \mathbb{R}$, with $0 \in G$ we consider a biharmonic function $x \mapsto \phi(x)$. This means $\Delta\Delta\phi = 0$. Then there exist an analytic functions $\varphi, \chi : \mathbb{C} \mapsto \mathbb{C}$, such that

$$\phi(x) = \Re(\bar{z}\varphi(z) + \chi(z)), \qquad z = x + iy$$

Thank You!