Complex Analysis Questions

October 2012

Contents

1 Basic Complex Analysis

Question 1.1. *Use the del operator to reformulate the Cauchy–Riemann equations. State the generalized Cauchy–Riemann equations.*

Question 1.2. *What's the radius of convergence of the Taylor series of* $1/(x^2+1)$ *at* 100*?*

Question 1.3. *Define what it means to be Riemann integrable on the unit circle.*

Question 1.4. *What's the orientation of curves if you do a line integral on the boundary of an annulus?*

Question 1.5. *Prove Cauchy's theorem. What theorem of multivariable calculus is this similar to? What is Stokes' theorem?*

Question 1.6. *What is Goursat's theorem? How do you prove it? How do you use it to prove Cauchy's theorem?*

Question 1.7. *Prove the Cauchy integral formula.*

Question 1.8. *State Cauchy's integral formula for n^{th} derivatives.*

Question 1.9. *If you know a holomorphic function's values on the boundary of a region, how would you give estimates on its n^{th} derivative inside the region?*

Question 1.10. *Suppose we have a function whose first n derivatives vanish at the origin. What does it look like (geometrically)?*

Question 1.11. *Prove that a complex differentiable function has a power series.*

Question 1.12. *What does it mean for a function on $\mathbb{R}^2$ to be real-analytic? Why is a holomorphic function real-analytic?*

Question 1.13. *What are Laurent series expansions? How are these related to the Cauchy integral formula? How does one compute the Laurent coefficients?*

Question 1.14. *Where do Laurent series converge? Where do they converge uniformly?*

Question 1.15. *Suppose we have an otherwise entire function with poles at 1 and $2i$. Given a power series for this function about the origin, where does it converge? How many power series are there? Why must there be more than one? How can we compute the coefficients for each? What happens if we also allow negative powers; where does the Laurent series converge in that case?*

Question 1.16. *Calculate the Laurent series of* $\exp(1/z)$.

As

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

we have that

$$\exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k! z^k}.$$

Question 1.17. *How do you define the residue of a function at a pole? What's it good for? What other consequences are there?*

Question 1.18. *What is Cauchy's residue formula?*

Question 1.19. *Integrate $(1 + x^4)^{-1}$ over the real line.*

Question 1.20. *How would you integrate*

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^4)^2}?$$

Concepts, not calculations.

Question 1.21. *How would you compute*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^6+1}\,dx?$$

Question 1.22. *Do you know about elliptic functions? Tell about integrating*

$$\frac{1}{\sqrt{(1-x^2)(1+x^2)}}.$$

Concepts, not calculations.

Question 1.23. *Integrate* $(\sin x/x)^2$ *on the line.*

Question 1.24. *Integrate* $\sin x^2$*.*

Question 1.25. *What is a winding number? What sort of number is it? How do you prove it's an integer?*

Question 1.26. *If* f *is meromorphic, what is the meaning of the contour integral of* f'/f*?*

Question 1.27. *Prove the argument principle.*

Question 1.28. *Given the values of a meromorphic function on a curve, can you tell me the multiplicities of the zeroes and poles inside?*

Question 1.29. *Prove Rouché's theorem.*

Question 1.30. *Suppose* a *is a real constant bigger than* 1*. What can I say about solutions of the equation* $e^z = z^n e^a$*?*

Question 1.31. *What is Brouwer's fixed point theorem in the* 2*-dimensional case? What if, instead of that, I give you that* $|f| < 1$ *in the unit disc? Can you prove a fixed point theorem using complex analysis?*

Question 1.32. *Show that the zeroes of a polynomial are continuous functions of its coefficients.*

Question 1.33. *State the open mapping theorem.*

Question 1.34. *Given a holomorphic function from the unit disc to itself, if it has* 50 *zeroes of radius less than* $1/3$*, what can you say about its value at* 0*?*

Question 1.35. *State/prove the identity/uniqueness theorem.*

Question 1.36. *How many zeroes can a holomorphic function have in the disc? Can the zeroes have a limit point on the boundary? Give an example.*

Question 1.37. *You are given a holomorphic function in a (connected) domain in $\mathbb{C}$ vanishing up to infinite order at some point. What can you say about this function? Prove it.*

Question 1.38. *Suppose f, a complex-valued function on the unit disc, has the value 5 on the line $x = y$. If f is holomorphic, what is f? How do you prove this? What if f is harmonic? Give a nonconstant example.*

Question 1.39. *State Morera's theorem. What is Morera's theorem used for?*

Question 1.40. *State Cauchy's theorem for a triangle. Can you give a converse to this theorem?*

Question 1.41. *Take a sequence of holomorphic functions converging uniformly to some function f. Is f holomorphic? What happens in the case of C^∞ real functions?*

Question 1.42. *Suppose that you have a sequence of holomorphic functions on a region and they converge pointwise on some simple closed curve inside the region. What can you say about the sequence of derivatives on the interior of the curve?*

Question 1.43. *Consider the series $\sum_{n\geq 0}(-1)^n(2n+1)^{-s}$. Where does it converge? Can it be continued beyond that? How do you show that? What is its value at 1? Why? Can you do it with a contour integral? Can you arrive at the value at 1 in some other way?*

Question 1.44. *Under what conditions is the pointwise limit of holomorphic functions holomorphic?*

Question 1.45. *State/prove the Vitali convergence theorem.*

Question 1.46. *Let $F_n : \mathbb{D} \to \mathbb{C}$ a sequence of holomorphic functions (say $\mathbb{D}$ is the unit disc) with L^1 norms bounded by 1. Show that it has a subsequence converging uniformly on compact subsets of a disc of radius $1/2$.*

Question 1.47. *What do you know about several complex variables?*

Question 1.48. *Prove that, for a sequence of holomorphic functions on a compact set, convergence in L^2 implies uniform convergence.*

Question 1.49. *Let U be a region, bounded if you like. Let H denote the set of all square-integrable holomorphic functions on U. Is H a Banach space? Why?*

Question 1.50. *If a power series has radius of convergence 1, can it be continuous on the whole unit circle? Analytic?*

Question 1.51. *Does there exist a holomorphic f defined on the unit disc which has zeroes at all points $n/(n+1)$ where n is a positive integer? Can such an f be bounded? Do you know a condition involving Blaschke products?*

Question 1.52. *Consider a function holomorphic in the unit disc satisfying* $f(0) = 0$ *and*

$$f(2z) = \frac{f(z)}{1 + f(z)^2}$$

If such a function exists, can it be continued to a meromorphic function on $\mathbb{C}$*? Does such a function exist?*

Question 1.53. *When* α *is irrational, what can you say about this sum?*

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \alpha n^2}?$$

2 Entire Functions

Question 2.1. *What is an entire function?*

Question 2.2. *What is Liouville's theorem? Proof?*

Question 2.3. *What is your favorite proof of Liouville's theorem?*

Question 2.4. *Prove a sharper version of Liouville's theorem where a polynomial bound implies that the function is a polynomial of lower degree.*

Question 2.5. *Prove the fundamental theorem of algebra.*

Question 2.6. *Suppose you have an entire function whose growth is bounded by* $\log |z|$*. Does it have to be constant? What about a harmonic function?*

Question 2.7. *What can you say about a function that grows like* $\sqrt{|z|}$ *for large* z*?*

Question 2.8. *If we know that* f *is entire and that* $|f(z)| \leq |z|^{100}$*, what can I say about this function?*

Question 2.9. *If* f *is entire and* $\Re(f)$ *is bounded, does* f *have to be constant?*

Question 2.10. *An entire function* f *has* $\Re(f) + \Im(f)$ *bounded. What can you say about* f*?*

Question 2.11. *Suppose an entire function is real on the unit circle. What does this say about the function?*

Question 2.12. *How many zeroes can an entire function have? How would you prove Jensen's inequality?*

Question 2.13. *How many zeroes must an entire function of order* $1/2$ *have?*

Question 2.14. *If* f *is entire and nowhere-vanishing, and* $|f(z)| < \exp(|z|)$*, what can you say?*

Question 2.15. *An entire function satisfies* $|f(z)| < e^{|z|^{100}}$. *What can you say about it?*

Question 2.16. *What are the rank, order, and genus of an entire function? What is the order of the zeta function? Why does it have to have infinitely many zeroes? How do you bound it in the critical strip? Give an elementary bound for* $\sum(-1)^n/n^s$ *using summation by parts on the line* $\Re(s) = 1/2$.

Question 2.17. *Prove that all zero-free entire functions of finite order are exponentials of polynomials.*

Question 2.18. *How does the complex function* $\sin z$ *differ from the real function* $\sin x$*? On the real line,* $\sin^2 x + \cos^2 x = 1$*, but in the complex plane, is* $\sin^2 z + \cos^2 z$ *even bounded? Are the given definitions for* $\sin z$ *and* $\cos z$ *the right ones? Why?*

Question 2.19. *Can you have a holomorphic function from the plane to the plane minus the origin? To the plane minus two points?*

Question 2.20. *What can you say about an entire function that maps into the plane minus the negative real axis? What about the plane minus a segment? Can you generalize the above cases?*

Question 2.21. *Prove the little Picard theorem.*

Question 2.22. *Can the zeroes of a holomorphic function be ANY set with no limit point? How is this related to the order and genus?*

Question 2.23. *What can you say about the ring of functions that are holomorphic in some neighborhood of the origin? What about multiple variables? What about the ring of entire functions?*

Question 2.24. *Take the Fourier transform of* $e^{-|x|^3}$. *Now consider it as a function of a complex variable. Is it an entire function? Give an estimate for its order. What are its zeroes (just kidding)? When we write this entire function as a product of exp(some polynomial) with a product over the zeroes, what is the degree of the polynomial? Seriously, does this function have ANY zeroes? Does it have infinitely many? Could you give a proof along these lines that the Riemann zeta function has infinitely many zeroes?*

Question 2.25. *Do you know the growth of the Riemann zeta function on the line where* $\Re(s) = 1$*?*

3 Singularities

Question 3.1. *Classify the types of singularities. Describe each.*

Question 3.2. *Prove Riemann's theorem on removable singularities.*

Question 3.3. *Define "meromorphic function". Give an example of a non-rational meromorphic function. What are the poles and residues of* $1/\sin z$*?*

Question 3.4. *What happens in the neighborhood of a pole? In which directions does it go to infinity?*

Question 3.5. *Which meromorphic functions have meromorphic antiderivatives?*

Question 3.6. *If f holomorphic on $\mathbb{C} \setminus \{0\}$, when is it a derivative of some F holomorphic in $\mathbb{C} \setminus \{0\}$?*

Question 3.7. *Can a meromorphic function have infinitely many poles in a compact set?*

Question 3.8. *What is the behavior of the gamma function at 0?*

Question 3.9. *How does a function behave near an essential singularity?*

Question 3.10. *Prove the Casorati–Weierstrass theorem.*

Question 3.11. *Can you prove the big Picard theorem?*

Question 3.12. *On an annulus, if a function's Laurent series has infinitely many terms in the principal part, can you say it has an essential singularity?*

Question 3.13. *Talk about the singularity $\log z$ has at 0.*

Question 3.14. *What happens near $z = 0$ for the function $(\log z)/z$? Is $z = 0$ a singularity? What is it?*

Question 3.15. *Consider all germs of meromorphic functions at 0. What kind of algebraic structure is this? Is it algebraically closed?*

4 Infinite Products

Question 4.1. *Suppose I give you a sequence of points converging to infinity. How would you construct an entire function with those points as zeroes?*

Question 4.2. *State/prove the Weierstrass factorisation theorem.*

Question 4.3. *Let's say a function has zeroes at the integers. What are the Weierstrass factors?*

Question 4.4. *What is the Weierstrass product theorem? What are those extra factors for? Give an example where the product wouldn't converge.*

Question 4.5. *What is the infinite product for the reciprocal of the gamma function? What if we want zeroes on all integers? Do we need convergence factors this time?*

Question 4.6. *Prove the basic functional equation of the gamma function directly from the product identity.*

Question 4.7. *Prove the Stirling formula.*

Question 4.8. *What is the relation between the gamma function and the sine function?*

Question 4.9. *Why was Hadamard interested in rates of growth of entire functions? Do you know the prime number theorem? How is this related to factoring an entire function into an infinite product?*

5 Analytic Continuation

Question 5.1. *Talk about the analytic continuation of your favorite transcendental function. What about the zeta function? Talk about zeta and L-functions in arbitrary number fields.*

Question 5.2. *Do you know the functional equation for the Riemann zeta function? Talk about its analytic continuation.*

Question 5.3. *If you have a holomorphic function that maps a triangle into a disc (that is, a bounded function in the triangle), can you analytically continue it to a slightly larger domain?*

Question 5.4. *Suppose you have a function holomorphic on the top half of the unit disc, continuous on the real interval $[-1, 1]$. Can you define a holomorphic extension to the whole disc? How do you prove it?*

Question 5.5. *State and prove Schwarz reflection.*

Question 5.6. *Let $f : [0, 1] \to \mathbb{R}$. For $\Im(z) > 0$, define*

$$G(z) = \int_0^1 \frac{f(x)}{z - x}\, dx.$$

What can you say about G? What is the largest domain to which this function can naturally be extended? Can it be extended to all of $\mathbb{C}$?

Question 5.7. *Given a complex function f on the boundary of the unit circle, can you tell when it can be analytically extended inside? If f is real on the boundary, when can it be represented as $|g(z)|^2$ where g is holomorphic in the unit disc?*

Question 5.8. *Several complex variables: Can you describe an open region of $\mathbb{C}^2$ such that every holomorphic function defined there extends to a larger region?*

Question 5.9. *Why can't $1/z$ on the unit circle extend to a holomorphic function on the disc?*

Question 5.10. *If you have a real function on the boundary of a domain, how would you extend it to be the real part of a holomorphic function on the domain?*

Question 5.11. *Give an example of a doubly connected region and a holomorphic function on it that doesn't extend to a bounded component of the complement.*

Question 5.12. *You are given a domain in $\mathbb{C}$. Can you construct a holomorphic function in this domain which cannot be extended to any larger domain? How?*

6 Doubly Periodic Functions

Question 6.1. *Why there are no doubly periodic functions of order one?*

Question 6.2. *Can doubly periodic functions be entire? Assume they have a pole of order* 1 *in a fundamental region. Can it be the only singularity?*

Question 6.3. *Talk about doubly periodic functions on* $\mathbb{C}$*. Prove that the sum of the residues of such a function in a period parallelogram is* 0*.*

Question 6.4. *Discuss the Weierstrass* $\wp$*-function and the differential equation it satisfies.*

Question 6.5. *Do you know any function with poles at a lattice?*

7 Maximum Principles

Question 7.1. *What is the maximum modulus theorem? How do you prove it?*

Question 7.2. *Assume* f *has a complex derivative bounded by* 1 *on the unit circle. What can you say about the boundedness of* f *and its complex derivative on the unit disc? The circle?*

Question 7.3. *Suppose you have a holomorphic function on* $0 < \Re(z) < 1$*, continuous and bounded in absolute value by* 1 *on the boundary, and bounded in the interior of the strip. What can you say? Prove it.*

Question 7.4. *Consider* f*, a holomorphic function for* $0 < \Re(z) < 1$ *and continuous on* $0 \leq \Re(z) \leq 1$ *with* $|f(z)| \leq 1$ *for* $\Re(z) = 0$ *and* $|f| \leq 5$ *for* $\Re(z) = 1$*. What can you say about* $f(z)$ *for* $\Re(z) = 1/2$*? What if your function* f *is harmonic?*

Question 7.5. *Suppose you have a holomorphic function on the upper half of the unit disc, and you know that its absolute value is at most* 2 *on the semi-circle, and at most* 1 *on the real-axis part of the boundary. Bound the function in the interior.*

Question 7.6. *Suppose I have a function* f *which is holomorphic on the upper half-plane and which is continuous along the boundary. Suppose that* $|f(x)|$ *is bounded by* 5 *on the real axis between* -1 *and* 1 *and is bounded by* 1 *for* $|x| > 1$*. What can I say about* f*? As it turns out, the level sets of a function with the properties described above can be described very explicitly. They turn out to be circles which pass through the two points where the bounds change. The level sets at the integers look like a rising sun. What can you say about the image of these circles when I map them to the unit disc? Suppose that* g *is some holomorphic function which is not necessarily continuous on the boundary of the above region, but which actually satisfies* $g = 5$ *on the real line between* 1 *and* -1 *and* $g = 1$ *on the exterior of the region. What can I say about the relationship between* g *and the* f *above? Again I know that* g *is holomorphic on the upper half-plane.*

Question 7.7. *Give an example of a Phragmén–Lindelöf theorem.*

Question 7.8. *State/prove the Hadamard* 3*-circles theorem. Generalise to annuli with slits missing.*

Question 7.9. *State and prove the Borel–Carathéodory inequality.*

Question 7.10. *Draw three concentric circles. Suppose you have a holomorphic function in the annulus bounded by the innermost and outermost circles, and you have an upper bound for its absolute value on the innermost circle and another bound on the outermost circle. How would you bound the function on the middle circle? Now suppose you have a domain bounded by a Jordan curve and you cut out two regions from that domain bounded by Jordan curves inside the domain. We have a holomorphic function in the remaining triply connected domain and have various bounds —* 3, 5, 7 *— on the boundary components. How would you bound the function in the domain?*

8 Harmonic Functions

Question 8.1. *What is a harmonic function? If it is defined around the origin, when can it be extended to the origin? Why? Given a smooth function on the boundary of a disc, can it be extended inside as a harmonic function? Is the extension unique? Prove it.*

Question 8.2. *When can a function on the interior of the unit disc be represented by a Poisson integral?*

Question 8.3. *What are harmonic functions? What is their relation to holomorphic functions?*

Question 8.4. *When is a harmonic function the real part of a holomorphic function? Why does the harmonic conjugate exist locally and how do you construct it?*

Question 8.5. *Describe how to solve the Dirichlet problem on the disc. On an arbitrary domain?*

Question 8.6. *State/prove the Poisson formula.*

Question 8.7. *Tell us about the Dirichlet principle. Does it relate to a variational problem?*

Question 8.8. *Given several quarter-circles with various holes chopped out, and values assigned to different pieces of the boundary, do there exist harmonic functions on these domains with these boundary values?*

Question 8.9. *Give a harmonic function on the upper half-plane with limit equal to* 1 *on* $[a, b]$ *and* 0 *on the rest of the real axis. Write it in the form of an integral.*

Question 8.10. *How do you explicitly find the harmonic function inside a domain bounded by a line segment and a half-circle with boundary value* 0 *on the line and boundary value* 1 *on the semicircle?*

Question 8.11. *Suppose* f *is harmonic on the unit disc,* $|f|$ *bounded by* 5 *on the top half of the boundary, by* 4 *on the bottom. How well can you bound* f *in the disc?*

Question 8.12. *Talk about Harnack's principle, first for a ball, then for a general domain. Give a bound for the ratio between the maximum and the minimum of a harmonic function.*

Question 8.13. *Consider the powers of the complex function f. Make some statements about the convergence of these powers using Harnack's theorems.*

Question 8.14. *Is a bounded subharmonic function in the complex plane necessarily constant?*

9 Conformal Mappings

Question 9.1. *What is special about a conformal map?*

Question 9.2. *What is the relation between conformal mappings and holomorphic functions?*

Question 9.3. *Is a holomorphic map always conformal? If $f'(z) = 0$, how does it look locally? Is a conformal map always holomorphic?*

Question 9.4. *What does it mean geometrically when a function's derivative is 0 at some point?*

Question 9.5. *What are the possible images of a circle under a Möbius transformation?*

Question 9.6. *Suppose I have a finite group of Möbius transformations acting on the unit disc. Show there is a global fixed point.*

Question 9.7. *What are the conformal mappings from S^2 to S^2?*

Question 9.8. *Construct an explicit conformal map between the upper half-plane and the disc. What 19^{th} century mathematician's name is associated to this map? How do you know this map does what you say it does?*

Question 9.9. *Write down a conformal mapping of a horizontal strip onto the unit disc.*

Question 9.10. *Write down a conformal mapping of the upper half-plane onto a horizontal strip.*

Question 9.11. *Map a vertical strip conformally onto the unit disc. Where does a line through 0 go?*

Question 9.12. *Write down a conformal map from the upper right quadrant to the unit disc.*

Question 9.13. *How do you find a conformal map that transforms a quarter of a circle into a half-circle? Does the squaring function do it? Can you find a Möbius transformation?*

Question 9.14. *Construct the conformal map from a fan (a section of annulus) to the unit disc.*

Question 9.15. *Consider mapping a square onto the upper half-plane conformally. What freedom do you have to choose where the corners go?*

Question 9.16. *What about mapping the unit disc to a rectangle?*

Question 9.17. *Describe the Schwarz–Christoffel formula for a triangle.*

Question 9.18. *What's the formula for a conformal map from the upper half-plane to a polygon?*

Question 9.19. *Why are all automorphisms of the unit disc linear fractional transformations? What classical group do the automorphisms form? Discuss this group acting on the upper half-plane instead of the unit disc.*

Question 9.20. *Talk about isometries of the hyperbolic plane.*

Question 9.21. *Can you map an open square to an open disc analytically? What about continuous extension to the boundary?*

10 Riemann Mapping Theorem

Question 10.1. *Why can't you conformally map the whole plane onto the disc?*

Question 10.2. *What is special about simply connected regions in the plane?*

Question 10.3. *Which simply-connected domains are NOT equivalent to the unit disc?*

Question 10.4. *State/prove the Riemann mapping theorem. What can you say about continuity up to the boundary? About remaining one-one at the boundary? Give an example of a domain conformally equivalent to the disc where uncountably many points on the unit circle correspond to a single point on the boundary.*

Question 10.5. *State/prove Montel's theorem.*

Question 10.6. *What is a normal family?*

Question 10.7. *What is Schwarz' lemma? Prove it.*

Question 10.8. *Give a geometric interpretation of the Schwarz lemma involving the hyperbolic metric.*

Question 10.9. *Suppose you have a bounded injective holomorphic map f from the unit disc to $\mathbb{C}$, $f(0) = 0$, $f'(0) = 1$. Can you bound f?*

Question 10.10. *Prove that non-vanishing functions on simply connected domains have holomorphic square roots.*

Question 10.11. *What was Riemann's original (flawed) proof of his mapping theorem?*

Question 10.12. *If I knew how to solve the Dirichlet problem, could I prove Riemann Mapping from that?*

Question 10.13. *When are two annuli conformally equivalent? Prove it. How many parameters are there for annuli? Why are two annuli with different ratios not conformally equivalent?*

Question 10.14. *Use Schwarz reflection to prove that two annuli with different radial-ratios are not conformally equivalent.*

Question 10.15. *Suppose I give you two open sets in $\mathbb{C}$. When are they conformally equivalent? How many parameters? Do the moduli spaces have complex structure? What is a Green's function?*

Question 10.16. *Talk about multiply connected domains. How many degrees of freedom are there in choosing non-conformally-equivalent annuli with slits missing? Can you map multiply-connected domains onto a circle with circular holes punched?*

Question 10.17. *Is there a conformal map from the plane minus a point to the plane minus any other point? What about two points? Three points?*

Question 10.18. *Give a hyperbolic metric on, say, $\mathbb{C} \setminus \{0, 1\}$.*

Question 10.19. *What is the universal cover of $\mathbb{C}$ minus a point? Minus two points?*

Question 10.20. *What is the uniformisation theorem for Riemann surfaces and how is it proved?*

Question 10.21. *State the uniformisation theorem. What do you mean by universal cover? Is it topological or analytic? How would you prove it?*

11 Riemann Surfaces

Question 11.1. *Define "Riemann surface". Define "branch point". Define "sheet". What happens when you wind around a branch point?*

Question 11.2. *You are given a holomorphic function in a neighborhood of a point. What is the Riemann surface of this function? What are the points of the Riemann surface? What is the topology on it?*

Question 11.3. *Let F, G be meromorphic functions such that $F^n + G^n = 1$, assuming $n \geq 4$ if necessary. Prove that F and G are constants.*

Question 11.4. *What is the Riemann surface of $\sqrt{z}$, over the disc and over the Riemann sphere? Give an example of a map from the torus to the Riemann sphere branched over 4 points.*

Question 11.5. *Talk about conformal isomorphisms of compact Riemann surfaces. Which surfaces have infinite groups of conformal automorphisms?*

Question 11.6. *Talk about the field of meromorphic functions on a compact surface. Do a sphere and a torus have isomorphic fields of meromorphic functions?*