A Tutorial on Differential Geometry

Elham Sakhaee

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1 Introduction

I prepared this technical report as part of my preparation for Computer Vision PhD qualifying exam. Here we discuss two properties of surfaces known as **"First"** and **"Second"** fundamental forms and their applications in computer vision.

2 Background

A parametric surface is defined as $X(u,v) = [x(u,v), y(u,v), z(u,v)]$ where u and v are the parameters. as an example a cone as shown below can be either described as a parametrized surface being

$$X(u,v) = a.u.sin(v)\mathbf{i} + a.u.cos(v)\mathbf{j} + u\mathbf{k} \tag{1}$$

or in explicit representation as a surface z(x,y) being described as a function of x and y

$$z(x,y) = \sqrt{x^2 + y^2} \tag{2}$$

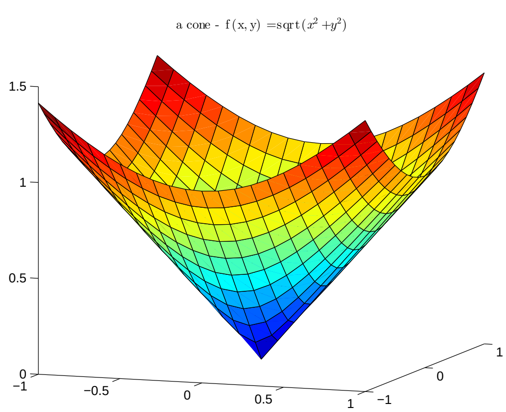

Figure 1: a cone drawn as a 3D surface z=f(x,y)

Having visualized a 3D surface, we recall that $X_u(u_1, v_1)$ and $X_v(u_1, v_1)$ are tangent vectors to a point lying on the surface described by parameters u and v. $X_u(u_1, v_1)$ and $X_v(u_1, v_1)$ being linearly independent, fully describe the tangent plane at the given point. The normal to the tangent plane can be simply found by $||X_u \times X_v||$. Keep in mind this vector needs to be normalized to be considered as unit normal vector to the tangent plane at point $[x(u_1, v_1), y(u_1, v_1), z(u_1, v_1)]$.

3 First Fundamental Form

First Fundamental Form of a surface is a property of the surface and is defined by

$$\mathbf{v}.\mathbf{v} =< \mathbf{v}, \mathbf{v} > \qquad \text{where} \qquad \mathbf{v} = X_u.du + X_v.dv \tag{3}$$

$\mathbf{v}$ can be thought of the velocity of particle moving on the surface, simply because the components of $\mathbf{v}$ indicate its speed along u and v which are linearly independent.
Question: what is the relation of the speed as given above and directional derivative?
It can been inferred form (3) that square root of First Fundamental Form $\sqrt{X_u^2 + X_v^2}$ is the length element, therefore integrating it along a space curve described as $C(u, v) = [x(u, v), y(u, v), z(u, v)]$ gives us the arc length.
Question: How does it correspond to Geodesic Curves? defined as minimum length space curve, lying on a surface and connecting two given points

It is common to represent (3) in terms of a matrix multiplied with the infinitesimal change vectors as below:

$$< \mathbf{v}, \mathbf{v} >= [du, dv]^T \begin{bmatrix} < X_u, X_u > & < X_u, X_v > \\ < X_u, X_v > & < X_v, X_v > \end{bmatrix} [du, dv] \tag{4}$$

we can further expand (4) and write it as a **quadratic** form

$$\begin{aligned} &< X_u, X_u > du^2 + 2 < X_u, X_v > dudv + < X_v, X_v > dv^2 \\ &E du^2 + 2F dudv + G dv^2 \end{aligned} \tag{5}$$

This is also a bilinear form, we leave it to the reader to prove. A peculiar application of First Fundamental Form which can be seen as the quadratic form in (5) is its application is computing surface area or arc length.
As discussed above arc length is the square root of First Fundamental Form :

$$ds^2 = E du^2 + 2F dudv + G dv^2 \tag{6}$$

The length element is useful to measure length of a space curve. Such curve lies on the surface which is mapping from uv space. Therefore, if we think of a curve in uv space described parametrically as $C(t) = [u(t), v(t)]$ then such curve is mapped to a space curve as $C(u(t), v(t)) = [x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))]$. Hence, we can benefit from (6) as follows:

$$\begin{aligned} ds^2 &= E du^2 + 2F dudv + G dv^2 \\ ds &= \sqrt{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2} \quad dt \\ ds &= \sqrt{Eu_t^2 + 2Fu_t v_t + Gv_t^2} \quad dt \end{aligned} \tag{7}$$

To utilize First Fundamental Form in finding surface area note that area of a planar parallelogram with sides being 3D vectors a and b is $a.b.sin(\theta)$ where θ is the planar angle between the two vectors. we can derive that $||a \times b|| = a.b.sin(\theta)$ and from that we know that for a 3D surface the area element is $dA = ||a \times b||$ if we assume a mapping $X(u, v)$ (shown in the image with r) takes that uv plane to the 3D-space surface, then $a = X_u du$ and $b = X_v dv$, (see the image below)

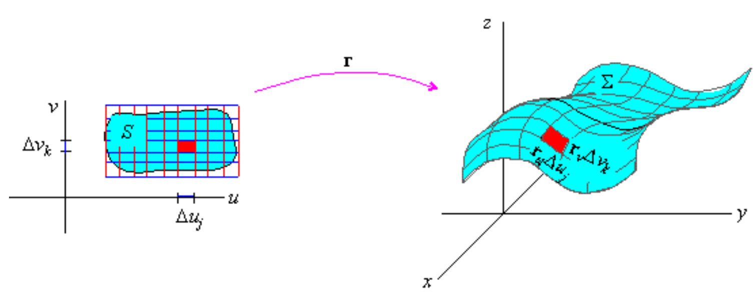

Figure 2: visualization of surface element from uv space

therefore

$$dA = ||X_u du \times X_v dv|| = ||X_u \times X_v||du.dv \tag{8}$$

from simple algebra we know that

$$\begin{aligned} &||X_u \times X_v||^2 + < X_u, X_v >^2 = ||X_u||^2.||X_v||^2 \\ &||X_u||^2 =< X_u, X_u >= E \qquad \text{and} \qquad ||X_v||^2 =< X_v, X_v >= G \end{aligned} \tag{9}$$

Therefore, using First Fundamental Form we can derive the surface element as to be:

$$dA^2 = EG - F^2 \tag{10}$$

4 Second Fundamental Form

We discussed that First Fundamental From is a quadratic form on the tangent space of a surface and is helpful in calculating surface area or arc length, now let's turn our attention to second Fundamental Form which is handy in finding

the curvature of a surface.
Given a point on a surface X(u,v), suppose we want to pull or push the surface along the normal vector. The deformed surface can be represented as $r(u,v,t) = X(u,v) - t.n(u,v)$. This is the equation of a family of surfaces, while the Second Fundamental From talks about one member of this family which is at t=0 and coincides with the given surface X(u,v). Basically Second Fundamental Form is about how First Fundamental Form changes as t changes as shown below.
Let's write the First Fundamental Form for the new surface $r(u,v,t)$. We have $r_u = X_u - t.n_u$ and $r_v = X_v - t.n_v$. the interesting observation here is that r_u and n_u both lie on the tangent plane (think about it geometrically).

$$\begin{aligned} &E(t) =< r_u, r_u >=< X_u - t.n_u, X_u - t.n_u >= X_u.X_u - 2t.n_u.X_u + t^2.n_u.n_u \\ &F(t) =< r_u, r_v >=< X_u - t.n_u, X_v - t.n_v >= X_u.X_v - t.n_u.X_v - t.n_v.X_u + t^2.n_u.n_v \\ &G(t) =< r_u, r_u >=< X_v - t.n_v, X_v - t.n_v >= X_v.X_v - 2t.n_v.X_v + t^2.n_v.n_v \end{aligned} \tag{11}$$

The Second Fundamental Form can be derived by differentiating First Fundamental Form with respect to t.

$$\frac{\partial E}{\partial t} du^2 + 2\frac{\partial F}{\partial t} dudv + \frac{\partial G}{\partial t} dv^2 | t = 0 \tag{12}$$

Let's do each part separately first, then put all together:

$$\begin{aligned} &\frac{\partial E(t)}{\partial t} = \frac{\partial}{\partial t} X_u.X_u - 2t.n_u.X_u + t^2.n_u.n_u \quad |t = 0 = -2n_u.X_u \\ &\frac{\partial H(t)}{\partial t} \quad |t = 0 = -n_u.X_v - n_v.X_u \\ &\frac{\partial G(t)}{\partial t} = \frac{\partial}{\partial t} X_v.X_v - 2t.n_v.X_v + t^2.n_v.n_v \quad |t = 0 = -2n_v.X_v \end{aligned} \tag{13}$$

We know that n is perpendicular to X_u and X_v therefore

$$n.X_u = 0 \to (n.X_u)_u = 0 \to n_u.X_u + n.X_{uu} = 0 \to -n_u.X_u = n.X_{uu} =< n, X_{uu} > \tag{14}$$

We can apply this for the other two equations. Eventually the Second Fundamental Form can be formulated as:

$$\begin{aligned} &< n, X_{uu} > du^2 + 2 < n, X_{uv} > dudv + < n, X_{vv} > dv^2 \\ &\text{or in matrix representation} \\ &[du, dv]^T \begin{bmatrix} < n, X_{uu} > & < n, X_{uv} > \\ < n, X_{uv} > & < n, X_{vv} > \end{bmatrix} [du, dv] \end{aligned} \tag{15}$$

Now the question is how to practically use this form? The answer is in the next section

5 Gaussian Curvature and Mean Curvature

5.1 Gaussian Curvature

The Gaussian curvature is the limit of surface area of A being transfered to Gauss unit sphere divided by the original surface area A on the surface. (see B.K.P Horn's book - chapter 16)
Using the above derivation we can verify that Gaussian curvature K is the determinant of Second Fundamental Form matrix divided by determinant of First Fundamental Form matrix. This is direct relation of matrix equation in (16)
as an easy example a plane has X_{uu}, X_{vv}, X_{uv} equal to zero, hence Second Fundamental Form for a plane is zero and hence the Gaussian curvature is also zero.
Note1
It can be simply derived that a surface of form $(x, y, z(x, y))$ has Gaussian curvature $K = \frac{z_{xx} \cdot z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2}$
Note2
Two surfaces that have same First Fundamental Form have same Gaussian curvature. keep in mind they might have different parametrization for First Fundamental Form.

5.2 Mean Curvature

We note that n_u and n_v are two vectors lying on the tangent space, hence can be written as linear combination of X_u and X_v. for example one can write $n_u = a_{11}X_u + a_{12}X_v$ and $n_v = a_{21}X_u + a_{22}X_v$. Therefore, we can summarize solving for a_{ij} in a matrix formulation as:

$$\begin{bmatrix} -< n_u, X_u > & -< n_u, X_v > \\ -< n_v, X_u > & -< n_v, X_v > \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} < X_u, X_u > & < X_u, X_v > \\ < X_v, X_u > & < X_v, X_v > \end{bmatrix} \tag{16}$$

which is a relation between First and Second Fundamental Forms. Having solved for $A = [a_{ij}]$ k_1 and k_2, the two *Principle Curvatures* are minus of eigenvalues of the matrix A. also
$H = MeanCurvature = \frac{1}{2}trace(A) = \frac{1}{2}(k_1 + k_2)$ and
$K = GaussianCurvature = det(A) = k_1.k_2$.

Now let's derive Mean Curvature for a surface given by $(x, y, z(x, y))$. we use the fact that $-< n_u, X_u >=< n, X_u u >$ and the other two counterparts and derive:

$$n = (1, 0, z_x) \times 0, 1, z_y \rightarrow normalize \rightarrow n = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}}[z_x, z_y, 1]^T \tag{17}$$

$$\frac{1}{\sqrt{1+z_x^2+z_y^2}}\begin{bmatrix} <[z_x,z_y,1]^T,[0,0,z_{xx}]> & <[z_x,z_y,1]^T,[0,0,z_{xy}]> \\ <[z_x,z_y,1]^T,[0,0,z_{xy}]> & <[z_x,z_y,1]^T,[0,0,z_{yy}]> \end{bmatrix} = \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} <[1,0,z_x]^T,[1,0,z_x]> & <[1,0,z_x]^T,[1,0,z_y]> \\ <[1,0,z_x]^T,[1,0,z_y]> & <[1,0,z_y]^T,[1,0,z_y]> \end{bmatrix} \tag{18}$$

This scary equation easily simplifies to:

$$\frac{1}{\sqrt{1+z_x^2+z_y^2}}\begin{bmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy}> \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} z_x^2+1 & z_x.z_y \\ z_x.z_y & z_y^2+1 \end{bmatrix} \tag{19}$$

To find the Gaussian and Mean curvature we note that
$det(AB) = det(A).det(B)$ and $trace(A.B) = trace(B.A) \neq trace(A)trace(B)$, therefore we need to do the following:

$$\frac{1}{\sqrt{1+z_x^2+z_y^2}}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{bmatrix}.\begin{bmatrix} z_x^2+1 & z_x.z_y \\ z_x.z_y & z_y^2+1 \end{bmatrix}^{-1} \tag{20}$$

we find out that

$$K = \frac{z_{xx}.z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2}$$
$$\text{and} \tag{21}$$
$$H = \frac{z_{xx}(1+z_y^2) - 2zxyz_xz_y + z_{yy}(1+z_x^2)}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}$$

We can visualize a surface using its Gaussian curvature, K, and Mean curvature, H, at each point. See the following figure.

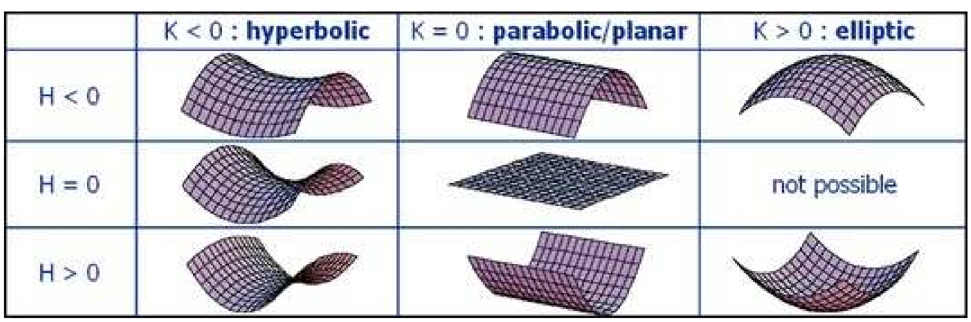

Figure 3: deciding the shape a surface at each point given its K and H