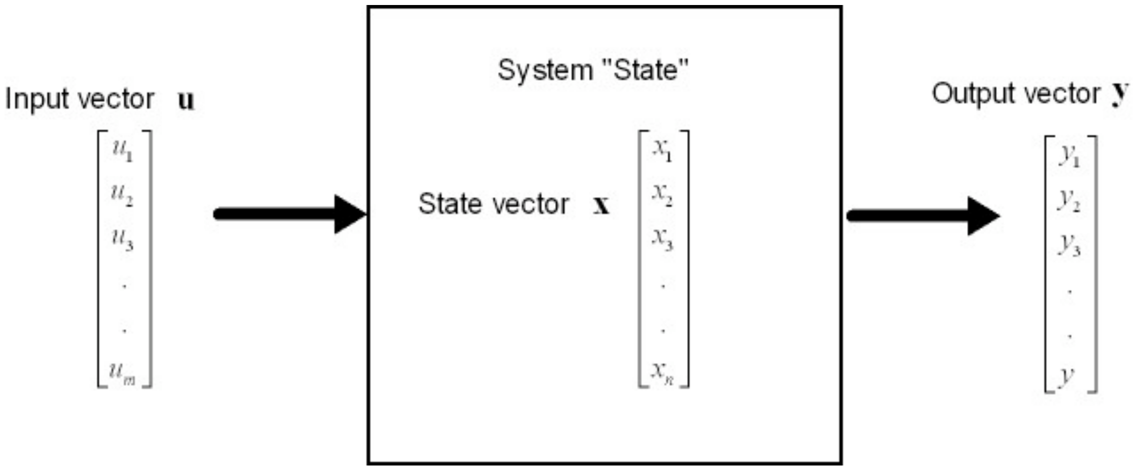
Modern Control

A. Stability, Controllability, Observability

The mathematical structure most naturally adapted to the description of systems is the state space representation. The state of a system is described at any instant by a set of state variables, for example the position and velocity of all the mass components of a mechanical system, or the current and voltage though and across every device in an electrical system or the temperature and pressure at every point of a thermodynamic system. The state of a system at any time can then be predicted from a knowledge of the initial state and the inputs that have acted upon the system over time.



*The **state** of a system is the smallest set of variables $x_1, x_2, \ldots x_n$ such that knowledge of these variables at t = 0, together with knowledge of all inputs $u_1, u_2, \ldots u_m$ for t ≥ 0, completely determines the outputs $y_1, y_2, \ldots y_r$ for t ≥ 0.*

The n-dimensional space whose coordinates consist of the x_1 axis, x_2 axis, ….x_n axis is called the **state space** of the system. Any state of a system is represented by a vector in state space.

A system that can be described by a finite number n of state variables is a lumped parameter system, governed by a set of n first-order differential equations:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
\dot{x}_3 &= f_3(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
&. \\
&. \\
\dot{x}_n &= f_n(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
\\
y_1 &= g_1(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
y_2 &= g_2(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t) \\
&. \\
&. \\
y_r &= g_r(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m, t)
\end{aligned}
$$

or in vector notation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u},t) \\
\mathbf{y} &= \mathbf{g}(\mathbf{x},\mathbf{u},t)
\end{aligned}
$$

The lumped system is linear if the functions $\mathbf{f}(\mathbf{x},\mathbf{u},t)$ and $\mathbf{g}(\mathbf{x},\mathbf{u},t)$ can be expressed as linear functions of the state and input vectors:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\
\mathbf{y} &= \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u}
\end{aligned}
$$

For linear, time invariant continuous systems, the matrices **A**, **B**, **C**, **D** are constant:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

where **A**: [n x n] system matrix
B: [n x m] input matrix
C: [r x n] output matrix
D: [r x m] transmission matrix

	n + m	
n + r	A n x n	B n x m
	C r x n	D r x m

In general, when the eigenvalues of the system matrix **A** are distinct, the state equations can be transformed into "modal canonical" form:

$$\dot{\mathbf{x}} = \mathbf{A_m}\mathbf{x} + \mathbf{B_m}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C_m}\mathbf{x} + \mathbf{D_m}\mathbf{u}$$

$$\mathbf{A_m} = \begin{bmatrix} \lambda_1 & 0 & . & . & 0 & 0 \\ 0 & \lambda_2 & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \lambda_{n\text{-}1} & 0 \\ 0 & 0 & . & . & 0 & \lambda_n \end{bmatrix}; \quad \mathbf{B_m} = \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1m} \\ b_{21} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & . & . & . & b_{nm} \end{bmatrix};$$

$$\mathbf{C_m} = \begin{bmatrix} c_{11} & c_{12} & . & . & . & c_{1n} \\ . & . & . & . & . & . \\ c_{r1} & . & . & . & . & c_{rn} \end{bmatrix}; \quad D_{\mathbf{m}} = \begin{bmatrix} d_{11} & d_{12} & d_{1m} \\ & & \\ d_{r1} & & d_{rm} \end{bmatrix};$$

where the system matrix $\mathbf{A_m}$ is the diagonal matrix of eigenvalues $(\lambda_i, i = 1..n)$, decoupling all the states. The eigenvalues of the system matrix are the poles of the system. If all the eigenvalues (system poles) have real parts less than zero, the system is stable.

A continuous LTI system is ***stable*** *if the real parts of the eigenvalues of the system matrix* ***A*** *are all less than zero.*

The $\mathbf{B_m}$ matrix of the modal canonical form tells us about the **controllability** of the system. Is it possible to find a set of inputs **u** to a system which can change the state **x** of a system from any initial state **x**(0) to any final state **x**(T) in finite time T?. The answer is yes, provided *all state variables are capable of being affected by the input.*

For any state x_i, $\dot{x}_i = \lambda_i x_i + \mathbf{b_i u}$

If any row $\mathbf{b_i}$ of the matrix $\mathbf{B_m}$ is <u>zero</u>, the corresponding state x_i cannot be affected by any of the inputs. The behavior of the state x_i is governed solely by the natural response of the system $\dot{x}_i = \lambda_i x_i$. The state x_i is *uncontrollable*.

*A system is **controllable** if **all** the state variables are capable of being affected by the system inputs.*

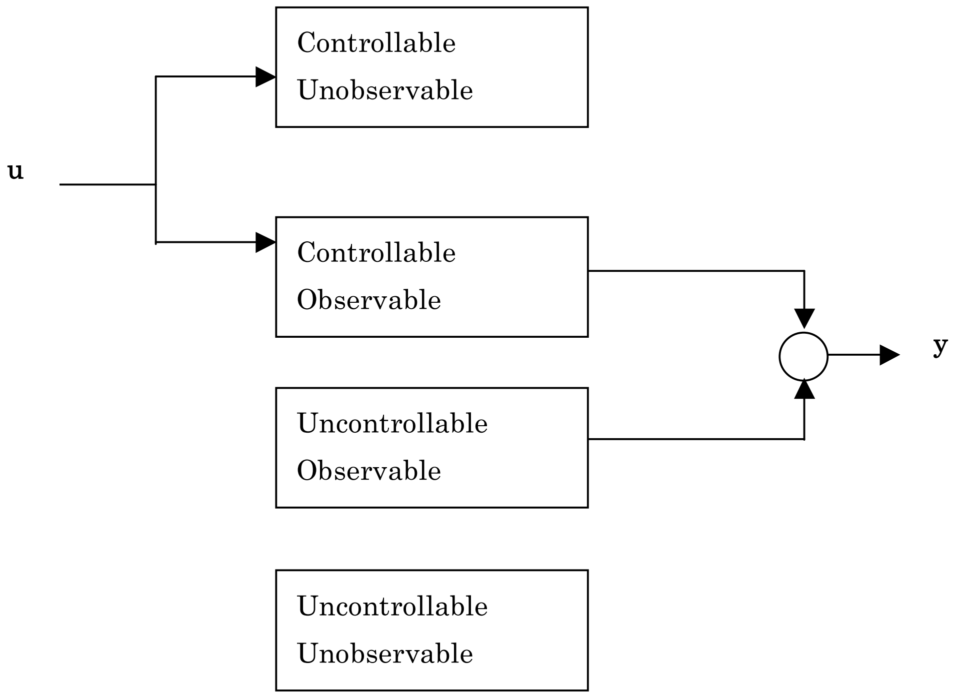
The $\mathbf{C_m}$ matrix of the modal canonical form tells us about the **observability** of the system. Is it possible to estimate any initial state $\mathbf{x}(0)$ of a system from a record of the outputs $\mathbf{y}(t)$ over a finite time?. The answer is yes, provided *all state variables influence the output.*

For any state x_j, $\mathbf{y} = \mathbf{c}_j x_j + \sum_{i \neq j} \mathbf{c}_i x_i + \mathbf{Du}$

If any column $\mathbf{c_j}$ of the output matrix $\mathbf{C_m}$ is <u>zero</u>, the corresponding state x_j cannot affect the output y_j. The state x_j is *unobservable.*

*A system is **observable** if **all** the state variables are capable of affecting the system outputs.*

On this basis, the state of a system can be divided into four categories:



A more general test for controllability which holds for systems with multiplicities in eigenvalues is to evaluate the rank of the "controllability matrix" $\mathbf{M_c}$ (n x nm)

$$\mathbf{M_c} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A^2B} & \cdots & \mathbf{A^{n-1}B} \end{bmatrix}$$

For a controllable system, the controllablity matrix must have rank n

Also we define the "observability matrix" $\mathbf{M_o}$ (nr x n) $\mathbf{M_o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA^2} \\ . \\ . \\ \mathbf{CA^{n-1}} \end{bmatrix}$

For an observable system, the observability matrix must have rank n

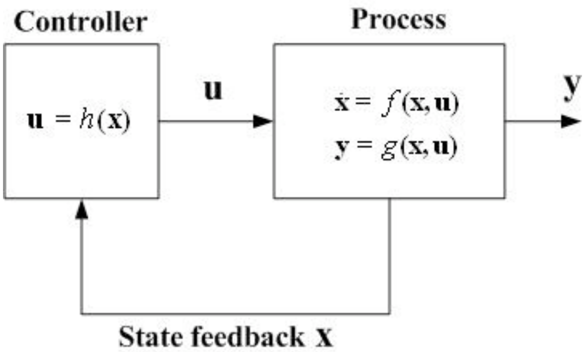
B State Feedback

We will use state space design to design control schemes to stabilize a system and to track a reference input command signal. We may have situations where all the state variables can be measured and used in feedback. Typically however only a limited number of output variables can be measured. In addition, there may be constraints on the transient response of the closed loop (overshoot, settling time) or restrictions on the control input. Optimal control approaches can be used to trade off the conflicting constraints.

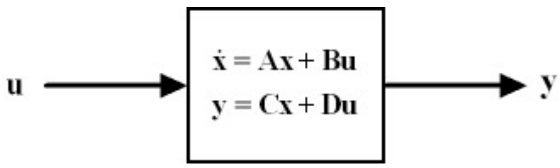
First consider the task of stabilizing a process described by the state equations:

$$\dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u})$$
$$\mathbf{y} = g(\mathbf{x},\mathbf{u})$$

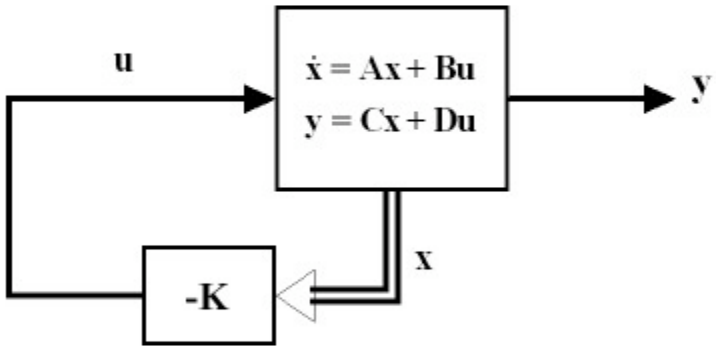
If the system is *controllable*, that is all state variables are capable of being affected by the input, then we can write a "control law" for the system $\mathbf{u} = h(\mathbf{x})$ by feeding back the state variables (or estimates of the state variables).



For a linear time-invariant single-input single-output process represented by the block diagram:



we can use a linear control law given by $\mathbf{u} = \mathbf{-Kx} = -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$



The overall closed loop state space representation can be found from the block diagram:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B(-Kx)}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{D(-Kx)}$$

or

$$\dot{\mathbf{x}} = \mathbf{A_f x}$$
$$\mathbf{y} = \mathbf{C_f x}$$

where $\mathbf{A_f} = \mathbf{A\text{-}BK}$ is the closed loop system matrix.
$\mathbf{C_f} = \mathbf{C\text{-}DK}$ is the closed loop output matrix

Modern control design is concerned with choosing the values of the feedback gain vector **K** to produce the desired closed loop response.

Pole Placement

Choose the values of the observer gain $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}$ to place the closed loop system poles (the eigenvalues of the closed loop system matrix $\mathbf{A_f}$) to satisfy the design constraints (e.g. percent overshoot, settling time etc.)

Assuming the desired poles are to be located at $\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$, then find **K** such that $eig(\mathbf{A\text{-}BK}) = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$

Optimal Control

Choose the values of $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}$ to minimize the performance index:

$$I = \int_0^\infty (\mathbf{x}^\mathbf{T}\mathbf{Q}\mathbf{x} + \mathbf{u}^\mathbf{T}\mathbf{R}\mathbf{u})dt$$

where **Q :** [nxn] symmetric state weighting matrix
R : [mxm] symmetric control weighting matrix

The coefficients in **Q** and **R** are chosen by the designer and represent weighting penalties on the state variables and control inputs respectively. The penalties are quadratic, so large deviations are penalized much more than small ones. For fast tracking, you want to penalize deviations of the state variables from their regulated values, so you would choose high coefficients in the **Q** matrix. At the same time, you do not want to use too much control effort, so you choose high values in the **R** matrix to penalize excessive control inputs. A reasonable rule of thumb is to initially choose diagonal matrices **Q** and **R** such that

$$Q_{ii} = 1/\text{ maximum acceptable value of } \left[x_i^2\right]$$

$$R_{ii} = 1/\text{ maximum acceptable value of } \left[u_i^2\right]$$

In optimal control, once the designer has selected the values of the **Q** and **R** matrices, the solution of the minimization problem can be calculated as:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\mathbf{T}\mathbf{S}$$

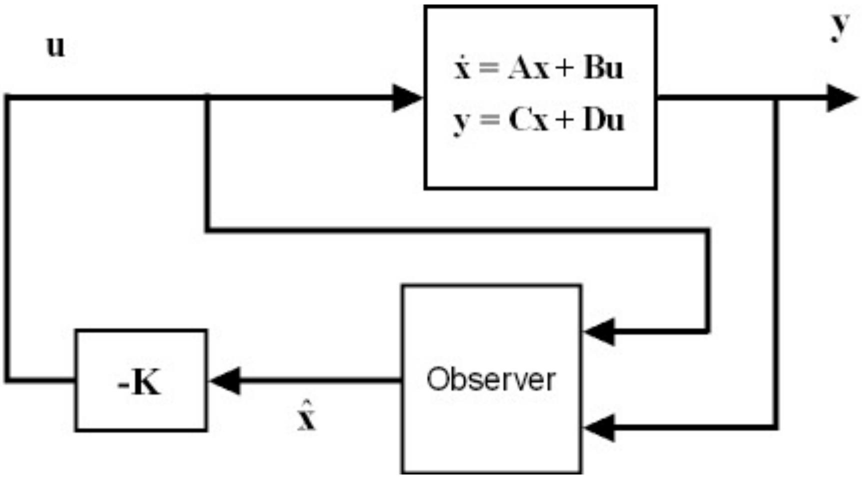
where S :[nxn] symmetric matrix satisfying the matrix Ricatti equation:

$$\mathbf{A}^\mathbf{T}\mathbf{S} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathbf{T}\mathbf{S} + \mathbf{Q} = \mathbf{0}$$

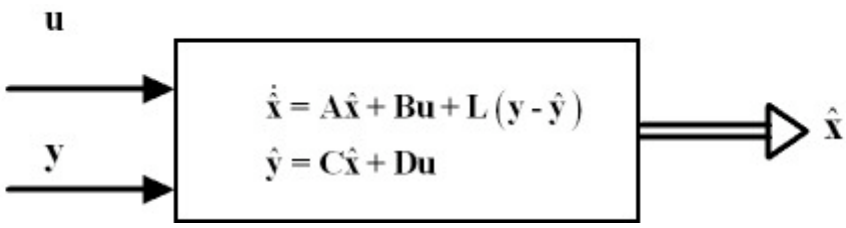
The proof is given in the Appendix

Observer Design

The control law we have been using so far has assumed that *all* the state variables $\mathbf{x}$ are available for feedback. In most cases, we cannot measure all the state variables of the system. To apply the control law, we need to make estimates of the state variables $\hat{\mathbf{x}}$ derived from the measured outputs of the system and the system inputs. This implies that the system is *observable*, that is, the measured output variables can capture the behavior of all the state variables. The subsystem that estimates the state variables is called an "observer" and its function is to output an estimate $\hat{\mathbf{x}}$ from measured output y and calculated input u.



The observer block operates on the difference between the measured output $\mathbf{y}$ and the estimated output $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} + \mathbf{Du}$



The [nx1] gain vector $\mathbf{L}$ is the observer gain and is set by the system designer.
The observer error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ between the actual and estimated state variables is described by:

$$\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{Ax} + \mathbf{Bu} - \left\{\mathbf{A}\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}})\right\} = \mathbf{A}\tilde{\mathbf{x}} - \mathbf{LC}\tilde{\mathbf{x}} = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}} = \mathbf{A_e}\tilde{\mathbf{x}}$$

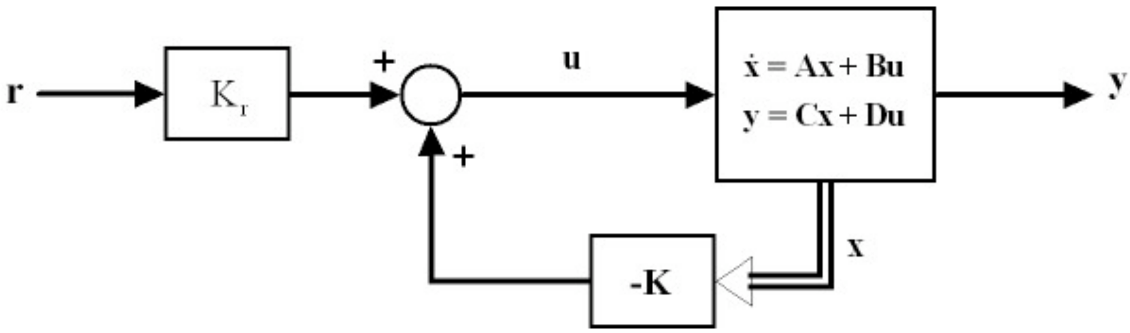
where $\mathbf{A_e} = \mathbf{A} - \mathbf{LC}$ is the observer system matrix.

To design an observer for a state space control system, you want to ensure that the observer error is brought to zero faster than the dominant time constant for the system. You choose the values of the observer gain $\mathbf{L} = \begin{bmatrix} L_1 & L_2 & . & . & . & L_n \end{bmatrix}^T$ to place the observer poles (the eigenvalues of the observer system matrix $\mathbf{A_e}$) 2-5 times faster than the system poles.

Assuming the desired observer poles are to be located at $\begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{bmatrix}$, then find **L** such that *eig*(**A-LC**)= $\begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{bmatrix}$

Reference Gain

The block diagram for a single-input single-output tracking system to track a reference input *r(t)* is shown below:



At steady state we want the output to track the input, that is: $e_{ss} = r - y_{ss} = 0$

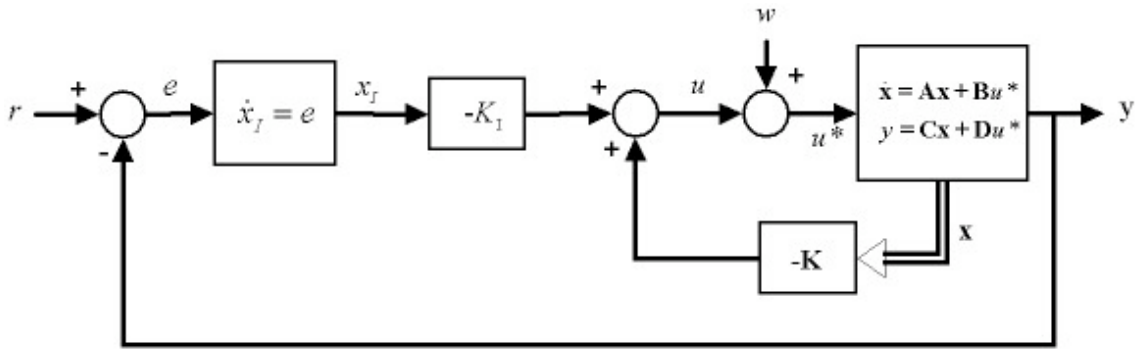
$$\dot{\mathbf{x}} = 0 = \left(\mathbf{A\text{-}BK}\right)\mathbf{x_{ss}} + \mathbf{B}K_r r \Rightarrow \mathbf{x_{ss}} = -\left(\mathbf{A\text{-}BK}\right)^{\mathbf{-1}}\mathbf{B}K_r r$$

$$\mathrm{y_{ss}} = \left(\mathbf{C\text{-}DK}\right)\mathbf{x_{ss}} + \mathbf{D}K_r r = -\left\{\left(\mathbf{C\text{-}DK}\right)\left(\mathbf{A\text{-}BK}\right)^{\mathbf{-1}}\mathbf{B\text{-}D}\right\}K_r\left(e_{ss} + y_{ss}\right)$$

For $e_{ss} = 0 \Rightarrow \; K_r = -\left\{\left(\mathbf{C\text{-}DK}\right)\left(\mathbf{A\text{-}BK}\right)^{\mathbf{-1}}\mathbf{B\text{-}D}\right\}^{\mathbf{-1}}$

Robust Control

When the system is subject to uncontrolled disturbances w, it is useful to augment the state vector with an extra state variable, the integral of the error signal:



The integral action attenuates the steady state error due to the uncontrolled disturbance

$$\dot{x}_I = e = r - y = r - \mathbf{Cx} - \mathbf{D}u - \mathbf{D}w$$

The augmented state equations become

$$\dot{\hat{\mathbf{i}}} = \begin{bmatrix} \mathbf{0} & \mathbf{-C} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \hat{\mathbf{i}} + \begin{bmatrix} \mathbf{-D} \\ \mathbf{B} \end{bmatrix} u + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} r + \begin{bmatrix} \mathbf{-D} \\ \mathbf{B} \end{bmatrix} w \qquad \text{where } \hat{\mathbf{i}} = \begin{bmatrix} x_I \\ \mathbf{x} \end{bmatrix}$$

and the feedback control law is

$$u = -\begin{bmatrix} K_I & \mathbf{K} \end{bmatrix} \hat{\mathbf{i}}$$

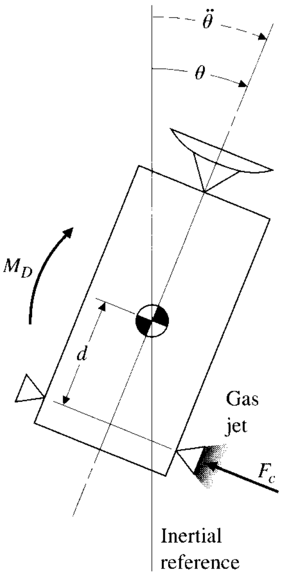
Pole Placement or Optimal Control techniques can then be used on the augmented state equations.

C Case Study of Modern Control Design

Satellite Attitude Control

MILSTAR Telecommunications Satellite

Satellites require attitude control for proper orientation of antennas and sensors with respect to the earth. A simplified one-axis model of a satellite is shown below with motion about an axis perpendicular to the page.



Franklin et al, Feedback Control of Dynamic Systems

4th Ed, Prentice-Hall, 2002

Gas jets provide the control moment for changing the orientation θ of the satellite. Uncontrolled disturbance torques M_D can occur as a result of solar pressure or orbit perturbations. The equation of motion about one axis of the satellite is given by

$$J\ddot{\theta} = F_c d + M_D;$$

where J is the moment of inertia of the satellite about its mass center, $F_c d$ is the control torque applied by the gas jets, and the disturbance moment M_D results from solar pressure acting on any asymmetry in the attached solar panels. θ is the angle of the satellite axis with respect to an inertial reference frame. Normalizing, we define:

$$u = \frac{F_c d}{J}; w = \frac{M_D}{J}$$

and obtain

$$\ddot{\theta} = u + w$$

with initial conditions $\theta(0) = 0; \dot{\theta}(0) = 0$, assuming that the initial satellite pointing angle and the initial satellite angular velocity are both zero.

Full State Feedback

Neglecting the disturbance moment, the motion about one axis of a communications satellite is given by

$$J\ddot{\theta} = F_c d$$

Convert to state space form with $x_1 = \theta; x_2 = \dot{\theta}; u = \frac{F_c d}{J};$

Then the state equation is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$

Find the optimal state feedback gain vector **K** to minimize the performance index:

$$I = \int_0^\infty (x_1^2 + wu^2) dt$$

<u>Solution</u>:

The weighting matrices are $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \mathbf{R} = \begin{bmatrix} w \end{bmatrix}$

Solve the matrix Ricatti equation

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s_1 & s_3 \\ s_3 & s_2 \end{bmatrix} + \begin{bmatrix} s_1 & s_3 \\ s_3 & s_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_1 & s_3 \\ s_3 & s_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} w^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s_1 & s_3 \\ s_3 & s_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$=> s_1 = \sqrt{2}w^{\frac{1}{4}}; s_2 = \sqrt{2}w^{\frac{3}{4}}; s_3 = w^{\frac{1}{2}}$$

Then $\mathbf{K} = \begin{bmatrix} w^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2}w^{\frac{1}{4}} & \sqrt{w} \\ \sqrt{w} & \sqrt{2}w^{\frac{3}{4}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{w}} & \sqrt{2}w^{-\frac{1}{4}} \end{bmatrix}$

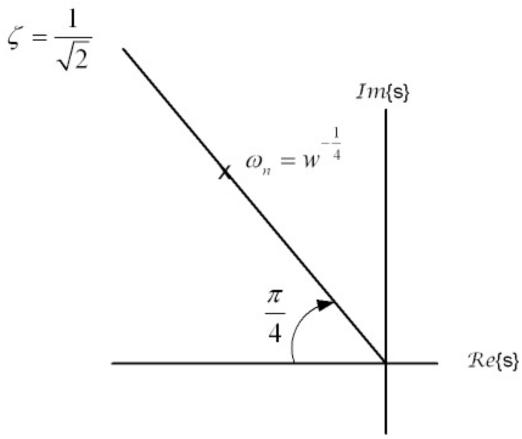
The optimal feedback gain **K** depends on the designers choice of the relative penalty w placed on the control effort u^2. Large values of w correspond to the design decision to conserve "fuel" and limit the accumulated control input. This results in low feedback gain values and correspondingly slow system response to a tracking command. The opposite is true for small values of w.

After the feedback gain **K** is computed, the system poles can be calculated as the eigenvalues of the closed loop system matrix $\mathbf{A_f}$ = **A-BK.** For the satellite example derived above, the system poles are found as follows:

$$\mathbf{A_f} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} w^{-\frac{1}{2}} & \sqrt{2}w^{-\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^{-\frac{1}{2}} & -\sqrt{2}w^{-\frac{1}{4}} \end{bmatrix}$$

$$\left|\lambda\mathbf{I} - \mathbf{A_f}\right| = \left|\begin{bmatrix} \lambda & -1 \\ w^{-\frac{1}{2}} & \lambda + \sqrt{2}w^{-\frac{1}{4}} \end{bmatrix}\right| = \lambda^2 + \sqrt{2}w^{-\frac{1}{4}}\lambda + w^{-\frac{1}{2}}$$

The poles are at $\lambda = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n$ where $\omega_n = w^{-\frac{1}{4}}; \zeta = \frac{1}{\sqrt{2}}$. This suggests that the optimum pole placements for the satellite design lie along the constant $\zeta = \frac{1}{\sqrt{2}}$ line in the complex plane at a radial distance from the origin $\omega_n = w^{-\frac{1}{4}}$



Large values of w correspond to "slow" poles close to the origin and a slow response to the tracking command. Small values of w correspond to "fast" poles with fast response to the tracking command.
The principal advantage of the modern control approach is that the design decisions are related directly to the balance between performance and cost.

Observer-based control
Design an observer based state space control scheme for a satellite with integral action to track a step input command $\theta_r = 1$. The control system must achieve the following specifications for the step response:

- Settling time < 5s
- Overshoot $< 10\%$
- Control effort $|u| < 10$
- Satellite angular velocity $|\dot{\theta}| < 1$ rad/s
- Zero steady state error to a disturbance $w = -2$

Solution:
Augment the state variables $\dot{x}_I = e = r - y = r - \mathbf{C}\mathbf{x}$

$$\dot{\hat{\mathbf{i}}} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{i}} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

for $\xi_1 = \int (r - \theta) dt; \xi_2 = \theta; \xi_3 = \dot{\theta};$

Find the optimal state feedback gain vector $\begin{bmatrix} K_I & K_1 & K_2 \end{bmatrix}$ to minimize the performance index:

$$I = \int_0^\infty (\mathbf{\hat{\imath}}^{\mathbf{T}}\mathbf{Q}\mathbf{\hat{\imath}} + \mathbf{u}^{\mathbf{T}}\mathbf{R}\mathbf{u})dt$$

Choose the weighting matrices corresponding to the given design constraints:

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \mathbf{R} = [\ 0.01\ ]$$

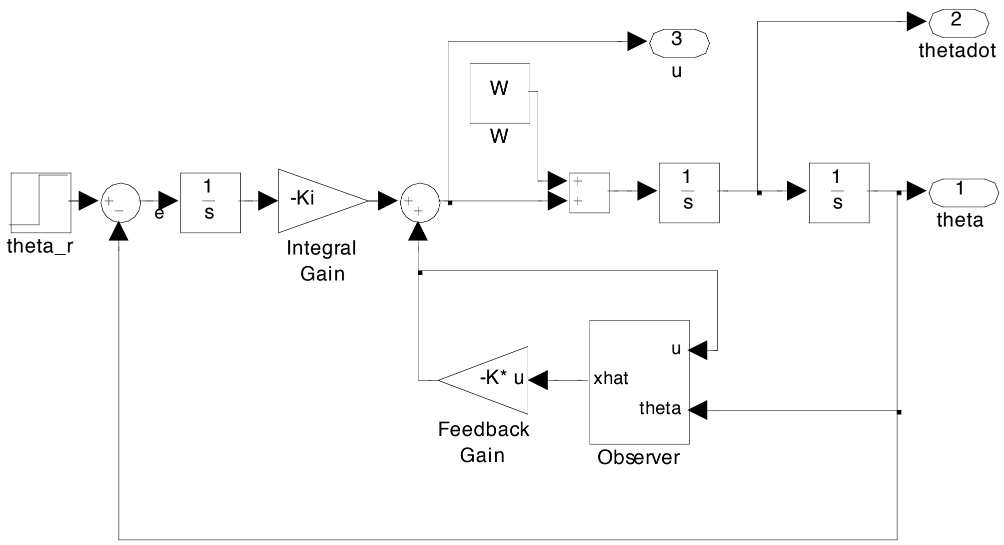
Solve the matrix Ricatti equation (using MATLAB's **lqr** command)
The optimal gains are:

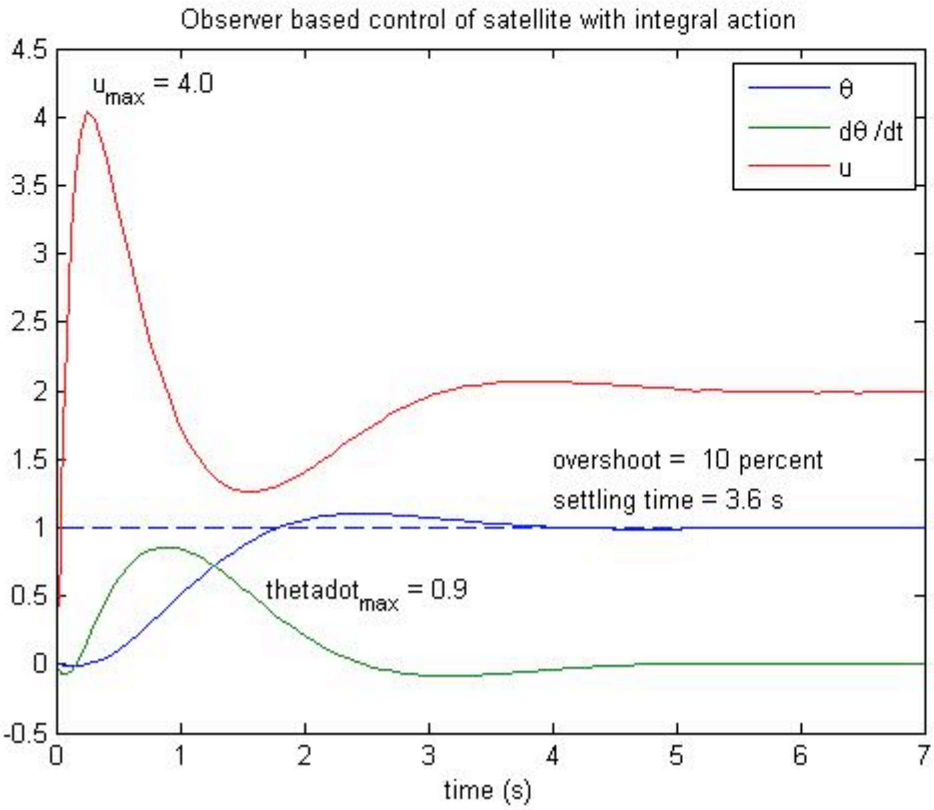
$$K_I = -20; \mathbf{K} = \begin{bmatrix} 24 & 12 \end{bmatrix}$$

Design an observer with poles 5 times faster than the system poles

$$\mathbf{L} = \begin{bmatrix} 101 \\ 2530 \end{bmatrix}$$

Simulate the satellite control system





Meets design specifications

Summary

The state space representation of systems allows for direct access to the internal behavior of a system. For LTI systems, linear algebra techniques can be used to design control systems involving:

- Determining the feedback gain vector **K** by pole placement or optimal control.
- Determining the observer gain vector **L** by pole placement
- Determining the reference gain K_r for zero steady state error
- Determining the augmented feedback gain vector $\left[K_I \quad \mathbf{-K}\right]$ for attenuating disturbances

Appendix The Optimal Linear Quadratic Regulator

Problem

Given a process $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ with initial state $\mathbf{x}(0)$; calculate the feedback gain vector $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}$ such that $\mathbf{u} = \mathbf{-Kx}$ minimizes the performance index $I = \int_0^\infty (\mathbf{x}^\mathbf{T}\mathbf{Qx} + \mathbf{u}^\mathbf{T}\mathbf{Ru})dt$ where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric weighting matrices.

Solution

With feedback $\dot{\mathbf{x}} = (\mathbf{A - BK})\mathbf{x} = \mathbf{A_f}\,\mathbf{x}$

Write $I = \int_0^\infty \mathbf{x}^\mathbf{T}\mathbf{Px}\,dt$ where $\mathbf{P} = \mathbf{Q} + \mathbf{K}^\mathbf{T}\mathbf{RK}$

We postulate the existence of an exact differential, such that

$$\frac{d}{dt}\left(\mathbf{x}^\mathbf{T}\mathbf{Sx}\right) = -\mathbf{x}^\mathbf{T}\mathbf{Px}$$

the differential is:

$$\dot{\mathbf{x}}^\mathbf{T}\mathbf{Sx} + \mathbf{x}^\mathbf{T}\mathbf{S}\dot{\mathbf{x}} = (\mathbf{A_f x})^\mathbf{T}\mathbf{Sx} + \mathbf{x}^\mathbf{T}\mathbf{S}(\mathbf{A_f x}) = \mathbf{x}^\mathbf{T}(\mathbf{A_f^T S} + \mathbf{SA_f})\mathbf{x}$$

$$\Rightarrow \; -\mathbf{P} = \mathbf{A_f^T S} + \mathbf{SA_f} \quad \cdots\cdots (1)$$

$$\Rightarrow \; I = \int_0^\infty -\frac{d}{dt}\left(\mathbf{x}^\mathbf{T}\mathbf{Sx}\right)dt = -\mathbf{x}^\mathbf{T}\mathbf{Sx}\Big|_0^\infty = \mathrm{x}^\mathrm{T}(0)\mathrm{Sx}(0)$$

Expand (1)

$$(\mathbf{A}^\mathbf{T} - \mathbf{K}^\mathbf{T}\mathbf{B}^\mathbf{T})\mathbf{S} + \mathbf{S}(\mathbf{A} - \mathbf{BK}) + \mathbf{Q} + \mathbf{K}^\mathbf{T}\mathbf{RK} = \mathbf{0} \quad \cdots\cdots (2)$$

Let $\hat{\mathbf{K}}$ be the optimal $\mathbf{K}$, then $\hat{\mathbf{K}} + \delta\mathbf{K}$ will be suboptimal.
Substitute $\hat{\mathbf{K}}$ in (2)

$$(\mathbf{A}^\mathbf{T} - \hat{\mathbf{K}}^\mathbf{T}\mathbf{B}^\mathbf{T})\mathbf{S} + \mathbf{S}\left(\mathbf{A} - \mathbf{B}\hat{\mathbf{K}}\right) + \mathbf{Q} + \hat{\mathbf{K}}^\mathbf{T}\mathbf{R}\hat{\mathbf{K}} = \mathbf{0} \quad \cdots\cdots (3)$$

Substitute $\hat{\mathbf{K}} + \delta\mathbf{K}$ in (2)

$$(\mathbf{A}^\mathbf{T} - \left(\hat{\mathbf{K}}+\text{ä}\hat{\mathbf{K}}\right)^\mathbf{T}\mathbf{B}^\mathbf{T})\mathbf{S} + \mathbf{S}\left(\mathbf{A} - \mathbf{B}\left(\hat{\mathbf{K}}+\text{ä}\hat{\mathbf{K}}\right)\right) + \mathbf{Q} + \left(\hat{\mathbf{K}}+\text{ä}\hat{\mathbf{K}}\right)^\mathbf{T}\mathbf{R}\left(\hat{\mathbf{K}}+\text{ä}\hat{\mathbf{K}}\right) = \mathbf{0}$$

Expanding and ignoring products of small quantities

$$\mathbf{(A^T - \hat{K}^T B^T)S + S\left(A - B\hat{K}\right) + Q + \hat{K}^T R\hat{K}}$$

$$\mathbf{-\delta K^T B^T S - SB\delta K + \hat{K}^T R\delta K + \delta K^T R\hat{K} + \hat{K}^T R\delta K = 0} \quad \ldots\ldots(4)$$

Subtract (3) from (4)

$$\mathbf{\delta K^T\left(R\hat{K} - B^T S\right) + \left(\hat{K}^T R - SB\right)\delta K = 0}$$

Set $\mathbf{R\hat{K} = B^T S}$ => $\mathbf{\hat{K} = R^{-1}B^T S}$

and set $\mathbf{\hat{K}^T R = SB}$ => $\mathbf{\hat{K}^T = SBR^{-1}}$ => $\mathbf{\hat{K} = R^{-1}B^T S}$ since **S** and **R** are symmetric

So: => optimal $\mathbf{\hat{K} = R^{-1}B^T S}$ $\cdots\cdots$ (5)

where **S** is found by substituting (5) into (2)

$$\mathbf{(A^T - SBR^{-1}B^T)S + S\left(A - BR^{-1}B^T S\right) + Q + SBR^{-1}RR^{-1}B^T S = 0}$$

=> $\mathbf{A^T S + SA - SBR^{-1}B^T S + Q = 0}$ the matrix Ricatti equation