# Where did the laws of physics come from? 

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#### Abstract

The laws of physics are constrained so that they select out no preferred coordinate system or reference frame. This is called the principle of covariance. This principle can be further generalized to include the coordinates in the abstract space of the functions used to formulate those laws. This is called global gauge invariance. When this symmetry applies independently at every point in space-time, it is called local gauge invariance. These symmetries are almost all that are needed to derive most of the familiar laws the law of physics, including classical mechanics, the great conservation laws, quantum mechanics, special and general relativity, and electromagnetism. Those structures that do not follow directly from coordinate invariance result from spontaneously broken symmetries.


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### 1.0 Introduction

Most laypeople think of the laws of physics as something like the Ten Commandments-rules governing the behavior of matter imposed by some great lawgiver in the sky. However, no stone tablet has ever been found upon which such laws were either naturally or supernaturally inscribed. On the contrary, the laws of
physics are human inventions-mathematical formulas that quantitatively describe the results of observations and measurements. These formulas are first inferred from and then tested against observations. If they hold up, they are eventually reformulated as part of general and universal theories that are derived from a minimum number of assumed fundamental principles. Very often, a "law" will turn out to be nothing more than a circular definition, such as Ohm's law which says that the voltage is proportional to the current in a resistor, where a resistor is defined as a device that obeys Ohm's law.

Since the time of Copernicus and Galileo it has been realized that the laws of physics should not single out any particular space-time reference frame, although a distinction between inertial and noninertial frames was maintained in Newtonian physics. That distinction was removed in 1916 by Einstein who formulated his general theory of relativity in a covariant way. That is, the form of Einstein's equations is the same in all reference frames, inertial or noninertial.

As this experience showed, physicists are highly constrained in the way they may formulate the laws of physics. Not only must they agree with the data, the equations that are used to describe that data should not be written in such a way as to specify a privileged coordinate system or reference frame. This principle of covariance generalizes other notions such as the Copernican and cosmological principles and the principle of Galilean relativity. The application of this principle is not a matter of choice; centuries of observations have shown that to do otherwise produces calculations that disagree with the data in some reference frames.

In 1918, Noether showed that coordinate independence was more than just a constraint on the mathematical form of physical laws.[1] She proved that some of the most important physics principles are, in fact, nothing more than tautologies that follow from space-time coordinate independence: energy conservation arises from time translation invariance, linear momentum conservation comes from space translation invariance, and angular momentum conservation is a consequence of space rotation invariance. These conserved quantities were simply the mathematical generators of the
corresponding symmetry transformation.
As the twentieth century progressed, invariance or symmetry principles became an increasingly dominant idea in physics. Not only were space-time coordinate symmetries built into theories, the notion of coordinate independence was extended to the abstract spaces physicists use to represent the other degrees of freedom of systems. Rotational symmetry was also applied to the space of quantum state vectors, resulting in derived properties of spin, isospin, charge, baryon number, and other observables that agreed with measurements.

Charge conservation, for example, was found to follow from the invariance of the Schrödinger equation to changes in the phase of the complex wave function. And then, a remarkable discovery was made. It was found that the Schrödinger equation could be made invariant to a local phase change in the wave function, that is, a change in phase that varies from point to point in space-time, provided that vector and scalar potentials were added. The potentials turned out to be exactly those that give the classical electric and magnetic fields. This local quantum phase symmetry was precisely related to the local classical gauge symmetry of electrodynamics. Maxwell's equations were derived from a single principle-local phase invariance.

If we think of the Schrödinger wave function as a "vector" in 2-dimensional complex space, then changing phase is equivalent to a rotation in that space and phase invariance, or gauge invariance, is equivalent to rotational invariance. Indeed, the generator of that transformation is the electric charge whose conservation follows from global gauge invariance.

In the standard model, the fields associated with the weak and strong nuclear forces are obtained by extending the idea of gauge symmetry to higher dimensions of abstract space. There the situation is complicated by the fact that all the symmetries are not exact at the "low temperatures" of current experimentation. Good thing. The diversity and complexity of the universe is a result of broken symmetries, without which we would not be here to do the experiments.

Twentieth century physics was also marked by the discovery that symmetries are often broken. In the 1950s, it was found that weak interactions maximally violated space reflection symmetry; that is, they were not invariant under the parity operation $P$ that changes the handedness or chirality of a system. In the 1960s certain rare decays were found to be noninvariant under the combined operation $C P$, where $C$ changes a particle to its antiparticle. The study of the origin of $C P$ violation remains a subject of considerable experimental and theoretical effort to this date.

In this paper, it will be shown that much of familiar physics can be derived from the generalized notion of coordinate invariance applied not only in space-time but in the spaces of other observables and the spaces of the functions that are used to mathematically describe physical phenomena. In order to make this result accessible to the greatest number of people, the mathematical level will be limited to that of an advanced undergraduate student in physics or mathematics. The equations will appear very familiar-just those found in physics textbooks, and it may appear that the author is using hindsight to make things come out the way they already are. However, the reader is asked to look carefully at how those equations are obtained. Certain familiar principles normally taken as axioms, such as the quantization of angular momentum and the invariance of the speed of light will be derived from the hypothesized symmetry principles without additional assumptions.

### 2.0 Gauge Symmetry

Let $q=\left(q_{0}, q_{1}, q_{2}, q_{3}, \ldots q_{n}\right)$ be the set of observables of a physical system such as a particle or group of particles and take them to be the coordinates of an $n$-dimensional vector $q$ in $q$-space. Spatial coordinates and time are included and placed on the same footing as the other observables. Thus a point in $q$-space, designated by the vector $q$, represents a particular set of measurements on a system. The generalized principle of covariance says that the laws of physics must be the same for any origin or orientation
of $q$, that us, any choice of coordinate system.
Let us define a vector $\square$ in another multidimensional space we will call $\square$-space.
Assume $\square=\square(\mathrm{q})$. The state vectors of quantum mechanics are familiar examples of $\square$ space vectors. We can imagine a set of coordinate axes in $\square$-space. Extending the notion of covariance to this space we will assume that the following principle holds: the laws of physics cannot depend on the orientation of the vector $\square$ in $\square$-space. This principle is called gauge symmetry.

## 3. Gauge Transformations and their Generators

To get started as simply as possible, let us take $\square(q)$ to be a complex function, that is, a 2-dimensional vector with coordinates $(\operatorname{Re}\{\square\}, \operatorname{Im}\{\square\})$. Let us perform a unitary transformation on $\square$ :

$$
\begin{equation*}
\square^{\prime}=U \square \tag{3.1}
\end{equation*}
$$

where $U+U=1$, so

$$
\begin{equation*}
\square^{+} \square^{\prime}=\square^{+} U^{+} U \square=\square^{+} \square \tag{3.2}
\end{equation*}
$$

This transformation does not change the magnitude of $\square$,

$$
\begin{equation*}
\left|\square^{\prime}\right|=\left(\square^{\prime}+\square^{\prime}\right)^{1 / 2}=\left(\nabla^{+} \square\right)^{1 / 2}=|\nabla| \tag{3.3}
\end{equation*}
$$

That is, $|\square|$ is invariant to the transformation, as required by gauge symmetry. We can write the operator $U$

$$
\begin{equation*}
U=\exp (i \square) \tag{3.4}
\end{equation*}
$$

where $\square^{+}=\square$, that is, $\square$ is a hermitian operator. Then,

$$
\begin{equation*}
\square^{\prime}=\exp (i \square \square \tag{3.5}
\end{equation*}
$$

So, $U$ simply changes the complex phase of $\square$. It could be called a "phase transformation," or just simply a unitary transformation. However, in the amplifications of this idea that we will discussing, the designation "gauge transformation" has become conventional. When $\square$ is a constant we have a global gauge transformation.When $\square$ is a not a constant but a function of position and time it is called a local gauge transformation.

Note also that the operation $U$ corresponds to a rotation in the complex space of $\square$. Later we will generalize these ideas to where $\square$ is a vector in higher dimensions and $\square$ will be represented by a matrix. But this basic idea of a gauge transformation as analogous to a rotation in an abstract function space will be maintained and gauge invariance viewed as an invariance under such rotations.

Let us write

$$
\begin{equation*}
\square=\square G \tag{3.6}
\end{equation*}
$$

where $\square$ is an infinitesimal number and $G$ is another operator. Then

$$
\begin{equation*}
U \approx 1+i \llbracket G \tag{3.7}
\end{equation*}
$$

where $G^{+}=G$ is hermitian and is called the generator of the transformation. Then,

$$
\begin{equation*}
\square^{\prime} \approx \square+i \square G \square \tag{3.8}
\end{equation*}
$$

Suppose we have a transformation that takes the variable $q_{\square}$ to $q^{\prime}{ }_{\square}=q_{\square}+\square_{\square}$. Then

$$
\begin{equation*}
\square^{\prime}\left(q^{\prime}{ }_{\square}\right)=\square\left(q_{\square}+\square_{\square}\right) \approx \square\left(q_{\square}\right)+\square_{\square} \partial \square / \partial q_{\square} \tag{3.9}
\end{equation*}
$$

It follows that the generator can be written

$$
\begin{equation*}
G_{\square}=-i \partial / \partial q_{\square} \tag{3.10}
\end{equation*}
$$

## Define

$$
\begin{equation*}
P_{\square} \equiv \hbar G=\square i \hbar \frac{\partial}{\partial q_{\square}} \tag{3.11}
\end{equation*}
$$

where $\hbar$ is an arbitrary constant introduced only if you want the units of $P_{\square}$ to be different from the reciprocal of the units of $q_{\square}$. The transformation operator can then be written

$$
\begin{equation*}
U=1+\frac{\partial}{\partial q_{\square}} \tag{3.12}
\end{equation*}
$$

For example, suppose that $q_{1}=x$, the $x$-coordinate of a particle. Then

$$
\begin{equation*}
P_{1} \equiv P_{x}=\square i \hbar \frac{\partial}{\partial x} \tag{3.13}
\end{equation*}
$$

which we recognize as the quantum mechanical operator for the $x$-component of momentum. Note that this association was not assumed but derived and no connection with mass and velocity has yet been made. This just happens to be the form of the generator of a space translation. Similarly, we can take $q_{2}=y, q_{3}=z$ and obtain the generators $P_{y}$ and $P_{z}$.

It may also be noted that $q$ might contain operationally defined momenta, in which case spatial coordinates would then be introduced in the manner of (3.13).

Of course, $\hbar$ will turn out to be the familiar quantum of action of quantum mechanics, $\hbar=h / 2 \pi$ where $h$ is Planck's constant. Physicists often take $\hbar=1$ in "natural units." We will leave $\hbar$ in our equations at this point to maintain familiarity, however it should be recognized that this constant, when expressed in non-dimensionless units, will turn out to be an arbitrary number determined only by that choice of units. No additional physical assumption about the "quantization of action" need be made and Planck's constant should not be viewed as a metric constant of nature. In particular, $\hbar$ cannot be zero. Once we have made the connection of $\left(P_{x}, P_{y}, P_{z}\right)$ with the 3momentum, quantization of action will already be in place.

We can also associate one of the variables, say $q_{0}$ with the time $t$. In order to provide a connection with the fully relativistic treatment we will make later, let $q_{o} \equiv i c t$, where $c$ is, like $\hbar$, another arbitrary conversion factor. Later we will associate it with the speed of light in a vacuum and find (not assume) that it is a Lorentz invariant. For now,

$$
\begin{equation*}
P_{o}=i \hbar \frac{\partial}{\partial q_{o}}=\square \frac{\hbar}{c} \frac{\partial}{\partial t} \tag{3.14}
\end{equation*}
$$

We can then define

$$
\begin{equation*}
H \equiv\left\lceil i P_{o} c=i \hbar \frac{\partial}{\partial t}\right. \tag{3.15}
\end{equation*}
$$

which we recognize as the quantum mechanical Hamiltonian (energy) operator. Note, again, that this familiar result was not assumed but derived. No connection with the physical quantity energy has yet been made. This just happens to be the form of the generator of a time translation.

## 4. Quantum Mechanics from Gauge Transformations

Suppose we have a complex function $\square(x, y, z, t)$ that describes, in some unspecified way, the state of a system. It will evolve with time according to

$$
\begin{equation*}
H \square=i \hbar \frac{\partial}{\partial t} \tag{4.1}
\end{equation*}
$$

This is the time-dependent Schrödinger equation of quantum mechanics, where $\square$ is interpreted as the wave function. If $H$ is independent of time, we have the solution

$$
\begin{equation*}
\square(t)=\square(0) \exp \square_{\square} \frac{i}{\hbar} H t \stackrel{\square}{[ } \tag{4.2}
\end{equation*}
$$

so

$$
\begin{equation*}
U(t)=\exp \overbrace{\hbar}^{i} \frac{i}{\hbar} H t \stackrel{\square}{\square} \tag{4.3}
\end{equation*}
$$

is the time evolution operator.
At this point, then, we have the makings of quantum mechanics with no physical assumptions whatsoever. That is, we have a mathematical theory that looks like quantum mechanics although we have not yet identified the operators $H$ and $P$ with the physics quantities energy and momentum. We have simply noted that these are generators of time and space translations respectively, which are themselves gauge transformations.

Let us proceed along these same lines, considering only the mathematics of gauge transformations and leaving the physics to later. This does not stop us from using the Dirac bra and ket notation for linear vectors and operators. Again, no physical assumption is being made. We are simply using a convenient mathematical formalism. So, let $|\square\rangle$ be a linear vector and $\langle\nabla|$ be its dual. For simplicity, we take our linear vectors to have unit norm,

$$
\begin{equation*}
\langle\square \mid \square\rangle=1 \tag{4.4}
\end{equation*}
$$

A unitary transformation on $|\square\rangle$ will preserve the norm.

$$
\begin{align*}
& |\overrightarrow{|q|}|=u| || \rangle  \tag{4.5}\\
& \langle\nabla| \mu^{+} u|\nabla|=1 \tag{4.6}
\end{align*}
$$

Let $A$ be a linear operator that gives another vector of unit norm

$$
\begin{equation*}
|\square\rangle=A|\square\rangle \tag{4.7}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\langle\square| A|D\rangle=1 \tag{4.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle\square| U^{+} A U|\nabla\rangle=\langle\square| A|\square\rangle=\langle\square| \angle| \rangle=1 \tag{4.9}
\end{equation*}
$$

We can define

$$
\begin{equation*}
A^{\prime}=U^{+} A U \tag{4.10}
\end{equation*}
$$

and write

$$
\begin{equation*}
\langle\square \mid A \phi\rangle=1 \tag{4.11}
\end{equation*}
$$

Let us consider the specific case where $U$ is the time evolution operator,

$$
\begin{equation*}
\square(t)\rangle=U \square(0)\rangle \tag{4.12}
\end{equation*}
$$

In that case,

$$
\begin{equation*}
\langle\square(0)| U^{+} A U|\square(0)\rangle=\langle\square(t) \mid A \square(t)\rangle=\langle\square t)|\square(t)\rangle=1 \tag{4.13}
\end{equation*}
$$

Alternatively, define

$$
\begin{equation*}
A(t)=U^{\dagger} A(0) U \tag{4.14}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\langle\square(0)| A(t)|\square(0)\rangle=1 \tag{4.15}
\end{equation*}
$$

This illustrates the two approaches to time evolution in quantum mechanics. In the Schrödinger picture, the state vector varies with time while the operators stay fixed. In the Heisenberg picture, the state vectors remain fixed while the operators evolve with time.

If we now interpret, in usual quantum mechanical fashion, the state vectors in terms of probabilities and the operators in terms of observables, the expectation value for the observable $A$, the mean value expected for an ensemble of measurements of $A$ when the system is in the state $|\square(0)\rangle$ is, in the Schrödinger picture,

$$
\begin{equation*}
A(0)=\langle\square(0) \mid A \square(0)\rangle \tag{4.16}
\end{equation*}
$$

It evolves with time according to

$$
\begin{equation*}
A(t)=\langle\square(t) \mid A \square(t)\rangle \tag{4.17}
\end{equation*}
$$

where

$$
\begin{equation*}
|\square(t)\rangle=U \square(0)\rangle \tag{4.18}
\end{equation*}
$$

In the Heisenberg picture we have

$$
\begin{equation*}
\langle A(0)\rangle=\langle\square(0)| A(0)|\square(0)\rangle \tag{4.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle A(t)\rangle=\langle\square(0)| A(t)|\nabla(0)\rangle \tag{4.20}
\end{equation*}
$$

where

$$
\begin{equation*}
A(t)=U^{\dagger} A(0) U=\exp \frac{i}{\hbar} H t \xrightarrow{\circ} A(0) \exp \frac{i}{\hbar} H t \tag{4.21}
\end{equation*}
$$

Let us look further at the time evolution of operators. Suppose we make an infinitesimal transformation in time $t \square t+d t$. Then

$$
\begin{gather*}
U(t)=1 \square \frac{i}{\hbar} H t  \tag{4.22}\\
A(t+d t)=\stackrel{\square}{\square}+\frac{i}{\hbar} H t \stackrel{\square}{\square} A(t) \square \frac{i}{\hbar} H t \cap=A(t) \square \frac{i}{\hbar}[A, H] \tag{4.23}
\end{gather*}
$$

Since

$$
\begin{equation*}
A(t+d t)=a(t)+d t \frac{\partial A}{\partial t} \tag{4.24}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
\frac{\partial A}{\partial t}=\square \frac{i}{\hbar}[A, H] \tag{4.25}
\end{equation*}
$$

The time rate of change of an observable then is

$$
\begin{equation*}
\frac{d A}{d t}=\frac{\partial A}{\partial t}+\square_{k \neq 0} \frac{\partial A}{\partial q_{k}} \frac{d q_{k}}{d t} \tag{4.26}
\end{equation*}
$$

or,

$$
\begin{equation*}
\frac{d A}{d t}=\square \frac{i}{\hbar}[A, H]+\square_{k=1}^{n} \frac{\partial A}{\partial q_{k}} \frac{d q_{k}}{d t} \tag{4.27}
\end{equation*}
$$

where the sum excludes the time variable.
Now we move to gauge transformations involving the non-temporal variables of a system. Consider the case where $A=P_{j}$. Then,

$$
\begin{equation*}
\frac{d P_{j}}{d t}=\square \frac{i}{\hbar}\left[P_{j}, H\right]+\square_{k=1}^{n} \frac{\partial P_{j}}{\partial q_{k}} \frac{d q_{k}}{d t} \tag{4.28}
\end{equation*}
$$

Next, let us look at the transformation of these non-temporal variables. Let the variable $q_{k} \square q_{k}+\square_{k}$, where $\square_{k}$ is infinitesimal. Then, as we saw above, the transformation
operator is

$$
\begin{equation*}
U=1+\frac{i}{\hbar} P_{k} \square k \tag{4.29}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left.\left|\square\left(q_{k}+\square_{k}\right)\right\rangle=\emptyset\left(q_{k}\right)\right\rangle+\frac{i}{\hbar} P_{k} \square_{k}\left|\square\left(q_{k}\right)\right\rangle \tag{4.30}
\end{equation*}
$$

Suppose we have an operator $A$ defined by

$$
\begin{equation*}
A\left|\square\left(q_{k}\right)\right\rangle=\left|\square\left(q_{k}\right)\right\rangle \tag{4.31}
\end{equation*}
$$

In our previous consideration of the time variable we derived the time evolution equation for an operator in the Heisenberg picture where the time dependence is carried by the operator rather than the state vector. Let us continue to work in that picture. The state vectors will then not depend explicitly on time, but they still can depend on the other variables. So,

$$
\begin{equation*}
\left.\left.\left.\left|\triangle\left(q_{k}+\square_{k}\right)\right\rangle=U\left|\square\left(q_{k}\right)\right\rangle=U A U^{+} U \square\left(q_{k}\right)\right\rangle=U A U^{+} \square\left(q_{k}+\square_{k}\right)\right\rangle=A \phi \square\left(q_{k}+\square_{k}\right)\right\rangle \tag{4.32}
\end{equation*}
$$

where

$$
\begin{equation*}
A^{\prime}=U A U^{+} \tag{4.33}
\end{equation*}
$$

We can write this

$$
\begin{equation*}
A \square=\frac{i}{\square}+\frac{i}{\hbar} P_{k} \square_{k} \overbrace{\square}^{\square} \square \frac{i}{\hbar} P_{k} \square_{k}=1 \square \frac{i}{\hbar}\left[A, P_{k}\right] \tag{4.34}
\end{equation*}
$$

so,

$$
\begin{equation*}
\frac{\partial A}{\partial q_{k}}=\square \frac{i}{\hbar}\left[A, P_{k}\right] \tag{4.35}
\end{equation*}
$$

From the differential form of the operators $P_{k}$,

$$
\begin{equation*}
\left[P_{j}, P_{k}\right]=0 \tag{4.36}
\end{equation*}
$$

for $j \neq k$, and so

$$
\begin{equation*}
\frac{\partial P_{j}}{\partial q_{k}}=0 \tag{4.37}
\end{equation*}
$$

Recall (4.28),

$$
\begin{equation*}
\frac{d P_{k}}{d t}=\square \frac{i}{\hbar}\left[P_{k}, H\right]+\square \frac{\partial P_{k}}{\partial q_{j}} \frac{d q_{j}}{d t} \tag{4.38}
\end{equation*}
$$

The summed terms are all zero, so

$$
\begin{equation*}
\frac{d P_{k}}{d t}=\square \frac{i}{\hbar}\left[P_{k}, H\right] \tag{4.39}
\end{equation*}
$$

We can also think of $q_{k}$ as an operator, so

$$
\begin{equation*}
\frac{\partial q_{k}}{\partial q_{k}}=1=\square \frac{i}{\hbar}\left[q_{k}, P_{k}\right] \tag{4.40}
\end{equation*}
$$

or,

$$
\begin{equation*}
\left[q_{k}, P_{k}\right]=i \hbar \tag{4.41}
\end{equation*}
$$

This can also be seen from

$$
\begin{equation*}
\left[q_{k}, P_{k}\right] \square=\square q_{k}, \frac{\hbar}{i} \frac{\partial}{\partial q_{k} \square}=\frac{\hbar}{i} q_{k} \frac{\partial \square}{\partial q_{k}} \square \frac{\hbar}{i} \frac{\partial}{\partial q_{k}} q_{k} \square=i \hbar \square \tag{4.42}
\end{equation*}
$$

For example,

$$
\begin{equation*}
\left[x, P_{x}\right]=i \hbar \tag{4.43}
\end{equation*}
$$

the familiar quantum mechanical commutation relation.
Now, we can also write

$$
\begin{equation*}
\frac{\partial H}{\partial q_{k}}=\frac{\hbar}{i}\left[H, P_{k}\right] \tag{4.44}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\frac{d P_{k}}{d t}=\square \frac{\partial H}{\partial q_{k}} \tag{4.45}
\end{equation*}
$$

which is the operator version of one of Hamilton's classical equations of motion and another way of writing Newton's second law of motion. Here we see that we have developed another profound concept, from gauge invariance alone. When the Hamiltonian of a system does not depend on a particular variable, then the observable corresponding to the generator of the gauge transformation of that variable is conserved. This is a version of Noether's theorem mentioned in the Introduction.

From this point, the rest of quantum mechanics can be developed. Observables $A$ are represented as hermitian operators and the expectation value of $A$ is

$$
\begin{equation*}
\langle A\rangle=\langle\square \mid A D\rangle \tag{4.46}
\end{equation*}
$$

The possible results of a measurement of $A$ is determined by the solutions of the eigenvalue equation

$$
\begin{equation*}
A|a\rangle=a|a\rangle \tag{4.47}
\end{equation*}
$$

where $|a\rangle$ is the eigenstate of A corresponding to eigenvalue $a$. When the state of a system is an eigenstate of an observable, the measurement of that observable will
always yield the eigenvalue corresponding to that state.
The symbol $|a\rangle\langle a|$ stands for an operator that projects $|\square\rangle$ onto the $|a\rangle$ axis. When the eigenvectors $|a\rangle$ form a complete set,

$$
\begin{equation*}
\square_{a}|a\rangle\langle a|=1 \tag{4.48}
\end{equation*}
$$

In that case, the state vector of a system will be the linear combination

$$
\begin{equation*}
|\square\rangle=\square_{a}|a\rangle\langle a \mid \square\rangle \tag{4.49}
\end{equation*}
$$

where $\left.\langle a \mid \square\rangle\right|^{2}$ is the probability for $|\square\rangle$ to be found in the eigenstate $|a\rangle$. The wave function is defined as the inner product

$$
\begin{equation*}
\square(q)=\langle q \nabla\rangle \tag{4.50}
\end{equation*}
$$

where $|q\rangle$ are the eigenstates of the spatial coordinates (or space-time coordinates relativistically) of the particles of the system. Momentum-space wave functions are also often used.

More generally, the eigenstates $|q\rangle$ are the basis states of a particular, arbitrary representation, like the unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ of the Cartesian coordinate axes $x, y, z$. $\square(q)$ is the projection of $|\square\rangle$ on $|q\rangle$.

We can represent $\square$ and $q$ as column matrices. Then

$$
\begin{equation*}
\square(q)=\square_{i} \square_{i}^{\dagger} q_{i} \tag{4.51}
\end{equation*}
$$

where $\square_{i}^{\dagger}$ is a row matrix.
In this representation, the observable $A$ is a square matrix,

$$
\begin{equation*}
\langle A\rangle=\square_{i, j} \square_{i}^{\dagger} A_{i j} \square_{j} \tag{4.52}
\end{equation*}
$$

or simply given by the matrix equation,

$$
\begin{equation*}
\langle A\rangle=\square^{\dagger} A \square \tag{4.53}
\end{equation*}
$$

Thus, the gauge transformation can be written

$$
\begin{equation*}
\square \square=U \square=\exp (i, \square) \square \tag{4.54}
\end{equation*}
$$

where $U$ and $\square$, and the corresponding generators, are square matrices.

### 5.0 Rotation and Angular Momentum

The variables $\left(q_{1}, q_{2}, q_{3}\right)$ can be identified with the coordinates $(x, y, z)$ of a particle and the corresponding momentum components are the generators of translations of these coordinates. (In this formulation, nothing prevents other particles being included with their space-time variables associated with other sets of four $q$ 's; note that by having each particle carry its own time coordinate we can maintain a fully relativistic scheme.) These coordinates can equally well be angular variables and the conjugate
momenta the corresponding angular momenta. These angular momenta will be conserved when the Hamiltonian is invariant to the gauge transformations that correspond to displacements of the corresponding angles. In this case, the displacements will be rotations about the spatial axes. For example, if we take ( $q_{1}, q_{2}, q_{3}$ ) $=\left(\square_{x}, \square_{y}, \square_{z}\right)$, where $\square_{k}$ is the angle of rotation about the $x$-axis, etc., then the generators of the rotations about these axes will be the angular momentum components ( $L_{x}, L_{y}, L_{z}$ ). Rotational invariance about any of these axes will lead to conservation of angular momentum about that axis.

Let us look for a moment at rotations in familiar 3-dimensional space. Suppose we have a vector $\mathbf{V}=\left(V_{x}, V_{y}\right)$ in the $x-y$ plane. Let is rotate it counter clockwise about the $z$-axis by an angle $\square$. We can write the transformation as a matrix equation:

Specifically, let us consider an infinitesimal rotation of the position vector $\mathbf{r}=(x, y)$ by $d \square$ about the $z$-axis. From above,


And so,

$$
\begin{equation*}
d x=-y d \square \tag{5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
d y=x d \square \tag{5.4}
\end{equation*}
$$

For any function $f(x, y)$,

$$
\begin{equation*}
f(x+d x, y+d y)=f(x, y)+d x \frac{\partial f}{\partial x}+d y \frac{\partial f}{\partial y} \tag{5.5}
\end{equation*}
$$

to first order. Or, we can write (reusing the function symbol $f$ ),

$$
\begin{aligned}
f(\square+d \square) & =f(\square) \square y d \square \frac{\partial f}{\partial x}+x d \square \frac{\partial f}{\partial y} \\
& =f(\square)+i d \square G f
\end{aligned}
$$

where

$$
\begin{equation*}
G=\square i\left\lceil x \frac{\partial}{\partial y} \square y \frac{\partial}{\partial x} \square=x P_{y} \square y P_{x}=L_{z}\right. \tag{5.7}
\end{equation*}
$$

the angular momentum about $z$. Similarly,

$$
\begin{equation*}
L_{x}=y P_{z}-z P_{y} \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{y}=z P_{x}-x P_{z} \tag{5.9}
\end{equation*}
$$

This result can be generalized as follows. If you have function that depends on a spatial
position vector $\mathbf{r}=(x, y, z)$, and you rotate that position vector by angle $\square$ about an arbitrary axis, then that function transforms as

$$
\begin{equation*}
f^{\prime}(\mathbf{r})=\exp (i \mathbf{L} \bullet \square) f(\mathbf{r}) \tag{5.10}
\end{equation*}
$$

where the direction of $\square$ is the direction of the axis of rotation. Once again this has the form of a gauge transformation, or phase transformation in $f$, where

$$
\begin{equation*}
U=\exp (i \mathbf{L} \bullet \square) \tag{5.11}
\end{equation*}
$$

From the previous commutation rules one can show that the generators $L_{x}, L_{y}$, and $L_{z}$ do not mutually commute. Rather,

$$
\begin{equation*}
\left[L_{x}, L_{y}\right]=i \hbar L_{z} \tag{5.12}
\end{equation*}
$$

and cyclic permutations. Thus the order of successive rotations is important.
Most quantum mechanics textbooks will contain the proof of the following result, although it is not always stated so generally: Any vector operator $\mathbf{J}$ whose components obey the angular momentum commutation rules,

$$
\begin{equation*}
\left[J_{x}, J_{y}\right]=i \hbar J_{z} \tag{5.13}
\end{equation*}
$$

will have the following eigenvalue equations:

$$
\begin{equation*}
J^{2}|j, m\rangle=j(j+1) \hbar^{2}|j, m\rangle \tag{5.14}
\end{equation*}
$$

where $J^{2}=J_{x}^{2}+J_{y}^{2}+J_{z}^{2}$ is the square of the magnitude of $\mathbf{J}$.

$$
\begin{equation*}
J_{z}|j, m\rangle=m \hbar|j, m\rangle \tag{5.15}
\end{equation*}
$$

where $m$ goes from $-j$ to +j in steps of one: $m=-j,-j+1, \ldots, j-1, j$. This implies that $j$ is an integer (including zero) or a half-integer.

## 6. Rotation and Gauge Transformations

We have already noted that the gauge transformation is like a rotation in the complex space of a function. Let us now generalize that concept.

Again, it is important not to confuse the two different spaces involved in our discussion. First we have the space spanned by the variables $\{q\}$ of a system. We have generally taken the first four of these to be the subspace of 4-dimensional space-time in which we describe events. If our "system" contains more than one event, then additional groups of 4-dimensional subspaces can be reserved for these. Other subspaces are left available for other variables.

Besides $q$-space, an additional abstract space have alread introudced as $\square \square$ space is used to describe the quantum state of a system. That space has coordinate axes that are defined by an arbitrary choice of basis vectors of the system $|q\rangle$, where if $Q$ is the operator corresponding to an observable,

$$
\begin{equation*}
Q|q\rangle=q|q\rangle \tag{6.1}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
|\square\rangle=\square_{i} c_{i}\left|q_{i}\right\rangle \tag{6.2}
\end{equation*}
$$

The basis states are usually taken to be orthonormal, that is,

$$
\begin{equation*}
\left\langle q_{i} \mid q_{j}\right\rangle=\square_{i j} \tag{6.3}
\end{equation*}
$$

so $\langle\square \mid \square\rangle=1$ and $\left.k_{i}\right|^{2}=\left\langle\left.\left\langle q_{i} \mid \square\right\rangle\right|^{2}\right.$ is the probability for a measurement of $Q$ giving the value $q_{i}$ when the system is in the state $|\square\rangle$.

For example, the basis states are frequently chosen to be $|x\rangle,|y\rangle$, and $|z\rangle$, where the observables are all the possible coordinates of a particle, that is, all the eigenstates of the eigenvalue equation

$$
\begin{equation*}
X|x\rangle=x|x\rangle \tag{6.5}
\end{equation*}
$$

The quantity $\square(x)=\langle x \mid\rangle$ is, for historical reasons, called the wave function, although it often has nothng to do with waves. Since $x$ is usually regarded as a continuous variable, $\square\lceil$ space is infinite dimensional. That is, $|x\rangle$ is not one axis but an infinite number of axes, one for every real number $x$. Even if we assume that $x$ is discrete in units of the Planck length, and space is finite, we still have an awfully large number of dimensions.

If the particle is an electron, then $\square \square$ space may also include the basis states $\left|+\frac{1}{2}\right\rangle$
and $\left|\square \frac{1}{2}\right\rangle$ that are the eigenstates of the $z$-component of spin of the electron. Even though spatial coordinates are more familiar than spins, 2-dimensional spin subspace is a lot easier to visualize than the subspace of spatial coordinate eigenstates.

In the 2-dimensional subspace spanned by the spin state vector of an electron, the basis states $\left|+\frac{1}{2}\right\rangle$ and $\left|\square \frac{1}{2}\right\rangle$ can be thought of as analogous to the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the more familiar 2-dimensional subspace $(x, y)$. The spin state $|\square\rangle$ is in general a 2dimensional vector oriented at some arbitrary angle. The basis vectors define two possible orientations of the spin angular momentum vector $\mathbf{S}$ in familiar 3-dimensional space, one along the $z$-axis and the other opposite. (The choice of $z$-axis here is arbitrary conventio). Thus, for example, if S points originally along the z-axis, a rotation of $180^{\circ}$ will take it to point along $-z$.

However, note that a rotation in $\square \square$ space of only $90^{\circ}$ takes the spin state from $\left|+\frac{1}{2}\right\rangle$ to $\left|\square \frac{1}{2}\right\rangle$. This implies that the unitary transformation matrix in this case is

$$
\begin{equation*}
U=\exp \frac{\square}{2} \cdot \frac{\square}{2} \tag{6.6}
\end{equation*}
$$

where $I$ is the unit $2 \times 2$ matrix.

More generally,

$$
\begin{equation*}
U=\exp \underset{=}{2} \tag{6.7}
\end{equation*}
$$

where the axial vector $\square$ points in the direction around which we rotate, and $\square$ is the Pauli spin vector whose components are conventionally written

We see that $U$ again has the form of a gauge transformation. The generator of the gauge transformation in the spin vector subspace of a spin $1 / 2$ particle is the spin angular momentum operator (in units of $\hbar$ ), $\mathbf{S}=\square / 2$. We could also have obtained this result from our previous proof that the gauge transformation for a rotation in 3-space is

$$
\begin{equation*}
U=\exp (i \mathbf{L} \cdot \square) \tag{6.9}
\end{equation*}
$$

where $\mathbf{L}$ is the angular momentum. Here $\mathbf{L}=\mathbf{S}=\square / 2$.

## 7. Special Relativity

Now we are ready to inject some familiar physics into the mix. It turns out to be most elegant to do this within the framework of special relativity. But note that, as was the case for quantum mechanics, the usual starting axioms will not be asserted. Rather they will be derived from the assumption of gauge invariance.

Let us consider the first four variables $\left(q_{0}, q_{1}, q_{2}, q_{3}\right.$ of our set $\{q\}$ which we have arbitrarily set to $\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=($ ict, $x, y, z)$, where $t$ is the time and $(x, y, z)$ are the spatial coordinates of an event. The constant $c$ is simply a factor that converts units of time to units of distance. It will turn out to be the invariant speed of light in a vacuum, but that is not being assumed at this point. Also, the assumption that $q_{0}$ is an imaginary number is not necessary; it just makes things easier to work out at this level of sophistication.

Let $x^{\prime}=\left(x^{\prime}{ }_{0}, x^{\prime}{ }_{1}, x^{\prime}{ }_{2}, x^{\prime}{ }_{3}\right)$ be the position of the event in reference frame moving at a speed $v=\square c$ along the $z$-axis with respect to the reference frame $x$, where

$$
\begin{equation*}
x \boxminus=L_{\square}^{\square} x_{\square} \tag{7.1}
\end{equation*}
$$

and the convention is used in which repeated Greek indices are summed from 0 to 3 . As is shown in many textbooks, the proper distance will be invariant if $L_{\square}^{\square}$ is the Lorentz transformation operator

$$
L_{\square}^{\square}=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{7.2}\\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \square & \sin \square \\
0 & 0 & \square \sin \square & \cos \square
\end{array}\right]
$$

where $\cos \square=\square \sin \square=i \square \square$ and $\square=(1-\square \not)^{1 / 2}$. By writing it this way, we see that the Lorentz transformation between reference frames moving at constant velocity with respect to one another along their respective $z$-axes is equivalent to a rotation by an angle $\square$ in the $\left(x_{3}, x_{0}\right)$-plane. That is, Lorentz invariance is analogous to rotational invariance in 3-space.

The complex angle $\square$ is a mathematical artifact of taking the zeroth component of the 4 -vector to be imaginary number and time a real number. We can make $\square$ real by using a non-Euclidean metric.

We have seen that the generators of space-time translation form a 4-component set:

$$
\begin{equation*}
P=\left(P_{0}, P_{1}, P_{2}, P_{3}\right)=\frac{H}{c}, P_{x}, P_{y}, P_{z} \tag{7.3}
\end{equation*}
$$

where we recall that $c$ is just a units-conversion constant. Quantum mechanically,

$$
\begin{equation*}
P_{k}\left|p_{k}\right\rangle=p_{k}\left|p_{k}\right\rangle \tag{7.4}
\end{equation*}
$$

where $p_{k}$ is the eigenvalue of $P_{k}$ when the system is in a state given by the eigenvector $\left|p_{k}\right\rangle$. Similarly,

$$
\begin{equation*}
H|E\rangle=E|E\rangle \tag{7.5}
\end{equation*}
$$

Let us work with these eigenvalues-which still have not been identified with familiar physical energy and momentum! But, that's coming up fast now. Write

$$
\begin{equation*}
p=\left(p_{0}, p_{1}, p_{2}, p_{3}\right)=\stackrel{\square}{\square} \frac{E}{c}, p_{x}, p_{y}, p_{z} \tag{7.6}
\end{equation*}
$$

The squared length of the 4 -vector

$$
\begin{equation*}
p_{\square} p_{\square}=p \nabla p \square \equiv \square m^{2} c^{2} \tag{7.7}
\end{equation*}
$$

is invariant to rotations in 4 -space. The invariant quantity $m$ is called the mass of the particle. Note that the length of the 4 -momentum vector is (in the metric we have chosen to use)

$$
\begin{equation*}
\left(p_{\square} p_{\square}\right)^{1 / 2}=i m c \tag{7.8}
\end{equation*}
$$

Defining 4-momentum in this way guarantees the invariance in the important result of an earlier section, namely the classical Hamilton equation of motion (4.45),

$$
\begin{equation*}
\frac{d P_{k}}{d t}=\square \frac{\partial H}{\partial q_{k}} \tag{7.9}
\end{equation*}
$$

This definition allows us to connect the operator $P_{k}$ with the operationally defined momentum $p_{k}$ and the operator $H$ with the operationally defined energy $E$.

Working with the operationally defined quantities, we can write (using boldface type for familiar 3-dimensional spatial vectors)

$$
\begin{equation*}
d \mathbf{p} \cdot d \mathbf{r}=\square d E d t \tag{7.10}
\end{equation*}
$$

Or, in terms of 4-vectors,

$$
\begin{equation*}
d p^{\square} d x_{\square}=0 \tag{7.11}
\end{equation*}
$$

which is Lorentz invariant.
Suppose we have a particle of mass $m$. Let $\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$ be the coordinate axes in the reference frame in which the particle is at rest, $\left|\mathbf{p}^{\prime}\right|=0$. Then its energy in that reference frame is

$$
\begin{equation*}
E^{\prime}=m c^{2} \tag{7.12}
\end{equation*}
$$

which is the rest energy. Next let us look at the particle in another reference frame $(x, y, z)$ in which the particle is moving along the $z$-axis at a constant speed $v$. Then, from the Lorentz transformation, the 3-momentum of the particle in that reference frame will be

$$
\begin{equation*}
p_{z}=\sqrt[\square]{\square}+\frac{\square}{c} E \square=\square(0+\square m c) \tag{7.13}
\end{equation*}
$$

We can write this in vector form

$$
\begin{equation*}
\mathbf{p}=\square m \mathbf{v} \tag{7.14}
\end{equation*}
$$

We note that $\mathbf{p} \square m \mathbf{v}$ when $v \ll c$, So, we have (finally) derived the well-known the relationship between momentum and velocity. Nowhere previously was it assumed that $\mathbf{p}=m \mathbf{v}$.

The energy of the particle in the same reference frame is

$$
\begin{equation*}
E=\square\left(E \square+\square p \square \square_{z}\right)=\square m c^{2} \tag{7.15}
\end{equation*}
$$

Note that, in general, the velocity of a particle is

$$
\begin{equation*}
\mathbf{v}=\frac{\mathbf{p} c^{2}}{E} \square \frac{\mathbf{p}}{m} \tag{7.16}
\end{equation*}
$$

when $v \ll c$ since, in that case, $E=m c^{2}$. We can also show that, for all $v$,

$$
\begin{equation*}
E=\|\left.\mathbf{p} c\right|^{2}+m^{2} c^{4} \theta^{-1 / 2} \tag{7.17}
\end{equation*}
$$

This is a "free particle" since

$$
\begin{equation*}
\mathbf{F}=\frac{d \mathbf{p}}{d t}=\square \bar{\square} E=0 \tag{7.18}
\end{equation*}
$$

More generally we can write

$$
\begin{equation*}
E=m c^{2}+T+V(\mathbf{r}) \tag{7.19}
\end{equation*}
$$

where $m c^{2}$ is the rest energy. The quantity

$$
\begin{equation*}
T=|\mathbf{p} c|^{2}+m^{2} c^{4} \theta^{1 / 2} \square m c^{2} \square \frac{1}{2} m v^{2} \tag{7.20}
\end{equation*}
$$

when $v \ll c$, is the kinetic energy, or energy of motion, and $V(\mathbf{r})$ is the potential energy. The force on the particle is then

$$
\begin{equation*}
\mathbf{F}=-\square V \tag{7.21}
\end{equation*}
$$

We are now in a position to interpret the meaning of $c$, which was introduced originally as a simple conversion factor. Suppose we have a particle of zero mass and 3momentum of magnitude $|\mathbf{p}|$. Then, the energy of that particle will be

$$
\begin{equation*}
E=|\mathbf{p}| c \tag{7.22}
\end{equation*}
$$

and the speed

$$
\begin{equation*}
v=\frac{p c^{2}}{E} \square \frac{p c^{2}}{p c}=c \tag{7.23}
\end{equation*}
$$

Thus $c$ is the speed of a zero mass particle, sometimes called "the speed of light." Since $c$ is the same constant in all references frames, the invariance of the speed of light, one of the axioms of special relativity, is thus seen to follow from 4 -space rotational symmetry.

So we have now shown that the generators of translations along the four axes of space-time are the components of the 4-momentum, which includes energy in the zeroth component and 3-momentum in the other components. These have their familiar connections with the quantities of classical physics. Mass is introduced as a Lorentz invariant quantity that is proportional to the length of the 4-momentum vector. The conversion factor $c$ is shown to be, as expected, the Lorentz-invariant speed of light in a vacuum.

## 8. Classical Mechanics

Except for specific laws of force for gravity and electromagnetism, all of classical mechanics can now be inferred from the above discussion. Conservation of energy, linear momentum, and angular momentum follow from global gauge invariance in space-time. Newton's first and third laws of motion follow from momentum conservation. Newton's second law basically defines the force on a body as the time rate of change of momentum,

$$
\begin{equation*}
\mathbf{F}=\frac{d \mathbf{p}}{d t} \tag{8.1}
\end{equation*}
$$

Above we saw that, for the operators $\mathbf{P}$ and $H$,

$$
\begin{equation*}
\frac{d \mathbf{P}}{d t}=\square \square H \tag{8.2}
\end{equation*}
$$

The classical observables will correspond to the eigenvalues of these and so

$$
\begin{equation*}
\frac{d \mathbf{p}}{d t}=\square \square E \tag{8.3}
\end{equation*}
$$

If $E=T+V$ and $T$ does not depend explicitly on spatial position,

$$
\begin{equation*}
\mathbf{F} \square \frac{d \mathbf{p}}{d t}=\square \square V \tag{8.4}
\end{equation*}
$$

as in the previous section. The more generalized and advanced formulations of classical mechanics, such as Lagrange's and Hamilton's equations of motion, can be now developed in the usual way.

## 9. Electromagnetism

In the following sections we will switch to the conventions used in slightly more advanced physics so that the resulting equations agree with the textbooks at that level. We have already seen that $\hbar$ and $c$ are arbitrary conversion factors, so we will work in units where $\hbar=c=1$. Furthermore, we will use a non-Euclidean (but still geometrically
flat) metric in defining our 4-vectors:

$$
\square_{\square}^{\square}=\begin{array}{cccc}
\frac{\square}{\square} & 0 & 0 & 0  \tag{9.1}\\
\square & \square 1 & 0 & 0 \\
\square & 0 & \square 1 & 0 \\
\square & 0 & 0 & \square 1
\end{array}
$$

where the space-time position 4 -vector is $x=(t, x, y, z)$, where we re-use x , the momentum 4 -vector is $p=\left(E, p_{x}, p_{y}, p_{z}\right)$, and

$$
\begin{equation*}
p_{\square} \square \square p^{\square}=E^{2} \square|\mathbf{p}|^{2}=m^{2} \tag{9.2}
\end{equation*}
$$

This choice of metric has the advantages of enabling us to directly identify the mass with the invariant length of the 4-momentum vector and eliminating the need for imaginary zeroth components.

In quantum mechanics, the state of a free particle is an eigenstate of energy and momentum. Consider the 4-momentum eigenvalue equation for a spinless particle (spin can be included, but this is sufficient for present purposes)

$$
\begin{equation*}
i \partial_{\square} \square=p_{\square} \square \tag{9.3}
\end{equation*}
$$

where we now use the convention $\partial_{\square} \square \equiv \frac{\partial \square}{\partial x_{\square}}$. The quantity $\square$ is the eigenfunction $\square(x)=\left\langle x \mid p_{\square}\right\rangle$ and can be thought of as having two abstract dimensions, its real and imaginary parts. If we rotate the axis in this space by an angle $\square$ we have the gauge transformation,

$$
\begin{equation*}
\square=\exp (i \square) \square \tag{9.4}
\end{equation*}
$$

The eigenvalue equation is unchanged, provided that $\square$ is independent of the space-time position $x$. This is the type of gauge invariance we have already considered, what we call global gauge invariance. The generator of the transformation, $\square$ is conserved. Below we will identify $\square$ as the negative of the charge of the particle.

Now suppose that $\square$ depends on $x$. In this case, we do a local gauge transformation and

$$
\begin{equation*}
\partial_{\square}\left[\boxed{C l}=\exp [i \square(x)]\left[\partial_{\square}+i\left(\partial_{\square} \square\right)\right] \square\right. \tag{9.5}
\end{equation*}
$$

and the eigenvalue equation is not invariant to this operation. Let us define a new operator, the covariant derivative,

$$
\begin{equation*}
\mathrm{D}_{\square}=\partial_{\square}+i q A_{\square} \tag{9.6}
\end{equation*}
$$

where $q$ is a constant and $A_{\square}$ transforms as

$$
\begin{equation*}
A_{\square}^{\prime}=A_{\square}+\partial_{\square} 【(x) \tag{9.7}
\end{equation*}
$$

where $\square(x)=-q \square(x)$. Then,

$$
\begin{align*}
& \mathrm{D}_{\square}^{\prime} \square \square=\left(\partial_{\square}+i q A_{\square}^{\prime}\right) \square=\left[\partial_{\square}+i q A_{\square}+i q\left(\partial_{\square} \square\right] \exp (i \square \square \square\right.  \tag{9.8}\\
& \quad=\exp (i \square)\left[\partial_{\square}+i q A_{\square}\right] \square+\exp (i \square) \square\left[i\left(\partial_{\square} \square\right)-i\left(\partial_{\square} \square\right)\right]
\end{align*}
$$

$$
\begin{gathered}
=\exp (i \square)\left[\partial_{\square}+i q A_{\square}\right] \square \\
=\exp (i \square) \mathrm{D}_{\square} \square
\end{gathered}
$$

Recall the the operator $P_{\square}$ associated with the relativistic 4-momentum is

$$
\begin{equation*}
P_{\square}=-i \partial_{\square} \tag{9.9}
\end{equation*}
$$

Let us define, analogously,

$$
\begin{equation*}
\mathrm{P}_{\square}=-i \mathrm{D}_{\square} \tag{9.10}
\end{equation*}
$$

Writing

$$
\begin{equation*}
\mathrm{P}_{\square}=P_{\square}+q A_{\square} \tag{9.11}
\end{equation*}
$$

we see that this operator $\mathrm{P}_{\square}$ is precisely the canonical 4-momentum in classical mechanics for a particle of charge $q$ interacting with an electromagnetic field described by the 4 -vector potential $A_{\square}=\left(A_{0}, \mathbf{A}\right)$, where $A_{o}=V / c$ in terms of the scalar potential $V$ and $\mathbf{A}$ is the 3 -vector potential. We will further justify this connection below. As already mentioned, $\square(x)=-q \square(x)$ and thus $q$ is conserved when $\square(x)$ is a constant. Also, note that for neutral particles $q=0$ and no fields need to be introduced.

In quantum mechanics, the canonical momentum must be used in place of the mechanical momentum in the presence of an electromagnetic field. For example, the Schrödinger equation for a non-relativistic particle of mass $m$ and charge $q$ in an electromagnetic field described by the 3-vector potential $\mathbf{A}$ and scalar potential $V$ is

$$
\begin{equation*}
\left.B-\left.q \mathbf{A}\right|^{2}+q V\right) \text { 雨 }=\left[i \hbar \square-\left.q \mathbf{A}\right|^{2}+q V\right) \neq i \hbar \frac{\partial \square}{\partial t} \tag{9.12}
\end{equation*}
$$

In quantum field theory, the basic quantity from which calculations proceed is the Lagrangian density. The Klein-Gordon Lagrangian density for a spinless particle of mass $m$ is

$$
\begin{equation*}
\mathrm{L}=-\frac{1}{2} \partial \square^{\partial^{\square}} \square+\frac{1}{2} m^{2} \square^{2} \tag{9.13}
\end{equation*}
$$

This is not locally gauge invariant. However, it becomes so if we write it

$$
\begin{equation*}
\mathrm{L}=-\frac{1}{2} \mathrm{D}_{D} \mathrm{D}^{\square} \square+\frac{1}{2} m^{2} \square^{2} \tag{9.14}
\end{equation*}
$$

The corresponding Klein-Gordon equation, the relativistic analogue of the Schrödinger equation for spinless particles, becomes

$$
\begin{equation*}
\mathbf{D}_{\square} \mathbf{D}^{\square} \square+m^{2} \square=0 \tag{9.15}
\end{equation*}
$$

Spin 1/2 particles of mass $m$ are described by the Dirac Lagrangian which similarly can be made gauge invariant by writing it, using conventional notation,

$$
\begin{equation*}
\mathrm{L}=i \nabla \square \mathrm{D}_{\square} \square \square m \square \square \tag{9.16}
\end{equation*}
$$

The corresponding Dirac equation

$$
\begin{equation*}
i\left\lceil\mathrm{D}_{\square} \square \square m \square=0\right. \tag{9.17}
\end{equation*}
$$

also is gauge invariant. (Note: while I have not derived these equations, no additional physical assumptions are required in their derivation).

A spin 1 particle of mass $m_{A}$ is described by the Procca Lagrangian

$$
\begin{equation*}
\mathrm{L}=\square \frac{1}{16 \square} F^{\square \square} F_{\square \square}+m_{A}^{2} A^{\square} A_{\square} \tag{9.18}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{\square D}=\partial_{\square} A_{\square}-\partial_{\square} A_{\square} \tag{9.19}
\end{equation*}
$$

The first term in $L$ is gauge invariant while the second is not unless we set $m_{A}=0$. This leads to the deeply important result that particles with spin 1 whose Lagrangians are locally gauge invariant are necessarily massless. The photon is one such particle.

However, other spin 1 fundamental particles exist with nonzero masses. These masses result from broken gauge symmetry, as we will briefly discuss below.

In any case, the existence of a vector field $A_{\square}$ associated with a massless spin 1 particle is implied by the assumption of local gauge invariance. It is a field introduced to maintain local gauge invariance. That field can be identified with the classical electromagnetic fields E and B and the particle with the photon. That is, the photon is the quantum of the field $A_{\square}$ which itself is associated with the classical 4-vector electromagnetic potential.

To see the classical connection, note that

$$
\begin{equation*}
A^{\prime}{ }_{k}=A_{k}+\partial_{k} \square(x) \tag{9.20}
\end{equation*}
$$

where $k=1,2,3$, or, in 3-vector notation

$$
\begin{equation*}
\mathbf{A}^{\prime}=\mathbf{A}+\square \square \tag{9.21}
\end{equation*}
$$

It follows that the 3-vector

$$
\begin{equation*}
\mathbf{B}^{\prime}=\square \times \mathbf{A}^{\prime}=\square \times \mathbf{A}-\square \times \square \square=\square \times \mathbf{A}=\mathbf{B} \tag{9.22}
\end{equation*}
$$

is locally gauge invariant. Furthermore,

$$
\begin{equation*}
\square \cdot \mathbf{B}=\square \bullet(\square \times \mathbf{A})=0 \tag{9.23}
\end{equation*}
$$

Thus, B may be interpreted as the familiar classical magnetic field 3-vector; the above equation is Gauss's law of magnetism, one of Maxwell's equations. The zeroth component of the 4 -vector potential,

$$
\begin{equation*}
A \square=A_{o}+\frac{\partial \square}{\partial x_{o}} \tag{9.24}
\end{equation*}
$$

can be written

$$
\begin{equation*}
V \square=V \square \frac{\partial \square}{\partial t} \tag{9.25}
\end{equation*}
$$

which implies that the 3-vector

$$
\begin{gather*}
\mathbf{E} \square \square \square V \square \square \frac{\partial \mathbf{A}[ }{\partial t} \\
=\square \square V+\square \frac{\partial \square}{\partial t} \square \frac{\partial \mathbf{A}}{\partial t} \square \frac{\partial \square \square}{\partial t}  \tag{9.26}\\
=\square \square V \square \frac{\partial \mathbf{A}}{\partial t}=\mathbf{E}
\end{gather*}
$$

is also locally gauge invariant. Furthermore,
an
so

$$
\begin{equation*}
\square \square \mathbf{E}=\square \frac{\partial(\square \square \mathbf{A})}{\partial t}=\square \frac{\partial \mathbf{B}}{\partial t} \tag{9.28}
\end{equation*}
$$

which is Faraday's law of induction, another of Maxwell's equations, with E interpreted as the classical electric field.

Summarizing, we have found that the motion of a free charged particle is not invariant under a local gauge transformation. However, we can make it invariant by adding a term to the canonical momentum that corresponds to the 4 -vector potential of the electromagnetic field. Thus the electromagnetic force can be thought of as a fictitious force that is introduced to preserve local gauge symmetry. Conservation of charge follows from global gauge symmetry.

## 10. The Subnuclear Forces

The gauge transformation just described corresponds to a rotation in the abstract space of the 4-momentum eigenstate, which is the state of any particle of constant momentum. Here the transformation operator

$$
\begin{equation*}
U=\exp (i \square) \tag{10.1}
\end{equation*}
$$

can be trivially thought of as a1x1 matrix. The set of all such unitary matrices comprises the transformation group $U(1)$. The generators of the transformation, $\square$, form a set of $1 \times 1$ matrices that, clearly, mutually commute. Whenever the generators of a transformation group commute, that group is termed abelian. Electromagnetism is thus an abelian gauge theory.

Recall from our discussion of angular momentum that the unitary operator

$$
\begin{equation*}
U=\exp \stackrel{\square}{\frac{\square}{2}} i \square \cdot \square \tag{10.2}
\end{equation*}
$$

operates in the space of spin state vectors. In this case $U$ is represented by a $2 \times 2$ matrix. The set of all such matrices comprises the transformation group $\operatorname{SU}(2)$, where the prefix $S$ specifies that the matrices of the group are unimodular, that is, have unit determinant. This follows from the fact that, for any matrix $U$,

$$
\begin{equation*}
U=\exp (i A) \tag{10.3}
\end{equation*}
$$

we have

$$
\begin{equation*}
\operatorname{det} U=\exp (\operatorname{Tr} A) \tag{10.4}
\end{equation*}
$$

Since the Pauli matrices are traceless, $\operatorname{det} U=1$.
Following a procedure similar to what was done above for $U(1)$, let us write

$$
\begin{equation*}
U=\exp \frac{1}{2} i g \square \square \square \tag{10.5}
\end{equation*}
$$

where the three components of $\square$ form a set of matrices identical to the Pauli spin matrices and we use a different symbol just to avoid confusion with spin. While the spin $\mathbf{S}=\square / 2$ is a vector in familiar 3-dimensional space, $\square$ is a 3-vector in some more abstract space we will call isospin space. The 3-vector $\mathbf{T}=\square / 2$ is called the isospin or isotopic spin. Global gauge invariance under $S U(2)$ implies conservation of isospin. The quarks and leptons of the standard model have $T=1 / 2$. The quantity $g$ is a constant analogous to the electric charge that measures the strength of the interaction.

Once again it is important not to confuse isospin space with the 2-dimensional subspace of the state vectors on which $U$ operates. When the isospin space 3 -vector $\Pi x$ ) depends on the space-time (yet another space) position 4 -vector $x$ we once more have a local gauge transformation. The generators being like angular momenta do not mutually commute, so the transformation group is non-abelian. This type of non-abelian gauge theory is called a Yang-Mills theory.

Let us attempt to make this clearer by rewriting $U$ with indices rather than boldface vector notion:

$$
\begin{equation*}
U=\exp \square^{\square} \frac{1}{2} i g \square_{k} \square^{k}(x)[ \tag{10.6}
\end{equation*}
$$

where the repeated Latin index $k$ is understood as summed from 1 to 3 .
Encouraged by our success in obtaining the electromagnetic force from local $U(1)$ gauge symmetry, let us see what we can get from local $S U(2)$ symmetry. Following the $U(1)$ lead, we define a covariant derivative

$$
\begin{equation*}
\mathrm{D}_{\square}=\partial_{\square}+\frac{1}{2} i g \square_{k} W_{\square}^{k} \tag{10.7}
\end{equation*}
$$

where $W_{\square}^{k}$ are three 4 -vector potentials analogous to to the electromagnetic 4-vector potential $A_{\square}$. As before, the introduction of the fields $W_{\square}^{k}$ maintains local gauge invariance. Or, we can say that local gauge invariance implies the presence of three 4vector potentials $W_{\square}^{k}$. In the standard model, these are interpreted as the fields of the weak interaction.

In quantum field theory, a particle is associated with every field, the so-called quantum of the field. The spin and parity of the particle, $J^{P}$, is determined by the transformation properties of the field. The quantum of a scalar field has $J^{P}=0^{+} ; \mathrm{a}$ vector field has $J^{P}=1$. For the electromagnetic field described by the potential $A_{\square}$ the quantum is the photon. Since $A_{\square}$ is a vector field, the photon has spin 1 . It is a vector gauge boson.

Similarly, the weak fields $W_{\square}^{k}$ will have three spin 1 particles as their quanta-three vector gauge bosons $W^{-}, W o$, and $W^{+}$, where the superscripts specify the electric charges of the particles. These can also be viewed as the three eigenstates of a particle with isospin $T=1$.

If the $U(1)$ symmetry of electromagnetism and the $S U(2)$ symmetry of the weak interaction were perfect, we would see the photon and three $W$ bosons above.

However, these symmetries are broken at the "low" energies at which most physical observations are made, including those at the current highest energy particle accelerators. This symmetry breaking leads to a mixing of the electromagnetic and weak forces. Here, briefly, is how this comes about in what is called unified electroweak theory.

The covariant derivative for electroweak theory (assumed, not derived) is written

$$
\begin{equation*}
\mathrm{D}_{\square}=\partial_{\square}+i g_{1} \frac{Y}{2} B_{\square}+i g_{2} \frac{\square_{k}}{2} W_{\square}^{k} \tag{10.8}
\end{equation*}
$$

where the $U(1)$ field is called $B_{\square}$ and the constant $g_{1}$ replaces the electric charge in that term. The quantity $Y$ is a constant called the hypercharge generator that can take on different values in different applications, a detail that need not concern us here. The $S U(2)$ term includes a constant $g_{2}$, the vector $\mathbf{T}=\square / 2$, or isospin, and the vector field $W_{\square}^{k}, k=1,{ }^{\prime},^{\prime}, 3$.

Neither $B$ nor $W^{o}$, the quanta of the fields $B_{\square}$ and $W_{\square}^{0}$, appear in experiments at current accelerator energies. Instead, the particles that do appear are the photon and Z , whose fields $A_{\square}$ and $Z_{\square}$ are mixtures of $B_{\square}$ and $W_{\square}^{0}$. These together with the $\mathrm{W}^{+}$and $\mathrm{W}^{-}$, the quanta of the fields $W_{\square}^{k}, ' k=1, ' 2$, constitute the vector gauge bosons of the electroweak sector of the standard model. Their mixing is also like a rotation, gauge symmetry being broken in this case,
where the rotation angle $\square_{\mathrm{w}}$ is called the Weinberg (or weak) mixing angle. This parameter is not determined by the theory and must be found from experiment. The current value of $\sin ^{2} \square_{\mathrm{N}}=0.23115$. The constants that determine the strength of the interaction are

$$
\begin{equation*}
g_{1}=e / \cos \rrbracket_{\mathrm{w}} \quad g_{2}=e / \sin \rrbracket_{\mathrm{w}} \tag{10.10}
\end{equation*}
$$

where $e$ is the unit electric charge.
As we have seen, the masses of gauge bosons are fundamentally zero. While the photon is massless, the $W^{ \pm}$and $Z$ bosons have large masses. These masses are shown to arise from another symmetry-breaking process called the Higgs mechanism. The symmetry-breaking is apparently spontaneous, that is, not determined by any known deeper physical principle. Spontaneous symmetry breaking describes a situation, like the ferromagnet, where the fundamental laws are symmetric and obeyed at higher energy, but the lowest energy state of the system breaks the symmetry.

Moving beyond the weak interactions and $S U(2)$, we have the strong interactions and $S U(3)$. In general, for $S U(n)$ there are $n^{2}-1$ dimensions in the subspace. Let us add the new term to the previous ones that included the electroweak forces

$$
\begin{equation*}
\mathrm{D}_{\square}=\partial_{\square}+i g_{1} \frac{Y}{2} B_{\square}+i g_{2} \frac{\square_{k}}{2} W_{\square}^{k}+i g_{3} \frac{\square_{a}}{2} G_{\square}^{a} \tag{10.11}
\end{equation*}
$$

where $\square=0,1,2,3$ for the four dimensions of space-time, the repeated index $k$ is summed from 1 to 3 in the $S U(2)$ term and the repeated index a is summed from 1 to 8 in the $S U(3)$ term. The $\square a$ are eight traceless $3 \times 3$ matrices analogous to the three Pauli
$2 \times 2$ isospin matrices $\left[k\right.$, and the $G_{\square}^{a}$ are eight spin 1 fields analogous to the singlet field $B_{\square}$ and the triplet field $W_{\square}^{k}$. of the electroweak interaction. The gauge bosons in this case are eight gluons. The symmetry is not broken, so they are massless. Global gauge invariance under $S U(3)$ implies the conservation of another quantity called color charge. While there is much more to the standard model, this should suffice to illustrate its basis in gauge symmetry and the importance of spontaneous broken symmetry.

## 11. General Relativity

General relativity can also be cast in the form of a gauge theory,[2] but to do so would take us well beyond the mathematical scope of this paper which we have tried to limit to the undergraduate level. However, we can outline how the principle of general covariance leads to general relativity. Typical treatments emphasize the role of two other principles, Mach's principle and the principle of equivalence. However, while Einstein acknowledged Mach's influence on his thinking, it is not clear that Mach's principle plays any significant part in deriving general relativity, especially since the principle itself is ill-defined. The principle of equivalence between gravity and acceleration is usually given great prominence, but that can also be seen as a consequence of the principle of covariance.

Consider the equation of motion for a freely falling body in terms of a coordinate system $y=\left(y_{0}, y_{1}, y_{2}, y_{3}\right)$, where $y_{o}=i c t$, falling along with the body. Since $d \mathbf{y}=0$,

$$
\begin{equation*}
\frac{d^{2} \mathbf{y}}{d t^{2}}=0 \tag{11.1}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\frac{d^{2} y_{o}}{d t^{2}}=0 \tag{11.2}
\end{equation*}
$$

## Furthermore,

$$
\begin{equation*}
(d \square)^{2}=(d t)^{2} \square \frac{1}{c^{2}}|d \mathbf{y}|^{2}=(d t)^{2} \tag{11.3}
\end{equation*}
$$

Let us work in units where $c=1$. We can write the above in 4 -vector form,

$$
\begin{equation*}
\frac{d^{2} y_{\square}}{d \square^{2}}=0 \tag{11.4}
\end{equation*}
$$

This the 4-dimensional equation of motion for a body not acted on by any force.This expresses the fact that a freely falling body experiences no external force.

Next, let us consider a coordinate system $x_{\square}$ fixed to a second body such as the earth, or any system that may be in relative acceleration. The equation of motion can be transformed to that coordinate system as follows:

$$
\begin{equation*}
\frac{d \square}{d \square} \frac{\partial y_{\square}}{d x_{\square}} \frac{d x_{\square} \square}{d \square}=0 \tag{11.5}
\end{equation*}
$$

from which, after some algebra,[3] we find

$$
\begin{equation*}
\frac{d^{2} x_{\square}}{d \square^{2}}+\square_{\square \square \square} \frac{d x_{\square}}{d \square} \frac{d x_{\square}}{d \square}=0 \tag{11.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\square_{\mathbb{T D}}=\frac{\partial x_{\square}}{\partial y_{\square}} \frac{\partial^{2} y_{\square}}{\partial x_{\square} \partial x_{\square}} \tag{11.7}
\end{equation*}
$$

is called the affine connection. An observer on earth witnesses a body accelerating toward the earth and interprets it as the action of a "gravitational force." The principle of equivalence thus merely defines gravity as the invisible force that produces the
 indices, it is not a tensor since it is not Lorentz invariant.

We can obtain the Newtonian picture in the limit of low speeds, $d x_{k} / d \square \approx 0$. In this case, $d t=d \square d^{2} x_{o} / d \square=0$, and

$$
\begin{equation*}
\frac{d^{2} x_{k}}{d t^{2}}=\square_{k o o}=g_{k} \tag{11.8}
\end{equation*}
$$

for $k=1,2,3$, where $g=\left(g_{1}, g_{2}, g_{3}\right)$ is the Newtonian field vector ("acceleration due to gravity"). Thus, the $\square_{\text {koo }}$ elements of the affine connection are just the Newtonian gravitational field components in the limit of low speeds. Additional elements then are needed to describe gravity at speeds near the speed of light.

The Newtonian field vector for any distribution of mass can be obtained from the gravitational potential $\square$ which is in general a solution of Poisson's equation

$$
\begin{equation*}
\square^{2} \square=4 \square G \square \tag{11.9}
\end{equation*}
$$

where $\square$ is the mass density and $g=-\square \square$. For example, suppose that we have a point
mass $m$ so that $\square(\mathbf{r})=m \square \mathbf{r})$. Then,

$$
\begin{equation*}
\square(r)=\square \frac{G m}{r} \tag{11.11}
\end{equation*}
$$

the familiar Newtonian result.
While the modified equation of motion (11.6) contains relativistic effects of gravity, it is not covariant. It has a different form in the two reference frames. Einstein sought to find equations to describe gravity that were covariant. He started with the Poisson equation (11.9) above, which is noncovariant since $\square$ is the mass or energy density and energy is the zeroth component of a 4 -vector. Let us search for a covariant quantity to replace density.

Suppose we have a dust cloud in which all the dust particles are moving slowly, that is, with $v \ll c$, in some reference fame. Let the energy density in that frame be $\square_{0}$. Let $E_{0}$ be the rest energy of each particle $(c=1)$ and $n_{0}$ be the number per unit volume. Then,

$$
\begin{equation*}
\square_{o}=E_{o} n_{o} \tag{11.12}
\end{equation*}
$$

In some other reference frame the energy density will be

$$
\begin{equation*}
\square=E n=\square E_{o} \square n_{o}=\square \square \square \tag{11.13}
\end{equation*}
$$

where $\square$ is the Lorentz factor, $E=\square E_{0}$, and $n=\square n_{0}$. To see the latter, note that $n=$ $d N / d V$, where $d N$ is the number of particles in the volume element $d V, d N_{o}=d N$, and $d V=d V_{o} / \square$ from Fitzgerald-Lorentz contraction.

Note that $\square$ is not simply the component of a 4 -vector because of the factor $[7$.

Rather it must be made part of a second-rank tensor. We can write

$$
\begin{equation*}
T_{\square \square}=\square_{o} v_{\square} v_{\square} \tag{11.14}
\end{equation*}
$$

where $v_{\square}$ is the 4 -velocity of the cloud. Then, since $v_{o}=d t / d \square=\square$

$$
\begin{equation*}
T_{o o}=\square_{o} v_{o} v_{o}=\left[7 \square_{o}=\square\right. \tag{11.15}
\end{equation*}
$$

$T_{\square \square}$ is the energy-momentum tensor, or stress-energy tensor. The other components comprise energy and momentum flows in various directions: $T_{o i}$ is the energy flow per unit area in the $i$-direction, that is, a heat flow; $T_{i i}$ is the flow of momentum component $i$ per unit area in their direction, the pressure across the i plane; $T_{i j}$ is the flow of momentum component $i$ per unit area in the $j$-direction, the viscous drag across the $j$ plane; $T_{i o}$ is the density of the $i$ component of momentum.

Einstein thus wrote, as the covariant form of Poisson's equation,

$$
\begin{equation*}
G_{\square D}=\square 8 \square G T_{\square \square} \tag{11.16}
\end{equation*}
$$

where $G$ is Newton's constant and the factor is chosen so that we get Poisson's equation in the Newtonian limit. Since the energy-momentum tensor $T_{\square \square}$ is covariant, the quantity $G_{\square D^{\prime}}$ is also a covariant tensor field.

By associating $G_{\square \square}$ with the energy-momentum tensor, Einstein was using, at most, a very weak form of Mach's principle that was much earlier proposed by Leibniz. Leibniz had objected to Newton's notion of an absolute space with respect to which bodies accelerate and argued that at least another body must be present for space and time concepts to be useful.[4]

In what has become the standard model of general relativity, Einstein related $G_{\square \square}$ to the curvature of space-time in a non-Euclidean geometry. In non-Euclidian geometry, the proper distance between two points in space-time is

$$
\begin{equation*}
(\square s)^{2}=\square x_{\square} g_{\square \mathbb{Z}} \square x_{\square} \tag{11.17}
\end{equation*}
$$

where $g_{\square \square}$ is the metric tensor.
Einstein assumed that $G_{\square \square}$ is a function of $g_{\square \square}$ and its first and second derivatives. In its usual form, Einstein's field equation is given as

$$
\begin{equation*}
R_{\square \square} \square \frac{1}{2} g_{\square \square} R+\square g_{\square \square}=\square 8 \square G T_{\square \square} \tag{11.18}
\end{equation*}
$$

where $R_{\square \square}$ and $R$ are contractions of the rank four Riemann curvature tensor. To see the explicit forms of these quantities, consult any textbook on general relativity.

The quantity $\square$ is the infamous cosmological constant. It is often reported in the media and in many books on cosmology that the cosmological constant was a "fudge factor" Einstein introduced to make things come out the way he wanted. Perhaps that was his motivation, but the fact is that unless one makes further assumptions, a cosmological constant is required by Einstein's equations of general relativity and should be kept in the equations until some principle is found that shows it to be zero.[5] For many years the measurements of the cosmological constant gave zero within measuring errors, but in the past two decades Einstein's fudge factor has resurfaced again in cosmology.

## 12. Conclusions

The sophisticated reader who at least has glanced at the equations in this paper will
recognize them as very familiar. Almost every one can be found in standard textbooks. What has been attempted here is to show that those equations do not follow from very unique or very surprising physical properties of the universe. Rather, they arise from the very simple notion that whatever mathematical "laws" you write down to describe measurements, your equations cannot depend on the origin or direction of the coordinate systems you define in the space of those measurements or the space of the functions used to describe those laws. That is, they cannot reflect any privileged point of view. Except for the complexities that result from spontaneously broken symmetries, the laws of physics may be the way they are because they cannot be any other way. Or, at least they may have come about the simplest way possible. Table 11.1 summarizes these conclusions.

## Acknowledgements

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Table 1. The laws and other basic ideas of physics and their origin.

Law/idea of Physics
Conservation of momentum
Conservation of angular momentum
Conservation of energy
(First law of thermodynamics)
Newton's 1st Law of Motion

Newton's 2nd Law of Motion
Newton's 3rd Law of Motion

Second law of thermodynamics Special relativity
Invariance of speed of light

## Origin

Space translation symmetry
Space rotation symmetry

Time translation symmetry
Conservation of momentum
(space translation symmetry)
Definition of force
Conservation of momentum (space translation symmetry)
Statistical definition of the arrow of time
Space-time rotation symmetry
Space-time rotation symmetry

General relativity Principle of covariance
Quantum time evolution
(time-dependent Schrödinger equation) Global gauge invariance
Quantum operator differential forms
Quantum operator commutation rules
Quantization of action
Quantization rules for angular momenta
Maxwell's equations of electromagnetism
Global gauge invariance
Global gauge invariance
Global gauge invariance
Global gauge invariance

Quantum Lagrangians for particles in presence of electromagnetic field
Conservation of electric charge
Masslessness of photon
Conservation of weak isospin
Electroweak Lagrangian

Conservation of color charge
Strong interaction Lagrangian
Masslessness of gluon
Structure of the vacuum (Higgs particles)
Doublet structure of quarks and leptons

Masses of particles

Local gauge invariance under $\mathrm{U}(1)$
Global gauge invariance under $\mathrm{U}(1)$
Local gauge invariance under $\mathrm{U}(1)$
Global gauge invariance under $\mathrm{SU}(2)$
Mixing of $U(1)$ and $S(2)$ local gauge
symmetries (spontaneous symmetry
breaking)
Global gauge invariance under $\mathrm{SU}(3)$
Local gauge invariance under $\operatorname{SU}(3)$
Local gauge invariance under $\operatorname{SU}(3)$
Spontaneous symmetry breaking
Conservation of weak isospin
(global gauge invariance under $\mathrm{SU}(2)$ )
Higgs mechanism
(spontaneous symmetry breaking)

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