Complex Analysis Exam 1

October 29, 2018

Name: ____________________

1. Let \( f(z) \) be a branch of \( \log \frac{1+z}{1-z} \). Find the power series of this function centered at the point \( z = 0 \) and find its radius of convergence.

Proof: \( f(z) = \log(1+z) - \log(1-z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} \). The radius of convergence is

\[
R = \lim_{n \to \infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n \to \infty} \frac{1}{\sqrt[n]{2n+1}} = \lim_{n \to \infty} \frac{\sqrt[n]{2n+1}}{n} = 1. \quad \square
\]

2. Find a linear transformation \( f(z) \) that maps the points 1, 2 and \( i \) to the points \( i, 1, 2 \), respectively.

Solution: Consider the cross ratio \( (z, 1, 2, i) = (w, i, 1, 2) \) where \( w = f(z) \), i.e.,

\[
\frac{z - 1}{z - i} \cdot \frac{2 - i}{2 - 1} = \frac{w - i}{w - 2} \cdot \frac{1 - 2}{1 - i}.
\]

i.e., \((z - 1)(2 - i)(1 - i)(w - 2) = -(w - i)(z - i)\), i.e., \([(z - 1)(2 - i)(1 - i) + (z - i)]w = i(z = i) + 2(2 - i)(1 - i)(z - 1)\), i.e.,

\[
w = f(z) = \frac{(-5i + 2)z - 1 + 6i}{(2 - 3i)z - 1 + 2i}.
\]

3. Evaluate the integral \( \oint_C \frac{z}{|z|} \, dz \) where \( C = \partial \Delta(4) \) is the circle with counterclockwise orientation.

Solution: The curve \( C \) is defined by \( \phi : [0, 2\pi] \to \mathbb{C}, t \mapsto 4 \cos t + 4i \sin t = 4e^{it} \). Then

\[
\int_C \frac{z}{|z|} \, dz = \int_0^{2\pi} \frac{4e^{-it}}{4} 4ie^{it} \, dt = \int_0^{2\pi} 4idt = 8\pi i.
\]
4. Compute

\[ \oint_C \frac{dz}{z^2 + 1} \]

where \( C = \partial \Delta(-i, 1) \) with counterclockwise.

\[ \text{Solution:} \quad \text{By Cauchy integral formula, by setting } f(\zeta) = \frac{1}{\zeta - i}, \]

\[ \oint_C \frac{dz}{z^2 + 1} = \oint_C \frac{f(\zeta)}{\zeta - (-i)} = 2\pi i f(-i) = -\pi. \]

5. Let \( f: \mathbb{C} \rightarrow \mathbb{C} \) be a linear transformation such that \( f(x + ix) \in \mathbb{R} \) for any \( x \in \mathbb{R} \). If \( f(2) = i \), find \( f(2i) = \) ?

\[ \text{Solution:} \quad \text{Since } 2i \text{ and } 2 \text{ are mutual reflection points with respect to the line } L = \{ x + ix \mid x \in \mathbb{R} \}, \text{ and since } f(L) = \mathbb{R}, \text{ we know that } f(2i) \text{ and } f(2) \text{ are mutual reflection points with respect to the real axis. Since } f(2) = i, \text{ it implies } f(2i) = -i. \]

6. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) has the radius of convergence \( R, 0 < R < \infty \). Show that there exists at least one point \( z_0 \in \partial \Delta(R) \) such that \( f \) cannot extend holomorphically on a neighborhood of \( z_0 \).

\[ \begin{align*}
\text{Proof:} & \quad \text{Suppose there is no such point } z_0. \text{ We seek a contradiction. In fact, for any point } z \in \partial \Delta(R), \text{ there exists a neighborhood } \Delta(z, \varepsilon_z) \text{ on which } f \text{ can extend holomorphically. Then we obtain an open covering } \mathcal{U} = \{ \Delta(z, \varepsilon_z) \} \text{ of } \partial \Delta(R). \text{ Since } \partial \Delta(R) \text{ is compact, there exists a finite subcover of } \mathcal{U}: \Delta(z_1, \varepsilon_{z_1}), \Delta(z_2, \varepsilon_{z_2}), \ldots, \Delta(z_m, \varepsilon_{z_m}). \text{ This implies that } f \text{ extend holomorphically on } \Delta(R') \text{ where } R' > R, \text{ which is a contradiction.}
\end{align*} \]

\[ \frac{1}{2} \text{By } f \text{ extends holomorphically at } z_0 \in \partial \Delta(R), \text{ it means that there is another holomorphic function } g(z) \text{ defined on a disk } \Delta(z_0, r) \text{ such that } g(z) = f(z) \text{ for all } z \in \Delta(R) \cap \Delta(z_0, r). \]