RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

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Rational DAHA

In this lecture we will define the Rational Double Affine Hecke Algebra (DAHA), its presentations, and some of its subalgebras and standard modules. For us let \( W \subset \text{GL}(\mathfrak{h}) \) be a complex reflection group, which requires \(|W| < \infty\) and a subset \( S = \{s \in W \mid \text{codim} \text{Fix}(s) = 1\} \) that generates \( W \). We call elements of \( S \) \textit{reflections} of \( W \). For each reflection \( s \in S \) we associate an element \( \alpha_s \in \mathfrak{h}^* \) satisfying \( \text{Fix}(s) = \ker(\alpha_s) \) and \( s\alpha_s = \lambda_s\alpha_s \) for some \( \lambda_s \neq 1 \). Note that this element is defined only up to multiplication by a nonzero scalar.

Next we need a parameter: a function \( c : S \to \mathbb{C} \) that is constant on conjugacy classes, that is to say we require \( c(ws^{-1}) = c(s) \) for all \( s \in S \) and \( w \in W \). With these data we can introduce the \textbf{Dunkl operator} on the space of polynomial functions \( \mathbb{C}[\mathfrak{h}] \).

\textbf{Definition.} Let \( y \in \mathfrak{h} \). The \textbf{Dunkl operator} associated to \( y \) is

\[
D_y = \partial_y - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s} \frac{\langle \alpha_s, y \rangle}{\alpha_s} (1 - s)
\]

where \( \partial_y \) is the directional derivative: \( \partial_y(x) = \langle y, x \rangle \) for \( x \in \mathfrak{h}^* \), and \( \partial_y(f) \) can be computed from any \( f \in \mathbb{C}[[\mathfrak{h}]] = \text{Sym}(\mathfrak{h}^*) \) using the Leibniz rule.

As written this is only an operator on \( \mathfrak{h}^*_{\text{reg}} := \mathfrak{h} \setminus \bigcup_{s \in S} \{\alpha_s = 0\} \) because it has pole set \( \{\alpha_s = 0\} \). In fact, it is an element of the algebra \( \mathcal{D}(\mathfrak{h}^*_{\text{reg}}) \rtimes W \). Here \( \mathcal{D}(X) \) is the space of differential operators on \( X \) and the algebra has the underlying vector space \( \mathcal{D}(X) \otimes \mathbb{C}W \) and the multiplication is defined to be

\[
(d_1 \otimes w_1)(d_2 \otimes w_2) = d_1w_1(d_2) \otimes w_1w_2
\]

However, even if the operator \( D_y \) has poles, it does act on \( \mathbb{C}[[\mathfrak{h}]] \), polynomials in \( \dim \mathfrak{h} \) variables, because given a polynomial \( f : \mathfrak{h} \to \mathbb{C} \) the result of \((1 - s)f[x] = f(x) - f(sx)\) is divisible by \( \alpha_s(x) \). Hence we may think of these operators as elements of \( \text{End}_C(\mathbb{C}[\mathfrak{h}]) \).

These operators generate part of our rational DAHA:

\textbf{Definition.} The \textbf{Rational DAHA} \( \mathcal{H}_c \) is the subalgebra of \( \text{End}_C(\mathbb{C}[\mathfrak{h}]) \) generated by

- \( \mathbb{C}[[\mathfrak{h}]] \) acting on itself by multiplication,
- \( \mathbb{C}W \),
- \( D_y \) for \( y \in \mathfrak{h} \).

Note that, by definition, \( \mathcal{H}_c \) acts faithfully on \( \mathbb{C}[[\mathfrak{h}]] \).

This algebra is a deformation of \( \mathcal{D}(\mathfrak{h}) \rtimes W \) in the sense that this algebra is recovered when \( c \equiv 0 \). We can also give presentations of \( \mathcal{H}_c \). For each \( s \in S \), take a nonzero element \( \alpha_s^\vee \in \mathfrak{h} \) such that \( s\alpha_s^\vee = \lambda_s^{-1}\alpha_s^\vee \). This element is again only defined up to a nonzero scalar, and we partially normalize so that \( \langle \alpha_s, \alpha_s^\vee \rangle = 2 \). Of course, this normalization is inspired by the case when \( W \) is the Weyl group of a root system, \( \alpha_s \) is a root and \( \alpha_s^\vee \) the corresponding coroot.
Theorem 1 (Etingof, Ginzburg). Below we assume $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$. There is an isomorphism

$$\mathcal{H}_c \xrightarrow{\cong} \mathcal{T}(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W/\mathcal{R}$$

where $\mathcal{T}$ is tensor algebra functor and

$$\mathcal{R} = \left\{ [x, x'] = [y, y'] = 0, [y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle \alpha_s, y \rangle \langle \alpha_s, x \rangle s \right\}$$

are the relations.

$\mathcal{H}_c$ has the following notable subalgebras:

1. $\mathbb{C}[\mathfrak{h}]$
2. $\mathbb{C}W$
3. $\mathcal{D}_c$, the subalgebra generated by $D_y$ for all $y \in \mathfrak{h}$.

The following theorem of Dunkl shows the relationships between these subalgebras and other algebras.

Theorem 2 (Dunkl). $\mathcal{D}_c$ is isomorphic to $\text{Sym}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, i.e. the multilinear forms on $\mathfrak{h}$. Moreover

1. The algebra generated by $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}W$ is isomorphic to $\mathbb{C}[\mathfrak{h}] \rtimes W$.
2. The algebra generated by $\mathbb{C}W$ and $D_y$ is isomorphic to $\mathbb{C}[\mathfrak{h}^*] \rtimes W$.

Another important property of $\mathcal{H}_c$ is that it has a basis akin to the PBW basis for the universal enveloping algebra $U(g)$ for a Lie algebra.

Theorem 3 (PBW: Etingof, Ginzburg). There is an isomorphism

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\cong} \mathcal{H}_c$$

given by $h \otimes w \otimes h^* \mapsto hw h^*$.

$\mathcal{H}_c$ has further filtered and graded properties.

1. $\mathcal{H}_c$ has a filtering in the following way. Let $\deg W = 0$ and $\deg \mathfrak{h} = \deg \mathfrak{h}^* = 1$. This induces a filtration on $\mathcal{H}_c$ and by the PBW theorem we can identify the associated graded

$$\text{gr} \mathcal{H}_c \xrightarrow{\cong} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W.$$ 

2. From the relations, we can see that $\mathcal{H}_c$ has a grading by setting $\deg W = 0$, $\deg \mathfrak{h} = -1$, and $\deg \mathfrak{h}^* = 1$. This grading is inner, meaning that there exists an element $h \in \mathcal{H}_c$ satisfying $[h, m] = \deg(m)m$ for a homogeneous element $m$. We can construct such an element (called the Euler element) as follows. Given a basis of $y_i$ of $\mathfrak{h}$ and a dual basis $x_i$ this element is

$$h = \sum_{i=1}^{\dim \mathfrak{h}} \frac{x_i y_i + y_i x_i}{2} = \sum_{i=1}^{\dim \mathfrak{h}} x_i y_i - \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s}.$$ 

This element is in fact independent of the choice of basis and satisfies

$$[h, w] = 0 \quad [h, x] = x \quad [h, y] = -y$$

where $w \in W, y \in \mathfrak{h}, x \in \mathfrak{h}^*$. 

Some Representation Theory of $\mathcal{H}_c$

Definition. Let $\mathcal{O} = \mathcal{O}_c$ be the category of finitely generated $\mathcal{H}_c$ modules which have a locally nilpotent action of $\mathfrak{h} \subset \mathcal{H}_c$.

An example object in this category is the polynomial representation $\mathbb{C}[\mathfrak{h}]$. Indeed, $\mathfrak{h}$ acts on $\mathbb{C}[\mathfrak{h}]$ by Dunkl operators, which decrease the degree of a polynomial by at least 1. Given an irreducible representation $\lambda$ of the group $W$ we can extend the action to one of algebra $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ by letting $\mathfrak{h}$ act by zero. This gives us our standard modules

$$\Delta_c(\lambda) := \text{Ind}_{\mathbb{C}[\mathfrak{h}^*] \rtimes W}^{\mathcal{H}_c} (\lambda) = \mathcal{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W} \lambda \overset{\text{PBW}}{=} \mathbb{C}[\mathfrak{h}] \otimes \lambda$$

The last equality is as a $\mathbb{C}[\mathfrak{h}]$ module and follows from the PBW theorem. As a simple example if we induce the trivial representation $\lambda = 1$ we have the polynomial representation $\Delta_c(1) = \mathbb{C}[\mathfrak{h}]$.

Generally we have a subspace $1 \otimes \lambda \subset \Delta_c(\lambda)$, on which $h$ acts by some scalar $c_\lambda$ because $h$ commutes with $W$. Hence the action of $h$ is diagonalizable with eigenvalues of the form $c_\lambda + k$ for $k \in \mathbb{Z}_{\geq 0}$ and weight spaces

$$\Delta_c(\lambda)_{c_\lambda + k} = \mathbb{C}[\mathfrak{h}]_k \otimes \mathfrak{h},$$

where $\mathbb{C}[\mathfrak{h}]_k$ are the homogeneous polynomials of degree $k$. This seemingly innocent fact has a couple of important consequences. First, it can be deduced that there is a unique irreducible quotient $L_c(\lambda)$ of $\Delta_c(\lambda)$. These irreducible quotients form a complete list of irreducibles in $\mathcal{O}_c$. Second, note that if $L_c(\mu)$ appears as a composition factor in $\Delta_c(\lambda)$, then $c_\mu = k + c_\lambda$ for some $k \geq 0$ (and, if $\mu \neq \lambda$, $k > 0$). It follows that if $c$ is a parameter so that $c_\lambda - c_\mu \notin \mathbb{Z}$ for any two irreducibles $\lambda \neq \mu$, then the category $\mathcal{O}_c$ is semisimple and equivalent to the category of representations of $W$.

The category $\mathcal{O}_c$ is intimately related to the category of finite-dimensional representations of a certain finite Hecke algebra, as follows. First, consider the element $\delta := \prod_{s \in S} \alpha_s \in \mathbb{C}[\mathfrak{h}] \subseteq \mathcal{H}_c$. Since $\mathcal{H}_c \leq D(\mathfrak{h}^\text{reg}) \rtimes W$ and $\mathfrak{h}^\text{reg}$ is precisely the principal open set defined by $\delta$, the non-commutative localization $\mathcal{H}_c[\delta^{-1}]$ makes sense and it follows from the formula defining the Dunkl operators that $\mathcal{H}_c[\delta^{-1}] = D(\mathfrak{h}^\text{reg}) \rtimes W$.

Now take $M \in \mathcal{O}_c$. By definition, it is finitely generated over $\mathcal{H}_c$. It is an exercise to see that, moreover, it is finitely generated over $\mathbb{C}[\mathfrak{h}]$. It follows that $M[\delta^{-1}] := \mathbb{C}[\mathfrak{h}][\delta^{-1}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$ is a $D(\mathfrak{h}^\text{reg}) \rtimes W$-module that is finitely generated over $\mathbb{C}[\mathfrak{h}^\text{reg}]$. From the theory of $D$-modules it follows that $M[\delta^{-1}]$ is a $W$-equivariant vector bundle on $\mathfrak{h}^\text{reg}$ with a flat connection. Taking $W$-invariants, we obtain a vector bundle on $\mathfrak{h}^\text{reg}/W$ with a flat connection. The monodromy representation then equips a fiber of this vector bundle with an action of the fundamental group $\pi_1(\mathfrak{h}^\text{reg}/W)$.

This analysis can be encoded as functors

$$\mathcal{O}_c \xrightarrow{\text{Rep}} \text{Rep}(\mathcal{H}_c[\delta^{-1}]) \xrightarrow{\text{Rep}} \text{Rep}(D(\mathfrak{h}^\text{reg}/W)) \xrightarrow{\text{Rep}} \text{Rep}(\pi_1(\mathfrak{h}^\text{reg}/W))$$

$$M \xrightarrow{\text{}} M[\delta^{-1}] \xrightarrow{\text{}} M[\delta^{-1}]_v \xrightarrow{\text{}} M[\delta^{-1}]_v^W$$

where we use the fact $\mathcal{H}_c[\delta^{-1}] = D(\mathfrak{h}^\text{reg}) \rtimes W$ to induce the second map, and the notation $M[\delta^{-1}]_v^W$ means the fiber of the bundle $M[\delta^{-1}]_v$ at a point $v \in \mathfrak{h}^\text{reg}/W$ (any choice of points yields isomorphic representations).
An amazing fact now is that the action of \( \pi_1(\mathfrak{h}^{\text{reg}}/W) \) factors through a much smaller quotient of the group algebra \( \mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W) \), known as the finite Hecke algebra. Here, we will only give details on the case when \( W = S_n \) is the symmetric group, acting on \( \mathfrak{h} = \mathbb{C}^n \) by permuting the coordinates. Note that in this case there is a single conjugacy class of reflections, and so our parameter is a single complex number \( c \in \mathbb{C} \). The set \( \mathfrak{h}^{\text{reg}} \) consists of points in \( \mathbb{C}^n \) with pairwise distinct coordinates, and \( \pi_1(\mathfrak{h}^{\text{reg}}/W) \) is the usual Artin braid group, generated by \( T_1, \ldots, T_{n-1} \) with relations \( T_i T_j = T_j T_i \) if \( |i - j| > 1 \), and \( T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \) (the element \( T_i \) represents a half loop around the hyperplane \( x_i = x_{i+1} \), that descends to a loop in the quotient \( \mathfrak{h}^{\text{reg}}/W \)). For a module \( M \in \mathcal{O}_c \), the action of \( \mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W) \) on \( M[\delta^{-1}]_w \) factors through the quotient

\[
H_q = \mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W)/\langle (T_i - 1)(T_i + e^{2\pi \sqrt{-1} c}) \rangle_{i=1,\ldots,n-1}
\]

that, up to a renormalization, coincides with the finite Hecke algebra that appeared in Monica Vazirani’s lectures. To summarize, we have a functor \( KZ: \mathcal{O}_c \to H_q \)-mod, known as the Knizhnik-Zamolodchikov functor (because the connection appearing in its definition coincides with the Knizhnik-Zamolodchikov connection). This functor is exact, and it is one of the most important tools in the representation theory of the rational DAHA \( H_c \).

**Examples**

**Example 1.** Let \( \ell = \sqrt{-1} \). Let \( W = \mathbb{Z}/\ell\mathbb{Z} = \langle s \mid s^\ell = 1 \rangle \) act on \( \mathfrak{h} = \mathbb{C} \) via multiplication by \( \eta = e^{2\pi i/\ell} \), i.e. \( s, z = \eta z \) which is the rotation of the complex plane by the angle \( 2\pi/\ell \). Our reflections are \( S = s^i \mid 1 \leq i \leq \ell - 1 \) and our function \( c \) is determined by the numbers \( c(s^i) = c_i \), or just the vector \( c = (c_1, \ldots, c_{\ell-1}) \). Pick \( x \in \mathfrak{h}^* \) and define \( \alpha_x = x \in \mathfrak{h}^* \); the number \( \lambda_x \) is \( \eta^{-1} \) since the \( W \) acts via the adjoint on \( \mathfrak{h}^* \). Then the Dunkl operator is

\[
D_y = \partial_y - \sum_{i=1}^{\ell-1} \frac{2c_i}{1 - \eta^{-i}} \frac{1 - s^i}{x}
\]

Then \( H_c \) can be presented as the algebra

\[
\mathbb{C}[x, y, s]/\mathcal{R}
\]

with relations

\[
\mathcal{R} = \left\langle s^\ell = 1, sx^{-1} = s x^{-1} = \eta x, s y^{-1} = \eta^{-1} y, [y, x] = 1 - \sum_{i=1}^{\ell-1} 2c_i s^i \right\rangle.
\]

**Example 2.** Let \( W = S_n \) act on \( \mathfrak{h} = \mathbb{C}^n \) by permutation of the coordinates and take

\[
S = \{(ij) \mid i < j\}.
\]

Then \( \alpha_{(ij)} = x_i - x_j \), the usual \( GL_n(\mathbb{C}) \) positive roots, and the numbers \( \lambda_x \) are all seen to be \(-1\). Since all elements of \( S \) are conjugate in \( S_n \) we need only specify a single complex number \( c \). Then the Dunkl operators take the form

\[
D_y = \partial_y - \sum_{j \neq i} \frac{c}{x_i - x_j} (1 - (ij))
\]

and \( H_c \) can be presented with the relations

\[
\mathcal{R} = \left\langle [x_i, x_j] = [y_i, y_j] = 0, [y_j, x_i] = c(ij) \text{ if } i \neq j, [y_i, x_i] = 1 - c \sum_{i \neq j} (ij) \right\rangle.
\]
as the algebra
\[ \mathcal{H}_c = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes S_n/\mathcal{R}. \]

Recall that the irreducible representations of \( S_n \) correspond exactly to partitions \( \lambda \) of \( n \). Let \( S(\lambda) \) be these modules. Then \( -s \sum_{s \in S} s \) acts on \( S(\lambda) \) by
\[
-c \sum_{i=1}^{n} JM_i
\]
where \( JM_i = \sum_{j<i} (ji) \) are the Jucys-Murphy’s elements. \( S(\lambda) \) has a basis consisting of standard Young tableaux of shape \( \lambda \) and \( JM_i \) acts by \( JM_i t = ct([i]) t \), where \( ct([i]) \) is the content of the cell \([i]\) which is defined to be its column coordinate minus its row coordinate,
\[
ct([i]) = \text{col}([i]) - \text{row}([i]).
\]
Therefore
\[
c_\lambda = -\frac{n}{2} - c \sum_{\Box \in \lambda} ct(\Box)
\]

We note that if \( L(\mu) \) appears in the Jordan-Hölder series for \( \Delta(\lambda) \) then \( c_\mu = c_\lambda + k \) for \( k \in \mathbb{Z}_{\geq 0} \). For generic complex numbers \( c \) the category \( \mathcal{O}_c \) is semi-simple.