VARIATIONAL PROBLEMS IN LAGRANGIAN GEOMETRY: $\mathbb{Z}_2$-CURRENTS

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1. **Introduction**

In this article we discuss aspects of the existence theory for extrema of the lagrangian variational problem introduced in [S-W] and described in the article of R. Schoen in this volume [Sc]. We will discuss questions about the regularity of the extrema only as they relate to the existence question.

Recall the set-up. We let (X, ω) be a compact symplectic manifold of dimension $2n$ equipped with a compatible metric g. In particular we could take (X, ω, g) to be Kähler. An n-dimensional submanifold Σ of X is called *lagrangian* if $\omega_{|\Sigma} = 0$. An n-dimensional cycle is lagrangian if each n-simplex is lagrangian. We say a homology class $\alpha \in H_n(X, \mathbb{Z})$ is lagrangian if it can be represented by a lagrangian cycle. We consider the variational problem of finding extrema of volume among the lagrangian cycles (lagrangian currents, lagrangian maps, etc.) representing the lagrangian homology class α. It is also possible to formulate other variational problems, such as boundary value problems, a homotopy problem, etc., but for simplicity we will emphasis the homology problem. To be rigorous it is necessary to specify precisely the class of lagrangians among which an extrema is sought.

Mapping Problem

When the domain manifold is 2-dimensional, because the energy of a conformal map equals the area of its image, it is possible to formulate a mapping problem. Let Σ be a Riemann surface and consider the maps $f : \Sigma \to X$ in $W^{1,2}(\Sigma, X)$. These are maps such that f and Df are in $L^2(\Sigma, \mathbb{R}^N)$ for some isometric embedding of X into $\mathbb{R}^N$. Note that $f^*\omega$ is an L^1-valued 2-form on Σ. We say that a map $f \in W^{1,2}(\Sigma, X)$ is *weakly lagrangian* if $f^*\omega = 0$ a.e. An important ingredient in formulating the lagrangian mapping problem is the following compactness result [S-W]: The set of weakly lagrangian maps in $W^{1,2}(\Sigma, X)$ satisfying a uniform energy bound is closed in the weak topology. As a consequence a minimizing sequence of lagrangian maps in $W^{1,2}(\Sigma, X)$ has a weakly lagrangian limit in $W^{1,2}(\Sigma, X)$. If the maps in the sequence represent a homology class then, as in the classical unconstrained case, the limit map may not represent the same class. However there are well-known techniques available to understand and handle this phenomenon.

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The main results for this problem were obtained in [S-W] and are surveyed in [Sc].

Integral Currents

In general it is necessary to use currents in the formulation of our variational problem. The class of currents that are the geometric measure theory generalization of oriented submanifolds are the integral currents. There are two natural topologies for integral currents, the weak topology and the flat norm topology. A sequence $\{T_i\}$ of n-currents converges to the n-current T in the weak topology if for every C^∞ n-form ϕ on X with compact support

$$\lim_{i\to\infty} T_i(\phi) = T(\phi).$$

The sequence converges to T in the flat norm topology if there is a sequence of $n+1$- integral currents P_i and a sequence of n-integral currents Q_i with

$$\lim_{i\to\infty}(|P_i| + |Q_i|) = 0,$$

satisfying:

$$T_i - T = \partial P_i + Q_i.$$

where $|P|$ denotes the mass (or volume) of the integral current P. Two currents are close in the flat norm topology if they are the boundary (up to a small n-volume current) of a current with small $(n+1)$-volume. We say an integral current T is lagrangian if for every C^∞ $(n-2)$-form ϕ on X with compact support,

$$T(\omega \wedge \phi) = 0.$$

It follows easily that if a sequence of lagrangian integral currents converges weakly to an integral current then the limit current is lagrangian. From the basic theory of currents (see [Si]) this implies the same result using the flat norm topology. Applying these compactness results to a minimizing sequence of lagrangian integral currents (representing a homology class) we can find a minimizer that is also a lagrangian integral current and represents the homology class. Unfortunately, in the unconstrained case, there are serious difficulties in the regularity theory of minimizers when the codimension of the minimizer is greater than one. These problems persist in the constrained (lagrangian) problem. Accordingly we seek a larger class of currents.

$\mathbb{Z}_2$-currents

The $\mathbb{Z}_2$-currents are the geometric measure theory generalization of the unoriented (in particular, non-orientable) submanifolds. Since there are more comparisons available than for integral currents the regularity theory for minimizers is better than that for integral currents [F]. Because the currents are unoriented, the weak topology is not available so we must use the flat norm topology. We say a $\mathbb{Z}_2$-current is lagrangian if its approximate tangent planes are lagrangian planes a.e.. To use the lagrangian $\mathbb{Z}_2$-currents in our variational problem we seek to establish a sequential compactness result in the flat norm topology. This problem is the subject of the next section.

2. Lagrangian $\mathbb{Z}_2$-Currents

In $\mathbb{R}^4$ set $\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_i dy_i - y_i dx_i)$. Then $d\lambda = \sum_{j=1}^{2} dx_i \wedge dy_i = \omega$ is the standard symplectic form. If γ is a closed curve in $\mathbb{R}^4$ the quantity $\int_\gamma \lambda$ is called the *period* of the curve. A well known fact in symplectic geometry is that a closed curve γ in $\mathbb{R}^4$ spans a lagrangian disc if and only if its period vanishes. The following isoperimetric inequality due to Allcock [A] and Gromov [G] is a quantitative version of this result.

Theorem 2.1. *Let γ be a closed curve in $\mathbb{R}^4$ satisfying $\int_\gamma \lambda = 0$. Then there is a lagrangian disc D spanning γ with,*

$$|D| \leq c|\gamma|^2,$$

where c is a universal constant.

Qiu [Q] proved the following non-orientable isoperimetric inequality using ideas based on Allcock's proof.

Theorem 2.2. *Let γ be any closed curve in $\mathbb{R}^4$. Then there is a lagrangian Möbius band M spanning γ with,*

$$|M| \leq c|\gamma|^2,$$

where c is a universal constant.

This isoperimetric inequality has the following interesting application. Recall ([Sc], [S-W]) that the 2-dimensional lagrangian cones in $\mathbb{R}^4 \simeq \mathbb{C}^2$ that are stationary under hamiltonian variations are parameterized by a pair of relatively prime integers $p, q \geq 1$. Explicitly the (p,q)-cone in $\mathbb{C}^2$ with coordinates (z_1, z_2) is:

$$C_{p,q} = \frac{1}{\sqrt{p+q}}\left(r\sqrt{q}e^{i\sqrt{\frac{p}{q}}s}, ir\sqrt{p}e^{-i\sqrt{\frac{q}{p}}s}\right), \tag{2.1}$$

where $0 \leq s \leq 2\pi\sqrt{pq}$ and $r \geq 0$. The difference $p - q$ is the Maslov index of the cone. If $|p-q| > 1$ then the cone is unstable for compactly supported hamiltonian variations and hence these cones are not tangent cones on lagrangian minimizers. However for at least one pair (p,q) with $|p-q| = 1$ the cone is a minimizer among oriented lagrangian comparisons and occurs as a tangent cone on a minimizer. (This fact will discussed in more detail below.) Using the non-orientable isoperimetric inequality the following result is obtained in [S-W]:

Theorem 2.3. *For any (p,q) with $(p,q) \neq (1,1)$, the cone $C_{p,q}$ does not minimize area among nonorientable lagrangian comparison surfaces.*

Outline of the Proof: Deform the annular region of the cone from $r = 1$ to $r = \rho$ by the mean curvature H. Since $H \approx 1/r$ this decreases area by $c \iint_1^\rho H^2 r dr ds = c' \ln \rho$. The mean curvature is a symplectic deformation so the deformed region remains lagrangian however, because H is not hamiltonian, the curves $r = 1$ to $r = \rho$ are deformed to curves that no longer have zero period. Thus the deformed region cannot be joined to the undeformed regions of the cone by annular lagrangian strips. However they can be joined by non-orientable lagrangians. Using the isoperimetric inequality such non-orientable lagrangians can be constructed having bounded area (independent of ρ). Therefore, for ρ sufficiently large, the new lagrangian surface has smaller area.

Suppose that the ambient manifold X is Kähler-Einstein (ie., $\mathrm{Ric} = \mathrm{R}\omega$). On a smoothly immersed lagrangian the mean curvature vector H is an infinitesimal symplectic motion and thus an admissible variation in the lagrangian variational problem. Using this observation and a first variation argument (see [Sc]) it follows that a *regular* lagrangian minimizer is a classical minimal submanifold ($H = 0$). In the mapping problem the regularity theorem for minimizers ([S-W]) shows that a minimizer is an immersion except at isolated points. These points are either branch points or singularities with tangent cones $C_{p,q}$ (2.1). The existence of non-planar tangent cones that are minimizing (among *oriented* lagrangian comparisons) implies that a minimizer for the mapping problem may have a singular point with such a tangent cone. For such a minimizer we are then unable to use H as a variation and therefore we cannot conclude that the minimizer is a classical minimal surface. This same problem occurs in the variational problem using 2-dimensional lagrangian integral currents. However, Theorem 2.3 implies that there are no non-planar tangent cones that are minimizing (among *unoriented* lagrangian comparisons). Using this observation and the regularity theory developed for the mapping problem [S-W], we conclude that a 2-dimensional lagrangian $\mathbb{Z}_2$ minimizer is regular (has no singularities) and therefore H is an admissible variation. What is lacking is a result establishing the existence of a lagrangian $\mathbb{Z}_2$-current minimizer. Unfortunately, the following result of Qiu [Q], also an application of the non-orientable isoperimetric inequality, shows that the existence theory for lagrangian $\mathbb{Z}_2$-current minimizers is problematic.

Theorem 2.4. *The set of lagrangian $\mathbb{Z}_2$-currents in $\mathbb{R}^4$ is dense in the flat norm topology in the set of all $\mathbb{Z}_2$-currents.*

Proof. Throughout the proof c will denote universal constants. Fleming [Fl] shows that the polyhedral chains are dense, in the flat norm topology, in the $\mathbb{Z}_2$-currents. Therefore, it suffices to prove the theorem for a planar unit square P in $\mathbb{R}^4$. Divide P into N^2 subsquares P_j, $j = 1, \dots, N^2$, so that each P_j has sides of length $\frac{1}{N}$. By Theorem 2.2 for each P_j there is a lagrangian

Möbius band M_j with $\partial M_j = \partial P_j$ and

$$|M_j| \leq c|\partial P_j|^2 = c(\frac{4}{N})^2.$$

Set $M = \sum_j M_j$. Since interior boundaries cancel we have $\partial P = \partial M$. Note that,

$$|M| \leq \sum_j |M_j| \leq N^2 c(\frac{4}{N})^2 = 16c.$$

Since $\partial(M_j - P_j) = 0$ by the classical isoperimetric inequality there is a 3-dimensional $\mathbb{Z}_2$-current T_j with $\partial T_j = M_j - P_j$ and

$$|T_j| \leq c|M_j - P_j|^{\frac{3}{2}}.$$

But $|M_j - P_j| \leq |M_j| + |P_j| \leq \frac{c}{N^2}$ so,

$$|T_j| \leq \frac{c}{N^3}.$$

Set $T = \sum_j T_j$. Then $\partial T = M - P$ and

$$|T| \leq N^2 \frac{c}{N^3} = \frac{c}{N}.$$

It follows that as $N \to \infty$, M approximates P in the flat norm topology. $\square$

Notice that the proof implies that a bounded domain in $\mathbb{C} \subset \mathbb{R}^4$ can be approximated by lagrangians satisfying a uniform area bound. Another curious consequence of Qiu's theorem is that in a symplectic 4-manifold X every class in $H_2(X, \mathbb{Z}_2)$ is lagrangian, that is, can be represented by a lagrangian cycle. By comparison a class $\alpha \in H_2(X, \mathbb{Z})$ is lagrangian if and only if $\int_\alpha \omega = 0$.

For our purposes the main consequence of Qiu's theorem is that the limit of a sequence of lagrangian $\mathbb{Z}_2$-currents with uniformly bounded volumes (masses) may not be lagrangian. It remains possible that a *minimizing* sequence of lagrangian $\mathbb{Z}_2$-currents is always lagrangian. However Qiu's result suggests otherwise. Note that the construction, in the proof of the theorem, introduces curves γ on the lagrangians for which $\int_\gamma \lambda \neq 0$, where $\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_i dy_i - y_i dx_i)$ is the Liouville form. An immersed lagrangian $\Sigma \subset \mathbb{R}^{2n}$ is called *exact* if $\int_\tau \lambda = 0$, for every closed curve τ on Σ. That is, the approximating lagrangians constructed in the proof of Theorem 2.4 are not exact. Thus it is possible that, restricting to exact lagrangians, sequential compactness holds.

Recall that an equivalent formulation of exactness, one more convenient for our purposes, is given as follows: Consider $\mathbb{R}^{2n+1} = \{(x, y, \varphi)\}$. Denote by π the projection $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ $(x, y, \varphi) \mapsto (x, y)$. On $\mathbb{R}^{2n+1}$ define the *contact* 1-form $\eta = d\varphi - \frac{1}{2}\sum_{j=1}^{n}(x_i dy_i - y_i dx_i)$. The pair $(\mathbb{R}^{2n+1}, \eta)$ is called a *contact manifold*. The hyperplane distribution defined by $\eta = 0$ is called the *contact distribution*. An n-dimensional submanifold S is called *legendrian* if $\eta_{|_S} = 0$ or equivalently if it is everywhere tangent to the contact distribution. A n-dimensional $\mathbb{Z}_2$-current is called *legendrian* if its approximate tangent planes lie in the contact distribution a.e.. It is easy to see that if S is legendrian then $\pi(S) \subset \mathbb{R}^{2n}$ is an exact lagrangian. Conversely, a smoothly immersed exact lagrangian $\Sigma \subset \mathbb{R}^{2n}$ has a legendrian lift.

For a point $p \in \mathbb{R}^{2n+1}$, denote the contact $2n$-plane at p by H_p. Then $d\pi_p : H_p \to \mathbb{R}^{2n}$ is an isomorphism. Define a metric g on the contact distribution by requiring that $d\pi_p$ is an isometry for every $p \in \mathbb{R}^{2n+1}$. The metric g is a Riemannian metric on the contact distribution but defines a degenerate (Carnot) metric on $\mathbb{R}^{2n+1}$. The importance of the legendrians for our purposes is due to the following theorem.

Theorem 2.5. *Let $\{\Sigma_i\}$ be a sequence of legendrian $\mathbb{Z}_2$-currents in $\mathbb{R}^{2n+1}$ that satisfy $|\Sigma_i| + |\partial\Sigma_i| < C$ (using the metric g). There is a subsequence that converges in the flat norm topology to a legendrian $\mathbb{Z}_2$-current.*

Proof. Introduce a Riemannian metric, depending on a parameter, on $\mathbb{R}^{2n+1}$ as follows. Choose a Reeb vector field V on L, that is, a vector field everywhere transverse to the contact distribution such that $\eta(V) = 1$. Define a metric g_1 on $\mathbb{R}^{2n+1}$ so that $d\pi$ restricted to the contact subspace $H \subset T\mathbb{R}^{2n+1}$ is an isometry and so that V is orthogonal to P and of unit length. Thus, $g_1 = g + \eta^2$. Introduce the family of metrics $g_\varepsilon = g + \varepsilon^{-2}\eta^2$.

Consider the sequence of legendrian $\mathbb{Z}_2$-currents Σ_i. Choose a sequence $\varepsilon_j \downarrow 0$. For each j, using the metric g_{ε_j}, choose a subsequence $\{\Sigma_{i_j}\}$ that converges in the flat norm topology to a $\mathbb{Z}_2$-current. Let Σ_0 denote the limit of a diagonal subsequence of $\{\Sigma_{i_j}\}$. It follows that, for any ε_j, $\mathrm{Vol}_{\varepsilon_j}(\Sigma_0)$ is bounded. If Σ_0 is not legendrian, then there is a set $E \subset \Sigma_0$ of positive measure whose tangent planes are not subspaces of the contact planes. This implies that $\mathrm{Vol}_{\varepsilon_j}(E) \to \infty$ as $\varepsilon_j \to 0$, a contradiction. □

Remark 1: The theorem shows that the set of legendrian $\mathbb{Z}_2$-currents has a suitable compactness property. It is remarkable that to formulate the monotonicity formula of [S-W] for lagrangian minimizers in $\mathbb{R}^{2n}$ it is necessary to introduce the legendrian lifts to the contact manifold $\mathbb{R}^{2n+1}$. There is no obvious reason for this coincidence.

Remark 2: The theorem is also true for sequences of legendrian integral currents. However in the case of legendrian integral currents convergence to a legendrian current can be easily proved in the weak topology. Convergence in the flat norm topology then follows.

An apparent conclusion of Theorem 2.5 is that minimizers among legendrian $\mathbb{Z}_2$-currents can be found. However, Theorem 2.5 is a local result and it is not clear how to generalize the notion of legendrian to manifolds. Indeed if ω is not an exact form, in general, periods cannot be defined. Of course, ω is locally exact and locally, by the Darboux theorem, every symplectic manifold looks like $\mathbb{R}^{2n}$. One could try to exploit this to define a class of "locally exact" lagrangians. However, fixing a covering of a symplectic manifold X by Darboux neighborhoods, it is not difficult to construct a sequence of lagrangians that are exact in every neighborhood of the cover but that converge in the flat norm topology to a lagrangian lying in a Darboux ball that is not exact. What is needed is a global version of exactness, that is, a global version of legendrian.

3. **Global Exactness**

Let (X, ω) be a compact symplectic manifold of dimension $2n$. Suppose that the symplectic form ω is integral, that is, the homology class $[\omega]$ is an integral class. Then there is an hermitian line bundle L over X with unitary connection η such that the curvature of η is ω [K]. Let L also denote the total space of the associated S^1 principal bundle and let $\pi : L \to X$ denote the bundle projection. On L the one-form η is globally well-defined and satisfies $d\eta = \pi^*\omega$. Thus η is a contact one-form. The horizontal distribution of the connection η is the contact distribution. Define a metric g on the horizontal distribution by requiring that $d\pi_p : H_p \to T_p(X)$ is an isometry for each $p \in L$, where H_p is the horizontal $2n$-plane at p. Then g is a degenerate (Carnot) metric on the contact manifold L. We say that an n-dimensional smooth submanifold S is *legendrian* if $\eta_{|S} = 0$ or, equivalently, if $T_p(S) \subset H_p$ for each $p \in S$. We say that a $\mathbb{Z}_2$-current is *legendrian* if its approximate tangent planes lie in the horizontal distribution a.e..

Using the same argument as in the proof of Theorem 2.5 it follows that a sequence of legendrian $\mathbb{Z}_2$-currents that satisfy a uniform mass bound for the metric g (as in Theorem 2.5) has a subsequence that converges in the flat norm topology to a legendrian $\mathbb{Z}_2$-current. Thus

Theorem 3.1. *Let $\mathcal{L}_\alpha$ be the set of legendrian $\mathbb{Z}_2$-cycles in L that represent the homology class $\alpha \in H_n(L, \mathbb{Z}_2)$. A sequence of currents in $\mathcal{L}_\alpha$ that minimizes volume has a subsequence that converges in the flat norm topology to a current in $\mathcal{L}_\alpha$.*

Theorem 3.1 shows that there is a global existence theory for legendrian $\mathbb{Z}_2$ minimizers. This existence theory can be applied to a class a in $H_n(X, \mathbb{Z}_2)$ that has a lift to a legendrian class α in $H_n(L, \mathbb{Z}_2)$. The push forward by π of the legendrian minimizer representing α is a lagrangian $\mathbb{Z}_2$-cycle that minimizes volume among $\mathbb{Z}_2$-cycles that represent a and that have legendrian lifts. We will call the lagrangian $\mathbb{Z}_2$ minimizer an "exact lagrangian" $\mathbb{Z}_2$ minimizer or, by abuse of nomenclature, a legendrian $\mathbb{Z}_2$ minimizer.

We remark that the minimizer of Theorem 3.1 is contact stationary in the sense of [S-W]. That is, the minimizer is stationary with respect to the contact transformations of L. These are the diffeomorphisms of L that preserve the contact distribution. In the case that the currents are 2-dimensional, the contact stationary currents satisfy a monotonicity inequality [S-W]. Using this it can be shown that the minimizers satisfy a regularity result similar to that satisfied by the minimizers of the mapping problem. In particular, the minimizers are (nonorientable) immersed surfaces except at isolated points which are singularities with $(p, p+1)$ tangent cone. The regularity question for the 2-dimensional legendrian minimizers is thus reduced to the question of whether or not the 2-dimensional $(p, p+1)$ cones are minimizing among $\mathbb{Z}_2$ legendrian comparisons. (We saw above in Theorem 2.3 that they are not minimizing among $\mathbb{Z}_2$ lagrangian comparisons. However the comparisons used in the proof of Theorem 2.3 have non-trivial periods and are therefore not legendrian comparisons.)

In [S-W] it is shown that at least one of the $(p, p+1)$ cones is minimizing among orientable lagrangian comparisons. Why such a cone exists and why it is pausible to believe that such minimizing cones do not exist among nonorientable legendrian comparisons can be seen as a consequence of the following topological reasoning: Let $\ell : \Sigma \to N$ be a lagrangian immersion where Σ is a surface (orientable or not) and N is a symplectic 4-manifold. Then there is a splitting $\ell^*TN \simeq T\Sigma \oplus T^*\Sigma$ of ℓ^*TN into a pair of lagrangian subbundles over Σ. But then,

$$\ell^*TN \simeq T\Sigma \oplus T^*\Sigma \simeq T\Sigma \otimes \mathbb{C}, \tag{3.1}$$

where the isomorphisms are of symplectic (or equivalently almost complex) vector bundles over Σ. It follows that $c_1(\ell^*TN) \in H^2(\Sigma, \mathbb{Z})$ is an element of order 2 and so vanishes if Σ is orientable. Suppose $\alpha \in H_2(N, \mathbb{Z})$ is a lagrangian homology class and $c_1(N)(\alpha) \neq 0$. Then the minimizer among orientable lagrangians representing α cannot be a (branched) immersion and hence at least one cone singularity must occur on the minimizer. Thus at least one of the $(p, p+1)$ cones must be a minimizer among oriented lagrangian comparisons. On the other hand if Σ is nonorientable then $c_1(\ell^*TN)$ need not vanish. In fact since it is an element of order 2 we can replace $c_1(N)$ with its mod 2 reduction, $w_2(N)$, the second Stiefel-Whitney class. Using (3.1) and the Whitney sum formula we have:

$$\begin{aligned} \ell^* w_2(N) &= w_2(T\Sigma \oplus T^*\Sigma) \\ &= w_2(T\Sigma) + w_1(T\Sigma) \cdot w_1(T^*\Sigma) + w_2(T^*\Sigma) \\ &= w_1^2(\Sigma). \end{aligned}$$

Now suppose $\alpha \in H_2(N, \mathbb{Z}_2)$ is a legendrian homology class and $w_2(N)(\alpha) \neq 0$. If the minimizer among $\mathbb{Z}_2$ legendrians representing α is immersed the Stiefel-Whitney number w_1^2 of the minimizer must be nonzero. What was an obstruction to regularity in the orientable case becomes a restriction on topology in the nonorientable case. Philosophically singularities in the orientable case are replaced with Möbius bands in the nonorientable case. The problem is to show that this philosophy is correct. If this can be done then the introduction of legendrian $\mathbb{Z}_2$-currents will be justified.

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