Chapter 1

The Trigonometric Functions

Trigonometry had its start as the study of triangles. The study of triangles can be traced back to the second millenium B.C.E. in Egyptian and Babylonian mathematics. In fact, the word trigonmetry is derived from a Greek word meaning "triangle measuring." We will study the trigonometry of triangles in Chapter 3. Today, however, the trigonometric functions are used in more ways. In particular, the trigonometric functions can be used to model periodic phenomena such as sound and light waves, the tides, the number of hours of daylight per day at a particular location on earth, and many other phenomena that repeat values in specified time intervals.

Our study of periodic phenomena will begin in Chapter 2, but first we must study the trigonometric functions. To do so, we will use the basic form of a repeating (or periodic) phenomena of travelling around a circle at a constant rate.

1.1 The Unit Circle

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the unit circle and why is it important in trigonometry? What is the equation for the unit circle?
- What is meant by "wrapping the number line around the unit circle?" How is this used to identify real numbers as the lengths of arcs on the unit circle?
- How do we associate an arc on the unit circle with a closed interval of real numbers?

Beginning Activity

As has been indicated, one of the primary reasons we study the trigonometric functions is to be able to model periodic phenomena mathematically. Before we begin our mathematical study of periodic phenomena, here is a little "thought experiment" to consider.

Imagine you are standing at a point on a circle and you begin walking around the circle at a constant rate in the counterclockwise direction. Also assume that it takes you four minutes to walk completely around the circle one time. Now suppose you are at a point P on this circle at a particular time t.

- Describe your position on the circle 2 minutes after the time t.
- Describe your position on the circle 4 minutes after the time t.
- Describe your position on the circle 6 minutes after the time t.
- Describe your position on the circle 8 minutes after the time t.

The idea here is that your position on the circle repeats every 4 minutes. After 2 minutes, you are at a point diametrically opposed from the point you started. After

4 minutes, you are back at your starting point. In fact, you will be back at your starting point after 8 minutes, 12 minutes, 16 minutes, and so on. This is the idea of periodic behavior.

The Unit Circle and the Wrapping Function

In order to model periodic phenomena mathematically, we will need functions that are themselves periodic. In other words, we look for functions whose values repeat in regular and recognizable patterns. Familiar functions like polynomials and exponential functions don't exhibit periodic behavior, so we turn to the trigonometric functions. Before we can define these functions, however, we need a way to introduce periodicity. We do so in a manner similar to the thought experiment, but we also use mathematical objects and equations. The primary tool is something called *the wrapping function*. Instead of using any circle, we will use the so-called **unit circle**. This is the circle whose center is at the origin and whose radius is equal to 1, and the equation for the unit circle is $x^2 + y^2 = 1$.

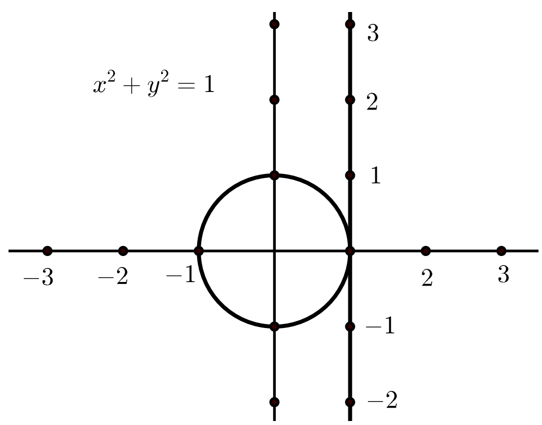


Figure 1.1: Setting up to wrap the number line around the unit circle

Figure 1.1 shows the unit circle with a number line drawn tangent to the circle at the point $(1, 0)$. We will "wrap" this number line around the unit circle. Unlike the number line, the length once around the unit circle is finite. (Remember that the formula for the circumference of a circle as $2\pi r$ where r is the radius, so the length once around the unit circle is 2π). However, we can still measure distances and locate the points on the number line on the unit circle by wrapping the number line around the circle. We wrap the positive part of this number line around the circumference of the circle in a counterclockwise fashion and wrap the negative part of the number line around the circumference of the unit circle in a clockwise direction.

Two snapshots of an animation of this process for the counterclockwise wrap are shown in Figure 1.2 and two such snapshots are shown in Figure 1.3 for the clockwise wrap.

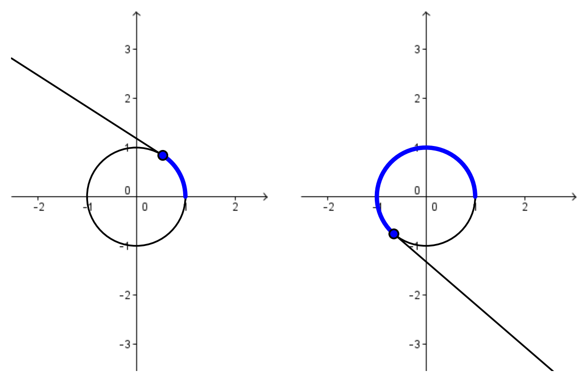


Figure 1.2: Wrapping the positive number line around the unit circle

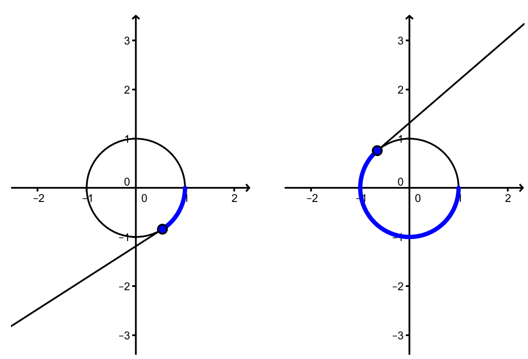


Figure 1.3: Wrapping the negative number line around the unit circle

Following is a link to an actual animation of this process, including both positive wraps and negative wraps.

`http://gvsu.edu/s/Kr`

Figure 1.2 and Figure 1.3 only show a portion of the number line being wrapped around the circle. Since the number line is infinitely long, it will wrap around the circle infinitely many times. A result of this is that infinitely many different numbers from the number line get wrapped to the same location on the unit circle.

- The number 0 and the numbers 2π, -2π, and 4π (as well as others) get wrapped to the point $(1, 0)$. We will usually say that these points get mapped to the point $(1, 0)$.
- The number $\dfrac{\pi}{2}$ is mapped to the point $(0, 1)$. This is because the circumference of the unit circle is 2π and so one-fourth of the circumference is $\dfrac{1}{4}(2\pi) = \dfrac{\pi}{2}$.
- If we now add 2π to $\dfrac{\pi}{2}$, we see that $\dfrac{5\pi}{2}$ also gets mapped to $(0, 1)$. If we subtract 2π from $\dfrac{\pi}{2}$, we see that $-\dfrac{3\pi}{2}$ also gets mapped to $(0, 1)$.

However, the fact that infinitely many different numbers from the number line get wrapped to the same location on the unit circle turns out to be very helpful as it will allow us to model and represent behavior that repeats or is periodic in nature.

Progress Check 1.1 (**The Unit Circle**.)

1. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(-1, 0)$ on the unit circle.
2. Describe all of the numbers on the number line that get wrapped to the point $(-1, 0)$ on the unit circle.
3. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(0, 1)$ on the unit circle.
4. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(0, -1)$ on the unit circle.

One thing we should see from our work in Progress Check 1.1 is that integer multiples of π are wrapped either to the point $(1, 0)$ or $(-1, 0)$ and that odd integer multiples of $\dfrac{\pi}{2}$ are wrapped to either to the point $(0, 1)$ or $(0, -1)$. Since the circumference of the unit circle is 2π, it is not surprising that fractional parts of π and the integer multiples of these fractional parts of π can be located on the unit circle. This will be studied in the next progress check.

Progress Check 1.2 (The Unit Circle and π).
The following diagram is a unit circle with 24 points equally spaced points plotted on the circle. Since the circumference of the circle is 2π units, the increment between two consecutive points on the circle is $\dfrac{2\pi}{24} = \dfrac{\pi}{12}$.

Label each point with the smallest nonnegative real number t to which it corresponds. For example, the point $(1, 0)$ on the x-axis corresponds to $t = 0$. Moving counterclockwise from this point, the second point corresponds to $\dfrac{2\pi}{12} = \dfrac{\pi}{6}$.

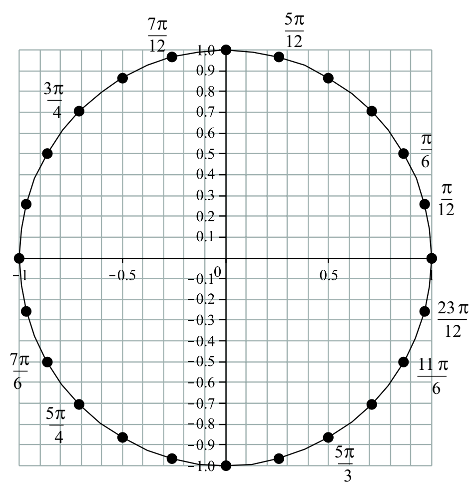


Figure 1.4: Points on the unit circle

Using Figure 1.4, approximate the x-coordinate and the y-coordinate of each of the following:

1. The point on the unit circle that corresponds to $t = \dfrac{\pi}{3}$.

2. The point on the unit circle that corresponds to $t = \dfrac{2\pi}{3}$.

3. The point on the unit circle that corresponds to $t = \dfrac{4\pi}{3}$.

4. The point on the unit circle that corresponds to $t = \dfrac{5\pi}{3}$.

5. The point on the unit circle that corresponds to $t = \dfrac{\pi}{4}$.

6. The point on the unit circle that corresponds to $t = \dfrac{7\pi}{4}$.

Arcs on the Unit Circle

When we wrap the number line around the unit circle, any closed interval on the number line gets mapped to a continuous piece of the unit circle. These pieces are called **arcs** of the circle. For example, the segment $\left[0, \dfrac{\pi}{2}\right]$ on the number line gets mapped to the arc connecting the points $(1, 0)$ and $(0, 1)$ on the unit circle as shown in Figure 1.5. In general, when a closed interval $[a, b]$ is mapped to an arc on the unit circle, the point corresponding to $t = a$ is called the **initial point of the arc**, and the point corresponding to $t = b$ is called the **terminal point of the arc**. So the arc corresponding to the closed interval $\left[0, \dfrac{\pi}{2}\right]$ has initial point $(1, 0)$ and terminal point $(0, 1)$.

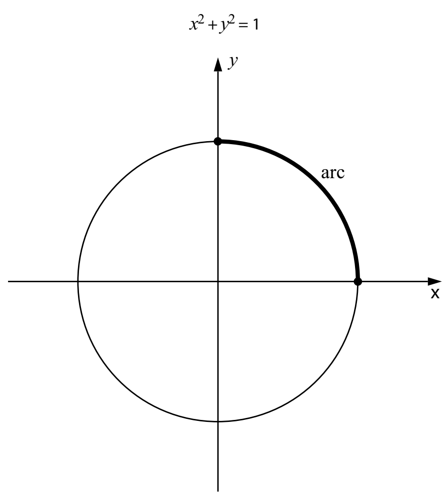


Figure 1.5: An arc on the unit circle

Progress Check 1.3 (Arcs on the Unit Circle).
Draw the following arcs on the unit circle.

1. The arc that is determined by the interval $\left[0, \dfrac{\pi}{4}\right]$ on the number line.

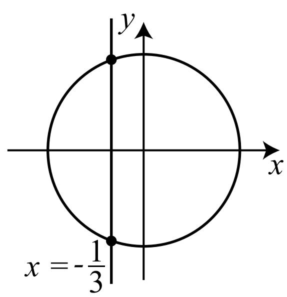
2. The arc that is determined by the interval $\left[0, \dfrac{2\pi}{3}\right]$ on the number line.

3. The arc that is determined by the interval $[0, -\pi]$ on the number line.

Coordinates of Points on the Unit Circle

When we have an equation (usually in terms of x and y) for a curve in the plane and we know one of the coordinates of a point on that curve, we can use the equation to determine the other coordinate for the point on the curve. The equation for the unit circle is $x^2 + y^2 = 1$. So if we know one of the two coordinates of a point on the unit circle, we can substitute that value into the equation and solve for the value(s) of the other variable.

For example, suppose we know that the x-coordinate of a point on the unit circle is $-\dfrac{1}{3}$. This is illustrated on the following diagram. This diagram shows the unit circle ($x^2 + y^2 = 1$) and the vertical line $x = -\dfrac{1}{3}$. This shows that there are two points on the unit circle whose x-coordinate is $-\dfrac{1}{3}$. We can find the y-coordinates by substituting the x-value into the equation and solving for y.



$$\begin{aligned} x^2 + y^2 &= 1 \\ \left(-\frac{1}{3}\right)^2 + y^2 &= 1 \\ \frac{1}{9} + y^2 &= 1 \\ y^2 &= \frac{8}{9} \end{aligned}$$

Since $y^2 = \dfrac{8}{9}$, we see that $y = \pm\sqrt{\dfrac{8}{9}}$ and so $y = \pm\dfrac{\sqrt{8}}{3}$. So the two points on the unit circle whose x-coordinate is $-\dfrac{1}{3}$ are

$$\left(-\frac{1}{3}, \frac{\sqrt{8}}{3}\right), \quad \text{which is in the second quadrant, and}$$

$$\left(-\frac{1}{3}, -\frac{\sqrt{8}}{3}\right), \quad \text{which is in the third quadrant.}$$

The first point is in the second quadrant and the second point is in the third quadrant. We can now use a calculator to verify that $\dfrac{\sqrt{8}}{3} \approx 0.9428$. This seems consistent with the diagram we used for this problem.

Progress Check 1.4 (Points on the Unit Circle.)

1. Find all points on the unit circle whose y-coordinate is $\dfrac{1}{2}$.
2. Find all points on the unit circle whose x-coordinate is $\dfrac{\sqrt{5}}{4}$.

Summary of Section 1.1

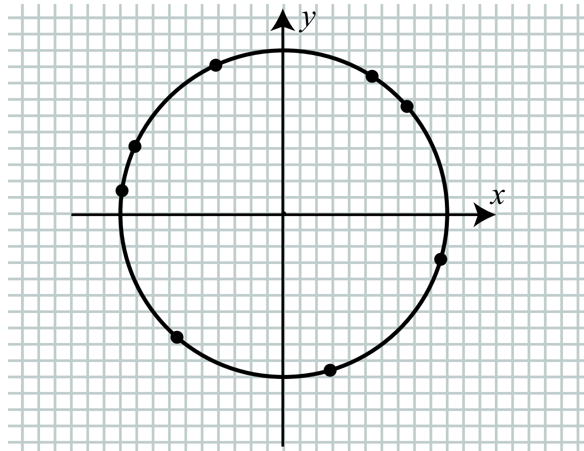
In this section, we studied the following important concepts and ideas:

- The **unit circle** is the circle of radius 1 that is centered at the origin. The equation of the unit circle is $x^2 + y^2 = 1$. It is important because we will use this as a tool to model periodic phenomena.
- We "wrap" the number line about the unit circle by drawing a number line that is tangent to the unit circle at the point $(1, 0)$. We wrap the positive part of the number line around the unit circle in the counterclockwise direction and wrap the negative part of the number line around the unit circle in the clockwise direction.
- When we wrap the number line around the unit circle, any closed interval of real numbers gets mapped to a continuous piece of the unit circle, which is called an arc of the circle. When the closed interval $[a, b]$ is mapped to an arc on the unit circle, the point correpsonding to $t = a$ is called **the initial point of the arc**, and the point corresponding to $t = b$ is called **the terminal point of the arc**.

Exercises for Section 1.1

1. The following diagram shows eight points plotted on the unit circle. These points correspond to the following values when the number line is wrapped around the unit circle.

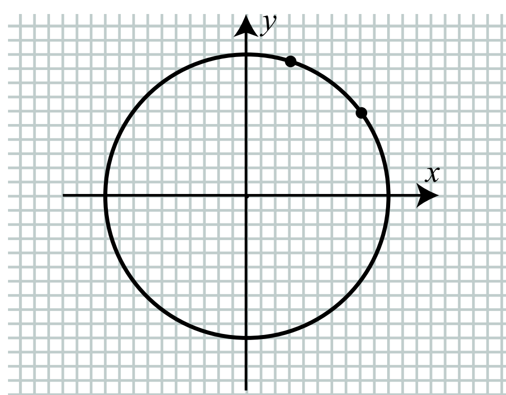
$$t = 1, t = 2, t = 3, t = 4, t = 5, t = 6, t = 7, \text{ and } t = 9.$$



(**a**) Label each point in the diagram with its value of t.

⋆ **(b)** Approximate the coordinates of the points corresponding to $t = 1$, $t = 5$, and $t = 9$.

2. The following diagram shows the points corresponding to $t = \dfrac{\pi}{5}$ and $t = \dfrac{2\pi}{5}$ when the number line is wrapped around the unit circle.



On this unit circle, draw the points corresponding to $t = \dfrac{4\pi}{5}$, $t = \dfrac{6\pi}{5}$, $t = \dfrac{8\pi}{5}$, and $t = \dfrac{10\pi}{5}$.

3. Draw the following arcs on the unit circle.

(**a**) The arc that is determined by the interval $\left[0, \frac{\pi}{6}\right]$ on the number line.

(**b**) The arc that is determined by the interval $\left[0, \frac{7\pi}{6}\right]$ on the number line.

(**c**) The arc that is determined by the interval $[0, -\frac{\pi}{3}]$ on the number line.

(**d**) The arc that is determined by the interval $[0, -\frac{4\pi}{5}]$ on the number line.

⋆ **4.** Determine the quadrant that contains the terminal point of each given arc with initial point $(1, 0)$ on the unit circle.

(**a**) $\frac{7\pi}{4}$

(**b**) $-\frac{7\pi}{4}$

(**c**) $\frac{3\pi}{5}$

(**d**) $\frac{-3\pi}{5}$

(**e**) $\frac{7\pi}{3}$

(**f**) $\frac{-7\pi}{3}$

(**g**) $\frac{5\pi}{8}$

(**h**) $\frac{-5\pi}{8}$

(**i**) 2.5

(**j**) -2.5

(**k**) 3

(**l**) $3 + 2\pi$

(**m**) $3 - \pi$

(**n**) $3 - 2\pi$

5. Find all the points on the unit circle:

⋆ **(a)** Whose x-coordinate is $\frac{1}{3}$.

⋆ **(b)** Whose y-coordinate is $-\frac{1}{2}$.

(c) Whose x-coordinate is $-\frac{3}{5}$.

(d) Whose y-coordinate is $-\frac{3}{4}$ and whose x-coordinate is negative.

1.2 The Cosine and Sine Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- If the real number t represents the (signed) length of an arc, how do we define $\cos(t)$ and $\sin(t)$?
- In what quadrants (of the terminal point of an arc t on the unit circle) is $\cos(t)$ positive (negative)? In what quadrants (of the terminal point of an arc t on the unit circle) is $\sin(t)$ positive (negative)?
- What is the Pythagorean Identity? How is this identity derived from the equation for the unit circle?

Beginning Activity

1. What is the unit circle? What is the equation of the unit circle?

2. Review Progress Check 1.4 on page 9.

3. Review the completed version of Figure 1.4 that is in the answers for Progress Check 1.2 on page 6.

4. (**a**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, \dfrac{\pi}{2}\right]$?

(**b**) What is the terminal point of the arc on the unit circle that corresponds to the interval $[0, \pi]$?

(**c**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, \dfrac{3\pi}{2}\right]$?

(**d**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, -\dfrac{\pi}{2}\right]$?

The Cosine and Sine Functions

We started our study of trigonometry by learning about the unit circle, how to wrap the number line around the unit circle, and how to construct arcs on the unit circle. We are now able to use these ideas to define the two major circular, or trigonmetric, functions. These circular functions will allow us to model periodic phenomena such as tides, the amount of sunlight during the days of the year, orbits of planets, and many others.

It may seem like the unit circle is a fairly simple object and of little interest, but mathematicians can almost always find something fascinating in even such simple objects. For example, we define the two major circular functions, the *cosine* and *sine*[1] in terms of the unit circle as follows. Figure 1.6 shows an arc of length t on the unit circle. This arc begins at the point $(1, 0)$ and ends at its terminal point $P(t)$. We then define the cosine and sine of the arc t as the x and y coordinates of the point P, so that $P(t) = (\cos(t), \sin(t))$ (we abbreviate the cosine as cos and the sine as sin). So the cosine and sine values are determined by the arc t and the cosine

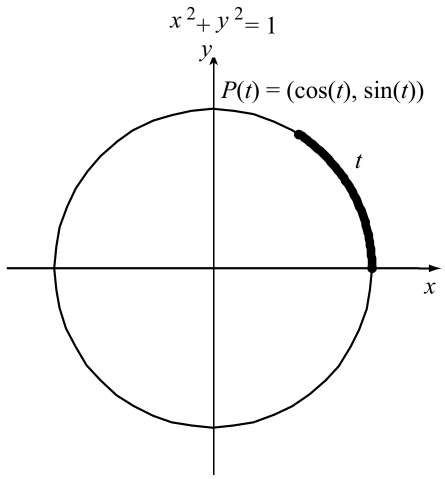


Figure 1.6: The Circular Functions

[1] According to the web site *Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html, the origin of the word sine is Sanskrit through Arabic and Latin. While the accounts of the actual origin differ, it appears that the Sanskrit work "jya" (chord) was taken into Arabic as "jiba", but was then translated into Latin as "jaib" (bay) which became "sinus" (bay or curve). This word was then anglicized to become our "sine". The word cosine began with Plato of Tivoli who use the expression "chorda residui". While the Latin word chorda was a better translation of the Sanskrit-Arabic word for sine than the word sinus, that word was already in use. Thus, "chorda residui" became "cosine".

and sine are *functions* of the arc t. Since the arc lies on the unit circle, we call the cosine and sine ***circular functions***. An important part of trigonometry is the study of the cosine and sine and the periodic phenomena that these functions can model. This is one reason why the circular functions are also called the ***trigonometric functions***.

Note: In mathematics, we always create formal definitions for objects we commonly use. Definitions are critically important because with agreed upon definitions, everyone will have a common understanding of what the terms mean. Without such a common understanding, there would be a great deal of confusion since different people who have different meanings for various terms. So careful and precise definitions are necessary in order to develop mathematical properties of these objects. In order to learn and understand trigonometry, a person needs to be able to explain how the circular functions are defined. So now is a good time to start working on understanding these definitions.

Definition. If the real number t is the directed length of an arc (either positive or negative) measured on the unit circle $x^2 + y^2 = 1$ (with counterclockwise as the positive direction) with initial point (1, 0) and terminal point (x, y), then the **cosine** of t, denoted $\cos(t)$, and **sine** of t, denoted $\sin(t)$, are defined to be

$$\cos(t) = x \qquad \text{and} \qquad \sin(t) = y.$$

Figure 1.6 illustrates these definitions for an arc whose terminal point is in the first quadrant.

At this time, it is not possible to determine the exact values of the cosine and sine functions for specific values of t. It can be done, however, if the terminal point of an arc of length t lies on the x-axis or the y-axis. For example, since the circumference of the unit circle is 2π, an arc of length $t = \pi$ will have it terminal point half-way around the circle from the point $(1, 0)$. That is, the terminal point is at $(-1, 0)$. Therefore,

$$\cos(\pi) = -1 \qquad \text{and} \qquad \sin(\pi) = 0.$$

Progress Check 1.5 (Cosine and Sine Values).
Determine the exact values of each of the following:

1. $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.
2. $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$.
3. $\cos(0)$ and $\sin(0)$.
4. $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$.
5. $\cos(2\pi)$ and $\sin(2\pi)$.
6. $\cos(-\pi)$ and $\sin(-\pi)$.

Important Note: Since the cosine and sine are functions of an arc whose length is the real number t, the input t determines the output of the cosine and sin. As a result, it is necessary to specify the input value when working with the cosine and sine. In other words, we ALWAYS write $\cos(t)$ where t is the real number input, and NEVER just cos. To reiterate, the cosine and sine are functions, so we MUST indicate the input to these functions.

Progress Check 1.6 (Approximating Cosine and Sine Values).
For this progress check, we will use the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle*. A web address for this applet is

http://gvsu.edu/s/JY

For this applet, we control the value of the input t with the slider for t. The values of t range from -20 to 20 in increments of 0.5. For a given value of t, an arc is drawn of length t and the coordinates of the terminal point of that arc are displayed. Use this applet to find approximate values for each of the following:

1. $\cos(1)$ and $\sin(1)$.
2. $\cos(2)$ and $\sin(2)$.
3. $\cos(-4)$ and $\sin(-4)$.
4. $\cos(5.5)$ and $\sin(5.5)$.
5. $\cos(15)$ and $\sin(15)$.
6. $\cos(-15)$ and $\sin(-15)$.

Some Properties of the Cosine and Sine Functions

The cosine and sine functions are called **circular functions** because their values are determined by the coordinates of points on the unit circle. For each real number t, there is a corresponding arc starting at the point $(1, 0)$ of (directed) length t that lies on the unit circle. The coordinates of the end point of this arc determines the values of $\cos(t)$ and $\sin(t)$.

In previous mathematics courses, we have learned that the domain of a function is the set of all inputs that give a defined output. We have also learned that the range of a function is the set of all possible outputs of the function.

Progress Check 1.7 (Domain and Range of the Circular Functions.)

1. What is the domain of the cosine function? Why?
2. What is the domain of the sine function? Why?
3. What is the largest x coordinate that a point on the unit circle can have? What is the smallest x coordinate that a point on the unit circle can have? What does this tell us about the range of the cosine function? Why?
4. What is the largest y coordinate that a point on the unit circle can have? What is the smallest y coordinate that a point on the unit circle can have? What does this tell us about the range of the sine function? Why?

Although we may not be able to calculate the exact values for many inputs for the cosine and sine functions, we can use our knowledge of the coordinate system and its quadrants to determine if certain values of cosine and sine are positive or negative. The idea is that the signs of the coordinates of a point $P(x, y)$ that is plotted in the coordinate plan are determined by the quadrant in which the point lies. (Unless it lies on one of the axes.) Figure 1.7 summarizes these results for the signs of the cosine and sine function values. The left column in the table is for the location of the terminal point of an arc determined by the real number t.

Quadrant	$\cos(t)$	$\sin(t)$
QI	positive	positive
QII	negative	positive
QIII	negative	negative
QIV	positive	negative

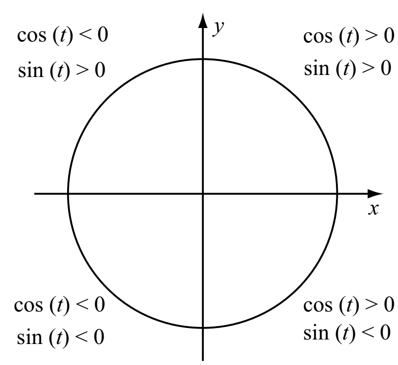


Figure 1.7: Signs of the cosine and sine functions

What we need to do now is to determine in which quadrant the terminal point of an arc determined by a real number t lies. We can do this by once again using the fact that the circumference of the unit circle is 2π, and when we move around the unit circle from the point $(1, 0)$ in the positive (counterclockwise) direction, we will intersect one of the coordinate axes every quarter revolution. For example, if $0 < t < \dfrac{\pi}{2}$, the terminal point of the arc determined by t is in the first quadrant and $\cos(t) > 0$ and $\sin(t) > 0$.

Progress Check 1.8 (Signs of $\cos(t)$ and $\sin(t)$.)

1. If $\dfrac{\pi}{2} < t < \pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
2. If $\pi < t < \dfrac{3\pi}{2}$, then what are the signs of $\cos(t)$ and $\sin(t)$?
3. If $\dfrac{3\pi}{2} < t < 2\pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
4. If $\dfrac{5\pi}{2} < t < 3\pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
5. For which values of t (between 0 and 2π) is $\cos(t)$ positive? Why?
6. For which values of t (between 0 and 2π) is $\sin(t)$ positive? Why?
7. For which values of t (between 0 and 2π) is $\cos(t)$ negative? Why?
8. For which values of t (between 0 and 2π) is $\sin(t)$ negative? Why?

Progress Check 1.9 (Signs of $\cos(t)$ and $\sin(t)$ (Part 2))
Use the results summarized in Figure 1.7 to help determine if the following quantities are positive, negative, or zero. (Do not use a calculator.)

1. $\cos\left(\dfrac{\pi}{5}\right)$
2. $\sin\left(\dfrac{\pi}{5}\right)$
3. $\cos\left(\dfrac{5\pi}{8}\right)$
4. $\sin\left(\dfrac{5\pi}{8}\right)$
5. $\cos\left(\dfrac{-9\pi}{16}\right)$
6. $\sin\left(\dfrac{-9\pi}{16}\right)$
7. $\cos\left(\dfrac{-25\pi}{12}\right)$
8. $\sin\left(\dfrac{-25\pi}{12}\right)$

The Pythagorean Identity

In mathematics, an **identity** is a statement that is true for all values of the variables for which it is defined. In previous courses, we have worked with algebraic identities such as

$$7x + 12x = 19x \qquad a + b = b + a$$
$$a^2 - b^2 = (a + b)(a - b) \qquad x(y + z) = xy + xz$$

where it is understood that all the variables represent real numbers. In trigonometry, we will develop many so-called trigonometric identities. The following progress check introduces one such identity between the cosine and sine functions.

Progress Check 1.10 (Introduction to the Pythagorean Identity)
We know that the equation for the unit circle is $x^2 + y^2 = 1$. We also know that if t is an real number, then the terminal point of the arc determined by t is the point $(\cos(t), \sin(t))$ and that this point lies on the unit circle. Use this information to develop an identity involving $\cos(t)$ and $\sin(t)$.

Using the definitions $x = \cos(t)$ and $y = \sin(t)$ along with the equation for the unit circle, we obtain the following identity, which is perhaps the most important trigonometric identity.

$$\text{For each real number } t, ((\cos(t))^2 + (\sin(t))^2 = 1.$$

This is called the **Pythagorean Identity**. We often use the shorthand notation $\cos^2(t)$ for $(\cos(t))^2$ and $\sin^2(t)$ for $(\sin(t))^2$ and write

$$\text{For each real number } t, \cos^2(t) + \sin^2(t) = 1.$$

Important Note about Notation. Always remember that by $\cos^2(t)$ we mean $(\cos(t))^2$. In addition, note that $\cos^2(t)$ is different from $\cos(t^2)$.

The Pythagorean Identity allows us to determine the value of $\cos(t)$ or $\sin(t)$ if we know the value of the other one and the quadrant in which the terminal point of arc t lies. This is illustrated in the next example.

Example 1.11 (Using the Pythagorean Identity)
Assume that $\cos(t) = \dfrac{2}{5}$ and the terminal point of arc (t) lies in the fourth quadrant. We will use this information to determine the value of $\sin(t)$. The primary tool we will use is the Pythagorean Identity, but please keep in mind that the terminal point

for the arc t is the point $(\cos(t), \sin(t))$. That is, $x = \cos(t)$ and $y = \sin(t)$. So this problem is very similar to using the equation $x^2 + y^2 = 1$ for the unit circle and substituting $x = \dfrac{2}{5}$.

Using the Pythagorean Identity, we then see that

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \left(\frac{2}{5}\right)^2 + \sin^2(t) &= 1 \\ \frac{4}{25} + \sin^2(t) &= 1 \\ \sin^2(t) &= 1 - \frac{4}{25} \\ \sin^2(t) &= \frac{21}{25} \end{aligned}$$

This means that $\sin(t) = \pm\sqrt{\dfrac{21}{25}}$, and since the terminal point of arc (t) is in the fourth quadrant, we know that $\sin(t) < 0$. Therefore, $\sin(t) = -\sqrt{\dfrac{21}{25}}$. Since $\sqrt{25} = 5$, we can write

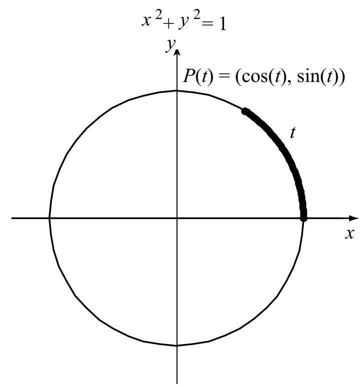
$$\sin(t) = -\frac{\sqrt{21}}{\sqrt{25}} = -\frac{\sqrt{21}}{5}.$$

Progress Check 1.12 (Using the Pythagorean Identity)

1. If $\cos(t) = \dfrac{1}{2}$ and the terminal point of the arc t is in the fourth quadrant, determine the value of $\sin(t)$.
2. If $\sin(t) = -\dfrac{2}{3}$ and $\pi < t < \dfrac{3\pi}{2}$, determine the value of $\cos(t)$.

Summary of Section 1.2

In this section, we studied the following important concepts and ideas:



- If the real number t is the directed length of an arc (either positive or negative) measured on the unit circle $x^2 + y^2 = 1$ (with counterclockwise as the positive direction) with initial point $(1, 0)$ and terminal point (x, y), then

  $$\cos(t) = x \qquad \text{and} \qquad \sin(t) = y.$$

- The signs of $\cos(t)$ and $\sin(t)$ are determind by the quadrant in which the terminal point of an arc t lies.

Quadrant	$\cos(t)$	$\sin(t)$
QI	positive	positive
QII	negative	positive
QIII	negative	negative
QIV	positive	negative

- One of the most important identities in trigonometry, called **the Pythagorean Identity**, is derived from the equation for the unit circle and states:

  For each real number t, $\cos^2(t) + \sin^2(t) = 1$.

Exercises for Section 1.2

* **1.** Fill in the blanks for each of the following:

(**a**) For a real number t, the value of $\cos(t)$ is defined to be the ______-coordinate of the __________ point of an arc t whose initial point is __________ on the __________ whose equation is $x^2 + y^2 = 1$.

(**b**) The domain of the cosine function is __________.

(**c**) The maximum value of $\cos(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \le t < 2\pi$. The minimum value of $\cos(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \le t < 2\pi$.

(**d**) The range of the cosine function is ______________.

2. (**a**) For a real number t, the value of $\sin(t)$ is defined to be the _______-coordinate of the ______________ point of an arc t whose initial point is ______________ on the ______________ whose equation is $x^2 + y^2 = 1$.

(**b**) The domain of the sine function is ______________.

(**c**) The maximum value of $\sin(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \le t < 2\pi$. The minimum value of $\sin(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \le t < 2\pi$.

(**d**) The range of the sine function is ______________.

3. (**a**) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
0	$(1, 0)$	1	0
$\dfrac{\pi}{2}$			
π			
$\dfrac{3\pi}{2}$			
2π			

(**b**) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
0	$(1, 0)$	1	0
$-\dfrac{\pi}{2}$			
$-\pi$			
$-\dfrac{3\pi}{2}$			
-2π			

(c) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
2π	$(1, 0)$	1	0
$\dfrac{5\pi}{2}$			
3π			
$\dfrac{7\pi}{2}$			
4π			

4. * **(a)** What are the possible values of $\cos(t)$ if it is known that $\sin(t) = \dfrac{3}{5}$?

(b) What are the possible values of $\cos(t)$ if it is known that $\sin(t) = \dfrac{3}{5}$ and the terminal point of t is in the second quadrant?

* **(c)** What is the value of $\sin(t)$ if it is known that $\cos(t) = -\dfrac{2}{3}$ and the terminal point of t is in the third quadrant?

* **5.** Suppose it is known that $0 < \cos(t) < \dfrac{1}{3}$.

(a) By squaring the expressions in the given inequalities, what conclusions can be made about $\cos^2(t)$?

(b) Use part (a) to write inequalities involving $-\cos^2(t)$ and then inequalities involving $1 - \cos^2(t)$.

(c) Using the Pythagorean identity, we see that $\sin^2(t) = 1 - \cos^2(t)$. Write the last inequality in part (b) in terms of $\sin^2(t)$.

(d) If we also know that $\sin(t) > 0$, what can we now conclude about the value of $\sin(t)$?

6. Use a process similar to the one in exercise (5) to complete each of the following:

(a) Suppose it is known that $-\dfrac{1}{4} < \sin(t) < 0$ and that $\cos(t) > 0$. What can be concluded about $\cos(t)$?

(b) Suppose it is known that $0 \leq \sin(t) \leq \dfrac{3}{7}$ and that $\cos(t) < 0$. What can be concluded about $\cos(t)$?

7. Using the four digit approximations for the cosine and sine values in Progress Check 1.6, calculate each of the following:

- $\cos^2(1) + \sin^2(1)$.
- $\cos^2(2) + \sin^2(2)$.
- $\cos^2(-4) + \sin^2(-4)$.
- $\cos^2(15) + \sin^2(15)$.

What should be the exact value of each of these computations? Why are the results not equal to this exact value?

1.3 Arcs, Angles, and Calculators

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we measure angles using degrees?
- What do we mean by the radian measure of an angle? How is the radian measure of an angle related to the length of an arc on the unit circle?
- Why is radian measure important?
- How do we convert from radians to degrees and from degrees to radians?
- How do we use a calculator to approximate values of the cosine and sine functions?

Introduction

The ancient civilization known as Babylonia was a cultural region based in southern Mesopatamia, which is present-day Iraq. Babylonia emerged as an independent state around 1894 BCE. The Babylonians developed a system of mathematics that was based on a sexigesimal (base 60) number system. This was the origin of the modern day usage of 60 minutes in an hour, 60 seconds in a minute, and 360 degrees in a circle.

Many historians now believe that for the ancient Babylonians, the year consisted of 360 days, which is not a bad approximation given the crudeness of the ancient astronomical tools. As a consequence, they divided the circle into 360 equal length arcs, which gave them a unit angle that was 1/360 of a circle or what we now know as a degree. Even though there are 365.24 days in a year, the Babylonian unit angle is still used as the basis for measuring angles in a circle. Figure 1.8 shows a circle divided up into 6 angles of 60 degrees each, which is also something that fit nicely with the Babylonian base-60 number system.

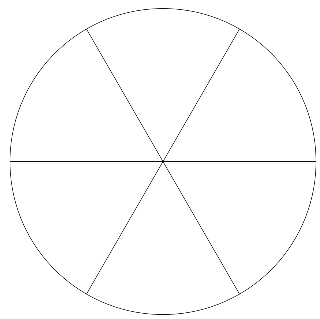


Figure 1.8: A circle with six 60-degree angles.

Angles

We often denote a line that is drawn through 2 points A and B by $\overleftrightarrow{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at the point A and continues indefinitely in the direction of the point B is called **ray *AB*** and is denoted by $\overrightarrow{AB}$. The point A is the **initial point** of the ray $\overrightarrow{AB}$. An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle**.

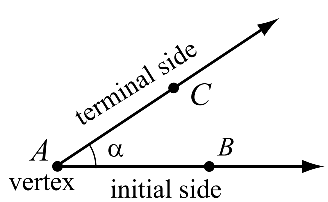


Figure 1.9: An angle including some notation.

Figure 1.9 shows the ray $\overrightarrow{AB}$ rotated about the point A to form an angle. The terminal side of the angle is the ray $\overrightarrow{AC}$. We often refer to this as angle BAC, which is abbreviated as $\angle BAC$. We can also refer to this angle as angle CAB or $\angle CAB$. If we want to use a single letter for this angle, we often use a Greek letter such as α (alpha). We then just say the angle α. Other Greek letters that are often used are β (beta), γ (*gamma*), θ (theta), ϕ (phi), and ρ (rho).

Arcs and Angles

To define the trigonometric functions in terms of angles, we will make a simple connection between angles and arcs by using the so-called standard position of an angle. When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**. The terminal side of the angle is then in one of the four quadrants or lies along one of the axes. When the terminal side is in one of the four quadrants, the terminal side determines the so-called quadrant designation of the angle. See Figure 1.10.

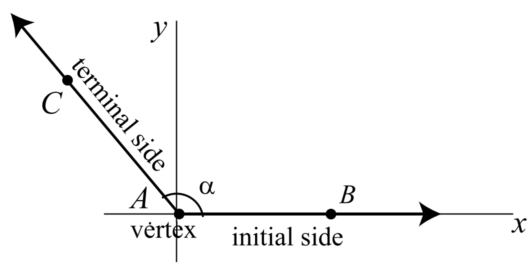


Figure 1.10: Standard position of an angle in the second quadrant.

Progress Check 1.13 (Angles in Standard Position)
Draw an angle in standard position in (1) the first quadrant; (2) the third quadrant; and (3) the fourth quadrant.

If an angle is in standard position, then the point where the terminal side of the angle intersects the unit circle marks the terminal point of an arc as shown in Figure 1.11. Similarly, the terminal point of an arc on the unit circle determines a ray through the origin and that point, which in turn defines an angle in standard position. In this case we say that the angle is *subtended* by the arc. So there is a natural correspondence between arcs on the unit circle and angles in standard position. Because of this correspondence, we can also define the trigonometric functions in terms of angles as well as arcs. Before we do this, however, we need to discuss two different ways to measure angles.

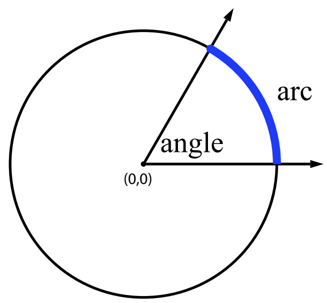


Figure 1.11: An arc and its corresponding angle.

Degrees Versus Radians

There are two ways we will measure angles – in degrees and radians. When we measure the length of an arc, the measurement has a dimension (the length, be it inches, centimeters, or something else). As mentioned in the introduction, the Babylonians divided the circle into 360 regions. So one complete wrap around a circle is 360 degrees, denoted 360°. The unit measure of 1° is an angle that is $1/360$ of the central angle of a circle. Figure 1.8 shows 6 angles of 60° each. The degree ° is a dimension, just like a length. So to compare an angle measured in degrees to an arc measured with some kind of length, we need to connect the dimensions. We can do that with the radian measure of an angle.

Radians will be useful in that a radian is a dimensionless measurement. We want to connect angle measurements to arc measurements, and to do so we will directly define an angle of 1 radian to be an angle subtended by an arc of length 1 (the length of the radius) on the unit circle as shown in Figure 1.12.

Definition. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).

This directly connects angles measured in radians to arcs in that we associate a real number with both the arc and the angle. So an angle of 2 radians cuts off an arc of length 2 on the unit circle, an angle of 3 radians cuts of an arc of length 3 on the unit circle, and so on. Figure 1.13 shows the terminal sides of angles with measures of 0 radians, 1 radian, 2 radians, 3 radians, 4 radians, 5 radians, and 6 radians. Notice that $2\pi \approx 6.2832$ and so $6 < 2\pi$ as shown in Figure 1.13.

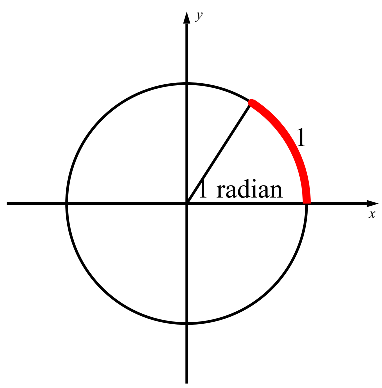


Figure 1.12: One radian.

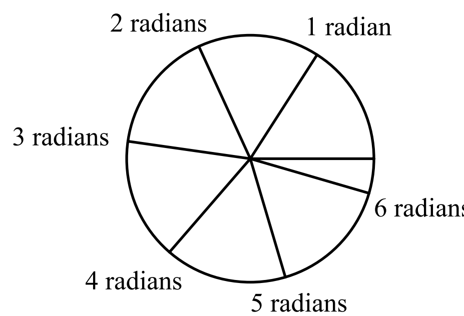


Figure 1.13: Angles with Radian Measure 1, 2, 3, 4, 5, and 6

We can also have angles whose radian measure is negative just like we have arcs with a negative length. The idea is simply to measure in the negative (clockwise) direction around the unit circle. So an angle whose measure is -1 radian is the angle in standard position on the unit circle that is subtended by an arc of length 1 in the negative (clockwise) direction.

So in general, an angle (in standard position) of t radians will correspond to an arc of length t on the unit circle. This allows us to discuss the sine and cosine of an angle measured in radians. That is, when we think of $\sin(t)$ and $\cos(t)$, we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

When we draw a picture of an angle in standard position, we often draw a small arc near the vertex from the initial side to the terminal side as shown in Figure 1.14, which shows an angle whose measure is $\dfrac{3}{4}\pi$ radians.

Progress Check 1.14 (Radian Measure of Angles)

1. Draw an angle in standard position with a radian measure of:

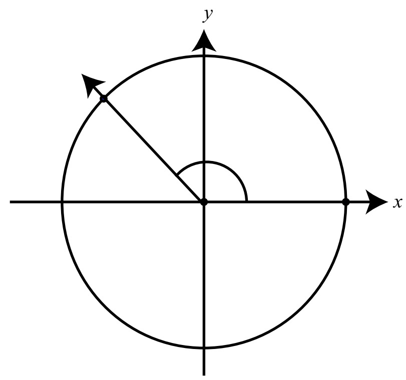


Figure 1.14: An angle with measure $\frac{3}{4}\pi$ in standard position

(**a**) $\dfrac{\pi}{2}$ radians.

(**b**) π radians.

(**c**) $\dfrac{3\pi}{2}$ radians.

(**d**) $-\dfrac{3\pi}{2}$ radians.

2. What is the degree measure of each of the angles in part (1)?

Conversion Between Radians and Degrees

Radian measure is the preferred measure of angles in mathematics for many reasons, the main one being that a radian has no dimensions. However, to effectively use radians, we will want to be able to convert angle measurements between radians and degrees.

Recall that one wrap of the unit circle corresponds to an arc of length 2π, and an arc of length 2π on the unit circle corresponds to an angle of 2π radians. An angle of $360°$ is also an angle that wraps once around the unit circle, so an angle of $360°$ is equivalent to an angle of 2π radians, or

- each degree is $\dfrac{\pi}{180}$ radians,
- each radian is $\dfrac{180}{\pi}$ degrees.

Notice that 1 radian is then $\dfrac{180}{\pi} \approx 57.3°$, so a radian is quite large compared to a degree. These relationships allow us to quickly convert between degrees and radians. For example:

- If an angle has a degree measure of 35 degrees, then its radian measure can be calculated as follows:

  $$35 \text{ degrees} \times \frac{\pi \text{ radians}}{180 \text{ degrees}} = \frac{35\pi}{180}\text{radians}.$$

  Rewriting this fraction, we see that an angle with a measure of 35 degrees has a radian measure of $\dfrac{7\pi}{36}$ radians.

- If an angle has a radian measure of $\dfrac{3\pi}{10}$ radians, then its degree measure can be calculated as follows:

  $$\frac{3\pi}{10}\text{radians} \times \frac{180 \text{ degrees}}{\pi \text{ radians}} = \frac{540}{10}\text{degrees}.$$

  So an angle with a radian measure of $\dfrac{3\pi}{10}$ has an angle measure of $54°$.

IMPORTANT NOTE: Since a degree is a dimension, we MUST include the degree mark ° whenever we write the degree measure of an angle. A radian has no dimension so there is no dimension mark to go along with it. Consequently, if we write 2 for the measure of an angle we understand that the angle is measured in radians. If we really mean an angle of 2 degrees, then we must write $2°$.

Progress Check 1.15 (Radian-Degree Conversions)
Complete the following table to convert from degrees to radians and vice versa.

Calculators and the Trignometric Functions

We have now seen that when we think of $\sin(t)$ or $\cos(t)$, we can think of t as a real number, the length of an arc, or the radian measure of an angle. In Section 1.5, we will see how to determine the exact values of the cosine and sine functions for a few special arcs (or angles). For example, we will see that $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$. However, the definition of cosine and sine as coordinates of points on the unit circle makes it difficult to find exact values for these functions except at very special arcs. While exact values are always best, technology plays an important role in allowing us to

Angle in radians	Angle in degrees
0	$0°$
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	$90°$
	$120°$
$\frac{3\pi}{4}$	$135°$
	$150°$
	$180°$

Angle in radians	Angle in degrees
$\frac{7\pi}{6}$	
$\frac{5\pi}{4}$	
$\frac{4\pi}{3}$	
$\frac{3\pi}{2}$	$270°$
	$300°$
	$315°$
	$330°$
2π	$360°$

Table 1.1: Conversions between radians and degrees.

approximate the values of the circular (or trigonometric)functions. Most hand-held calculators, calculators in phone or tablet apps, and online calculators have a cosine key and a sine key that you can use to approximate values of these functions, but we must keep in mind that the calculator only provides an *approximation* of the value, not the exact value (except for a small collection of arcs). In addition, most calculators will approximate the sine and cosine of angles.

To do this, the calculator has two modes for angles: Radian and Degree. Because of the correspondence between real numbers, length of arcs, and radian measures of angles, for now, we will always put our calculators in radian mode. In fact, we have seen that an angle measured in radians subtends an arc of that radian measure along the unit circle. **So the cosine or sine of an angle measure in radians is the same thing as the cosine or sine of a real number when that real number is interpreted as the length of an arc along the unit circle**. (When we study the trigonometry of triangles in Chapter 3, we will use the degree mode. For an introductory discussion of the trigonometric functions of an angle measure in degrees, see Exercise (4)).

Progress Check 1.16 (Using a Calculator)
In Progress Check 1.6, we used the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* at `http://gvsu.edu/s/JY` to approximate the values of the cosine and sine functions at certain values. For example, we found that

- $\cos(1) \approx 0.5403$,
  $\sin(1) \approx 0.8415$.
- $\cos(2) \approx -0.4161$
  $\sin(2) \approx 0.9093$.
- $\cos(-4) \approx -0.6536$
  $\sin(-4) \approx 0.7568$.
- $\cos(-15) \approx -0.7597$
  $\sin(-15) \approx -0.6503$.

Use a calculator to determine these values of the cosine and sine functions and compare the values to the ones above. Are they the same? How are they different?

Summary of Section 1.3

In this section, we studied the following important concepts and ideas:

- An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle**.
- When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**.
- There are two ways to measure angles. For degree meausre, one complete wrap around a circle is 360 degrees, denoted $360°$. The unit measure of $1°$ is an angle that is $1/360$ of the central angle of a circle. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).
- We convert the measure of an angle from degrees to radians by using the fact that each degree is $\dfrac{\pi}{180}$ radians. We convert the measure of an angle from radians to degrees by using the fact that each radian is $\dfrac{180}{\pi}$ degrees.

Exercises for Section 1.3

1. Convert each of the following degree measurements for angles into radian measures for the angles. In each case, first write the result as a fractional multiple of π and then use a calculator to obtain a 4 decimal place approximation of the radian measure.

$\star$ **(a)** $15°$
$\star$ **(b)** $58°$
(c) $112°$
(d) $210°$
$\star$ **(e)** $-40°$
(f) $-78°$

2. Convert each of the following radian measurements for angles into degree measures for the angles. When necessary, write each result as a 4 decimal place approximation.

$\star$ **(a)** $\dfrac{3}{8}\pi$ radians
$\star$ **(b)** $\dfrac{9}{7}\pi$ radians
(c) $-\dfrac{7}{15}\pi$ radians
$\star$ **(d)** 1 radian
(e) 2.4 radians
(f) 3 radians

3. Draw an angle in standard position of an angle whose radian measure is:

(a) $\dfrac{1}{4}\pi$
(b) $\dfrac{1}{3}\pi$
(c) $\dfrac{2}{3}\pi$
(d) $\dfrac{5}{4}\pi$
(e) $-\dfrac{1}{3}\pi$
(f) 3.4

4. In Progess Check 1.16, we used the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* to approximate values of the cosine and sine functions. We will now do something similar to approximate the cosine and sine values for angles measured in degrees.

We have seen that the terminal side of an angle in standard position intersects the unit circle in a point. We use the coordinates of this point to determine the cosine and sine of that angle. When the angle is measured in radians, the radian measure of the angle is the same as the arc on the unit circle subs-tended by the angle. This is not true when the angle is measure in degrees, but we can still use the intersection point to define the cosine and sine of the angle. So if an angle in standard position has degree measurement $a°$, then we define $\cos(a°)$ to be the x-coordinate of the point of intersection of the terminal side of that angle and the unit circle. We define $\sin(a°)$ to be the

y-coordinate of the point of intersection of the terminal side of that angle and the unit circle.

We will now use the Geogebra applet *Angles and the Unit Circle*. A web address for this applet is

http://gvsu.edu/s/VG

For this applet, we control the value of the input $a°$ with the slider for a. The values of a range from $-180°$ to $180°$ in increments of $5°$. For a given value of $a°$, an angle in standard position is drawn and the coordinates of the point of intersection of the terminal side of that angle and the unit circle are displayed. Use this applet to approximate values for each of the following:

* **(a)** $\cos(10°)$ and $\sin(10°)$.
(b) $\cos(60°)$ and $\sin(60°)$.
(c) $\cos(135°)$ and $\sin(135°)$.
* **(d)** $\cos(-10°)$ and $\sin(-10°)$.
(e) $\cos(-135°)$ and $\sin(-135°)$.
(f) $\cos(85°)$ and $\sin(85°)$.

5. Exericse (4) must be completed before doing this exercise. Put the calculator you are using in Degree mode. Then use the calculator to determine the values of the cosine and sine functions in Exercise (4). Are the values the same? How are they different?

1.4 Velocity and Angular Velocity

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is arc length?
- What is the difference between linear velocity and angular velocity?
- What are the formulas that relate linear velocity to angular velocity?

Beginning Activity

1. What is the formula for the circumference C of a circle whose radius is r?

2. Suppose person A walks along the circumference of a circle with a radius of 10 feet, and person B walks along the circumference of a circle of radius 20 feet. Also, suppose it takes both A and B 1 minute to walk one-quarter of the circumference of their respective circles (one-quarter of a complete revolution). Who walked the most distance?

3. Suppose both people continue walking at the same pace they did for the first minute. How many complete revolutions of the circle will each person walk in 8 minutes? In 10 minutes?

Arc Length on a Circle

In Section 1.3, we learned that the radian measure of an angle was equal to the length of the arc on the unit circle associated with that angle. So an arc of length 1 on the unit circle subtends an angle of 1 radian. There will be times when it will also be useful to know the length of arcs on other circles that subtend the same angle.

In Figure 1.15, the inner circle has a radius of 1, the outer circle has a radius of r, and the angle shown has a measure of θ radians. So the arc length on the unit

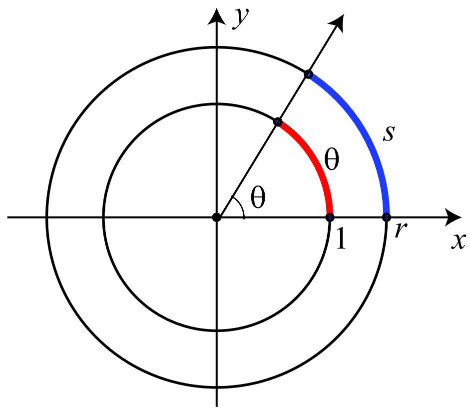


Figure 1.15: Arcs subtended by an angle of 1 radian.

circle subtended by the angle is θ, and we have used s to represent the arc length on the circle of radius r subtended by the angle.

Recall that the circumference of a circle of radius r is $2\pi r$ while the circumference of the circle of radius 1 is 2π. Therefore, the ratio of an arc length s on the circle of radius r that subtends an angle of θ radians to the corresponding arc on the unit circle is $\dfrac{2\pi r}{2\pi} = r$. So it follows that

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi}$$
$$s = r\theta$$

Definition. On a circle of radius r, the **arc length** s intercepted by a central angle with radian measure θ is

$$s = r\theta.$$

Note: It is important to remember that to calculate arc length[2], we must measure the central angle in radians.

[2] It is not clear why the letter s is usually used to represent arc length. One explanation is that the arc "subtends" an angle.

Progress Check 1.17 (Using the Formula for Arc Length)
Using the circles in the beginning activity for this section:

1. Use the formula for arc length to determine the arc length on a circle of radius 10 feet that subtends a central angle of $\dfrac{\pi}{2}$ radians. Is the result equal to one-quarter of the circumference of the circle?

2. Use the formula for arc length to determine the arc length on a circle of radius 20 feet that subtends a central angle of $\dfrac{\pi}{2}$ radians. Is the result equal to one-quarter of the circumference of the circle?

3. Determine the arc length on a circle of radius 3 feet that is subtended by an angle of $22°$.

Why Radians?

Degree measure is familiar and convenient, so why do we introduce the unit of radian? This is a good question, but one with a subtle answer. As we just saw, the length s of an arc on a circle of radius r subtended by angle of θ radians is given by $s = r\theta$, so $\theta = \dfrac{s}{r}$. As a result, a radian measure is a ratio of two lengths (the quotient of the length of an arc by a radius of a circle), which makes radian measure a dimensionless quantity. Thus, a measurement in radians can just be thought of as a real number. This is convenient for dealing with arc length (and angular velocity as we will soon see), and it will also be useful when we study periodic phenomena in Chapter 2. For this reason radian measure is universally used in mathematics, physics, and engineering as opposed to degrees, because when we use degree measure we always have to take the degree dimension into account in computations. This means that radian measure is actually more natural from a mathematical standpoint than degree measure.

Linear and Angular Velocity

The connection between an arc on a circle and the angle it subtends measured in radians allows us to define quantities related to motion on a circle. Objects traveling along circular paths exhibit two types of velocity: *linear* and *angular* velocity. Think of spinning on a merry-go-round. If you drop a pebble off the edge of a moving merry-go-round, the pebble will not drop straight down. Instead, it will continue to move forward with the velocity the merry-go-round had the moment the pebble was released. This is the *linear velocity* of the pebble. The linear velocity measures how the arc length changes over time.

Consider a point P moving at a constant velocity along the circumference of a circle of radius r. This is called **uniform circular motion**. Suppose that P moves a distance of s units in time t. The *linear velocity* v of the point P is the distance it travelled divided by the time elapsed. That is, $v = \dfrac{s}{t}$. The distance s is the arc length and we know that $s = r\theta$.

Definition. Consider a point P moving at a constant velocity along the circumference of a circle of radius r. The **linear velocity** v of the point P is given by

$$v = \frac{s}{t} = \frac{r\theta}{t},$$

where θ, measured in radians, is the central angle subtended by the arc of length s.

Another way to measure how fast an object is moving at a constant speed on a circular path is called *angular velocity*. Whereas the linear velocity measures how the arc length changes over time, the *angular velocity* is a measure of how fast the central angle is changing over time.

Definition. Consider a point P moving with constant velocity along the circumference of a circle of radius r on an arc that corresponds to a central angle of measure θ (in radians). The **angular velocity** ω of the point is the radian measure of the angle θ divided by the time t it takes to sweep out this angle. That is

$$\omega = \frac{\theta}{t}.$$

Note: The symbol ω is the lower case Greek letter "omega." Also, notice that the angular velocity does not depend on the radius r.

This is a somewhat specialized definition of angular velocity that is slightly different than a common term used to describe how fast a point is revolving around a circle. This term is **revolutions per minute** or **rpm**. Sometimes the unit **revolutions per second** is used. A better way to represent revolutions per minute is to use the "unit fraction" $\dfrac{\text{rev}}{\text{min}}$. Since 1 revolution is 2π radians, we see that if an object is moving at x revolutions per minute, then

$$\omega = x\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\,\text{rad}}{\text{rev}} = x(2\pi)\frac{\text{rad}}{\text{min}}.$$

Progress Check 1.18 (Determining Linear Velocity)
Suppose a circular disk is rotating at a rate of 40 revolutions per minute. We wish to determine the linear velocity v (in feet per second) of a point that is 3 feet from the center of the disk.

1. Determine the angular velocity ω of the point in radians per minute. **Hint**: Use the formula
$$\omega = x\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\,\text{rad}}{\text{rev}}.$$
2. We now know $\omega = \dfrac{\theta}{t}$. So use the formula $v = \dfrac{r\theta}{t}$ to determine v in feet per minute.
3. Finally, convert the linear velocity v in feet per minute to feet per second.

Notice that in Progress Check 1.18, once we determined the angular velocity, we were able to determine the linear velocity. What we did in this specific case we can do in general. There is a simple formula that directly relates linear velocity to angular velocity. Our formula for linear velocity is $v = \dfrac{s}{t} = \dfrac{r\theta}{t}$. Notice that we can write this is $v = r\dfrac{\theta}{t}$. That is, $v = r\omega$.

> Consider a point P moving with constant (linear) velocity v along the circumference of a circle of radius r. If the angular velocity is ω, then
>
> $$v = r\omega.$$

So in Progress Check 1.18, once we determined that $\omega = 80\pi\dfrac{\text{rad}}{\text{min}}$, we could determine v as follows:

$$v = r\omega = (3\text{ ft})\left(80\pi\frac{\text{rad}}{\text{min}}\right) = 240\pi\frac{\text{ft}}{\text{min}}.$$

Notice that since radians are "unit-less", we can drop them when dealing with equations such as the preceding one.

Example 1.19 (Linear and Angular Velocity)
The LP (long play) or $33\frac{1}{3}$ rpm vinyl record is an analog sound storage medium and has been used for a long time to listen to music. An LP is usually 12 inches or 10 inches in diameter. In order to work with our formulas for linear and angular velocity, we need to know the angular velocity in radians per time unit. To do this, we will convert $33\frac{1}{3}$ revolutions per minute to radians per minute. We will use the fact that $33\frac{1}{3} = \frac{100}{3}$.

$$\begin{aligned} \omega &= \frac{100}{3}\frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \\ &= \frac{200\pi}{3}\frac{\text{rad}}{\text{min}} \end{aligned}$$

We can now use the formula $v = r\omega$ to determine the linear velocity of a point on the edge of a 12 inch LP. The radius is 6 inches and so

$$\begin{aligned} v &= r\omega \\ &= (6 \text{ inches})\left(\frac{200\pi}{3}\frac{\text{rad}}{\text{min}}\right) \\ &= 400\pi\frac{\text{inches}}{\text{min}} \end{aligned}$$

It might be more convenient to express this as a decimal value in inches per second. So we get

$$\begin{aligned} v &= 400\pi\frac{\text{inches}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \\ &\approx 20.944\frac{\text{inches}}{\text{sec}} \end{aligned}$$

The linear velocity is approximately 20.944 inches per second.

Progress Check 1.20 (Linear and Angular Velocity)
For these problems, we will assume that the Earth is a sphere with a radius of 3959 miles. As the Earth rotates on its axis, a person standing on the Earth will travel in a circle that is perpendicular to the axis.

1. The Earth rotates on its axis once every 24 hours. Determine the angular velocity of the Earth in radians per hour. (Leave your answer in terms of the number π.)

2. As the Earth rotates, a person standing on the equator will travel in a circle whose radius is 3959 miles. Determine the linear velocity of this person in miles per hour.

3. As the Earth rotates, a person standing at a point whose latitude is 60° north will travel in a circle of radius 2800 miles. Determine the linear velocity of this person in miles per hour and feet per second.

Summary of Section 1.4

In this section, we studied the following important concepts and ideas:

- On a circle of radius r, the **arc length** s intercepted by a central angle with radian measure θ is
$$s = r\theta.$$
- Uniform circular motion is when a point moves at a constant velocity along the circumference of a circle. The **linear velocity** is the arc length travelled by the point divided by the time elapsed. Whereas the linear velocity measures how the arc length changes over time, the *angular velocity* is a measure of how fast the central angle is changing over time. The **angular velocity** of the point is the radian measure of the angle divided by the time it takes to sweep out this angle.
- For a point P moving with constant (linear) velocity v along the circumference of a circle of radius r, we have
$$v = r\omega,$$
where ω is the angular velocity of the point.

Exercises for Section 1.4

1. Determine the arc length (to the nearest hundredth of a unit when necessary) for each of the following.

⋆ **(a)** An arc on a circle of radius 6 feet that is intercepted by a central angle of $\dfrac{2\pi}{3}$ radians. Compare this to one-third of the circumference of the circle.

⋆ **(b)** An arc on a circle of radius 100 miles that is intercepted by a central angle of 2 radians.

⋆ **(c)** An arc on a circle of radius 20 meters that is intercepted by a central angle of $\dfrac{13\pi}{10}$ radians.

(d) An arc on a circle of radius 10 feet that is intercepted by a central angle of 152 degrees.

2. In each of the following, when it is possible, determine the exact measure of the central angle in radians. Otherwise, round to the nearest hundredth of a radian.

⋆ **(a)** The central angle that intercepts an arc of length 3π feet on a circle of radius 5 feet.

⋆ **(b)** The central angle that intercepts an arc of length 18 feet on a circle of radius 5 feet.

(c) The central angle that intercepts an arc of length 20 meters on a circle of radius 12 meters.

3. In each of the following, when it is possible, determine the exact measure of central the angle in degrees. Otherwise, round to the nearest hundredth of a degree.

⋆ **(a)** The central angle that intercepts an arc of length 3π feet on a circle of radius 5 feet.

⋆ **(b)** The central angle that intercepts an arc of length 18 feet on a circle of radius 5 feet.

(c) The central angle that intercepts an arc of length 20 meters on a circle of radius 12 meters.

(d) The central angle that intercepts an arc of length 5 inches on a circle of radius 5 inches.

(e) The central angle that intercepts an arc of length 12 inches on a circle of radius 5 inches.

4. Determine the distance (in miles) that the planet Mars travels in one week in its path around the sun. For this problem, assume that Mars completes one complete revolution around the sun in 687 days and that the path of Mars around the sun is a circle with a radius of 227.5 million miles.

5. Determine the distance (in miles) that the Earth travels in one day in its path around the sun. For this problem, assume that Earth completes one complete revolution around the sun in 365.25 days and that the path of Earth around the sun is a circle with a radius of 92.96 million miles.

6. A compact disc (CD) has a diameter of 12 centimeters (cm). Suppose that the CD is in a CD-player and is rotating at 225 revolutions per minute. What is the angular velocity of the CD (in radians per second) and what is the linear velocity of a point on the edge of the CD?

7. A person is riding on a Ferris wheel that takes 28 seconds to make a complete revolution. Her seat is 25 feet from the axle of the wheel.

(**a**) What is her angular velocity in revolutions per minute? Radians per minute? Degrees per minute?

(**b**) What is her linear velocity?

(**c**) Which of the quantities angular velocity and linear velocity change if the person's seat was 20 feet from the axle instead of 25 feet? Compute the new value for any value that changes. Explain why each value changes or does not change.

8. A small pulley with a radius of 3 inches is connected by a belt to a larger pulley with a radius of 7.5 inches (See Figure 1.16). The smaller pulley is connected to a motor that causes it to rotate counterclockwise at a rate of 120 rpm (revolutions per minute). Because the two pulleys are connected by the belt, the larger pulley also rotates in the counterclockwise direction.

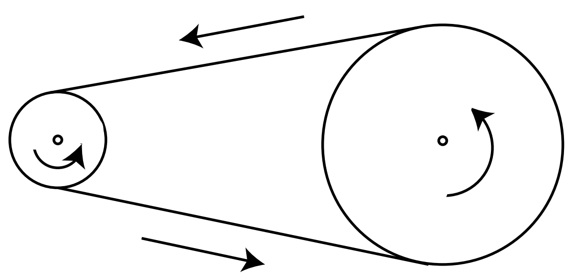


Figure 1.16: Two Pulleys Connected by a Belt

(**a**) Determine the angular velocity of the smaller pulley in radians per minute.

⋆ **(b)** Determine the linear velocity of the rim of the smaller pulley in inches per minute.

(c) What is the linear velocity of the rim of the larger pulley? Explain.

(d) Find the angular velocity of the larger pulley in radians per minute.

(e) How many revolutions per minute is the larger pulley turning?

9. A small pulley with a radius of 10 centimeters inches is connected by a belt to a larger pulley with a radius of 24 centimeters inches (See Figure 1.16). The larger pulley is connected to a motor that causes it to rotate counterclockwise at a rate of 75 rpm (revolutions per minute). Because the two pulleys are connected by the belt, the smaller pulley also rotates in the counterclockwise direction.

(a) Determine the angular velocity of the larger pulley in radians per minute.

⋆ **(b)** Determine the linear velocity of the rim of the large pulley in inches per minute.

(c) What is the linear velocity of the rim of the smaller pulley? Explain.

(d) Find the angular velocity of the smaller pulley in radians per second.

(e) How many revolutions per minute is the smaller pulley turning?

10. The radius of a car wheel is 15 inches. If the car is traveling 60 miles per hour, what is the angular velocity of the wheel in radians per minute? How fast is the wheel spinning in revolutions per minute?

11. The mean distance from Earth to the moon is 238,857 miles. Assuming the orbit of the moon about Earth is a circle with a radius of 238,857 miles and that the moon makes one revolution about Earth every 27.3 days, determine the linear velocity of the moon in miles per hour. Research the distance of the moon to Earth and explain why the computations that were just made are approximations.

1.5 Common Arcs and Reference Arcs

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we determine the values for cosine and sine for arcs whose endpoints are on the x-axis or the y-axis?
- What are the exact values of cosine and sine for $t = \dfrac{\pi}{6}$, $t = \dfrac{\pi}{4}$, and $t = \dfrac{\pi}{3}$?
- What is the reference arc for a given arc? How do we determine the reference arc for a given arc?
- How do we use reference arcs to calculate the values of the cosine and sine at other arcs that have $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ as reference arcs?

Beginning Activity

Figure 1.17 shows a unit circle with the terminal points for some arcs between 0 and 2π. In addition, there are four line segments drawn on the diagram that form a rectangle. The line segments go from: (1) the terminal point for $t = \dfrac{\pi}{6}$ to the terminal point for $t = \dfrac{5\pi}{6}$; (2) the terminal point for $t = \dfrac{5\pi}{6}$ to the terminal point for $t = \dfrac{7\pi}{6}$; (3) the terminal point for $t = \dfrac{7\pi}{6}$ to the terminal point for $t = \dfrac{11\pi}{6}$; and (4) the terminal point for $t = \dfrac{11\pi}{6}$ to the terminal point for $t = \dfrac{\pi}{6}$.

1. What are the approximate values of $\cos\left(\dfrac{\pi}{6}\right)$ and $\sin\left(\dfrac{\pi}{6}\right)$?

2. What are the approximate values of $\cos\left(\dfrac{5\pi}{6}\right)$ and $\sin\left(\dfrac{5\pi}{6}\right)$?

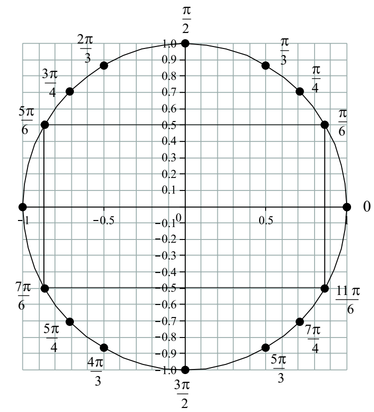


Figure 1.17: Some Arcs on the unit circle

3. What are the approximate values of $\cos\left(\dfrac{7\pi}{6}\right)$ and $\sin\left(\dfrac{7\pi}{6}\right)$?

4. What are the approximate values of $\cos\left(\dfrac{11\pi}{6}\right)$ and $\sin\left(\dfrac{11\pi}{6}\right)$?

5. Draw a similar rectangle on Figure 1.17 connecting the terminal points for $t = \dfrac{\pi}{4}$, $t = \dfrac{3\pi}{4}$, $t = \dfrac{5\pi}{4}$, and $t = \dfrac{7\pi}{4}$. How do the cosine and sine values for these arcs appear to be related?

Our task in this section is to determine the exact cosine and sine values for all of the arcs whose terminal points are shown in Figure 1.17. We first notice that we already know the cosine and sine values for the arcs whose terminal points are on one of the coordinate axes. These values are shown in the following table.

t	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
$\cos(t)$	1	0	-1	0	1
$\sin(t)$	0	1	0	-1	0

Table 1.2: Cosine and Sine Values

The purpose of the beginning activity was to show that we determine the values of cosine and sine for the other arcs by finding only the cosine and sine values for the arcs whose terminal points are in the first quadrant. So this is our first task. To do this, we will rely on some facts about certain right triangles. The three triangles we will use are shown in Figure 1.18.

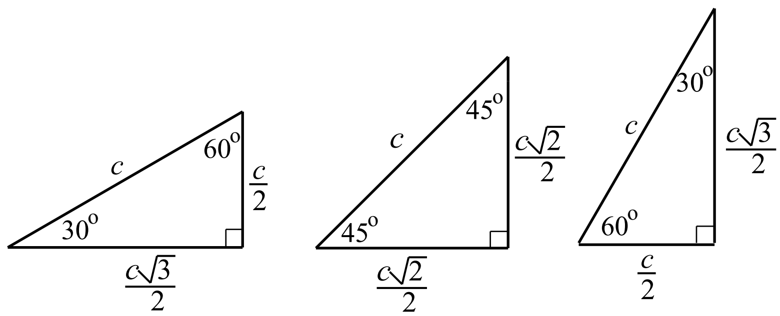


Figure 1.18: Special Right Triangles.

In each figure, the hypotenuse of the right triangle has a length of c units. The lengths of the other sides are determined using the Pythagorean Theorem. An explanation of how these lengths were determined can be found on page 420 in Appendix C. The usual convention is to use degree measure for angles when we work with triangles, but we can easily convert these degree measures to radian measures.

- A 30° angle has a radian measure of $\dfrac{\pi}{6}$ radians.
- A 45° angle has a radian measure of $\dfrac{\pi}{4}$ radians.
- A 60° angle has a radian measure of $\dfrac{\pi}{3}$ radians.

The Values of Cosine and Sine at $t = \dfrac{\pi}{6}$

Figure 1.19 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{6}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{6}\right), \sin\left(\dfrac{\pi}{6}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{6}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{6}\right).$$

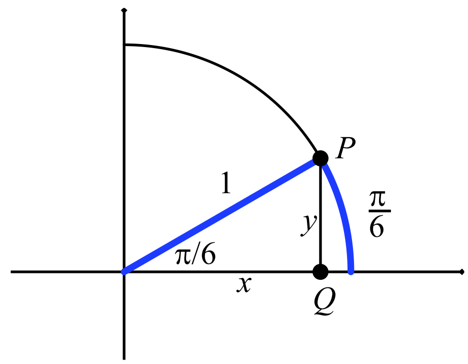


Figure 1.19: The arc $\dfrac{\pi}{6}$ and its associated angle.

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the angle associated with the arc is $\dfrac{\pi}{6}$ radians or $30°$. From what we know about this type of right triangle, the other acute angle in the right triangle is $60°$ or $\dfrac{\pi}{3}$ radians. We can then use the results shown in the triangle on the left in Figure 1.18 to conclude that $x = \dfrac{\sqrt{3}}{2}$ and $y = \dfrac{1}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\boxed{\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.}$$

Progress Check 1.21 (Comparison to the Beginning Activity)
In the beginning activity for this section, we used the unit circle to approximate the values of the cosine and sine functions at $t = \dfrac{\pi}{6}$, $t = \dfrac{5\pi}{6}$, $t = \dfrac{7\pi}{6}$, and $t = \dfrac{11\pi}{6}$. We also saw that these values are all related and that once we have values for the cosine and sine functions at $t = \dfrac{\pi}{6}$, we can use our knowledge of the four quadrants to determine these function values at $t = \dfrac{5\pi}{6}$, $t = \dfrac{7\pi}{6}$, and

$t = \dfrac{11\pi}{6}$. Now that we know that

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2},$$

determine the exact values of each of the following:

1. $\cos\left(\dfrac{5\pi}{6}\right)$ and $\sin\left(\dfrac{5\pi}{6}\right)$.

2. $\cos\left(\dfrac{7\pi}{6}\right)$ and $\sin\left(\dfrac{7\pi}{6}\right)$.

3. $\cos\left(\dfrac{11\pi}{6}\right)$ and $\sin\left(\dfrac{11\pi}{6}\right)$.

The Values of Cosine and Sine at $t = \dfrac{\pi}{4}$

Figure 1.20 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{4}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{4}\right), \sin\left(\dfrac{\pi}{4}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{4}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{4}\right).$$

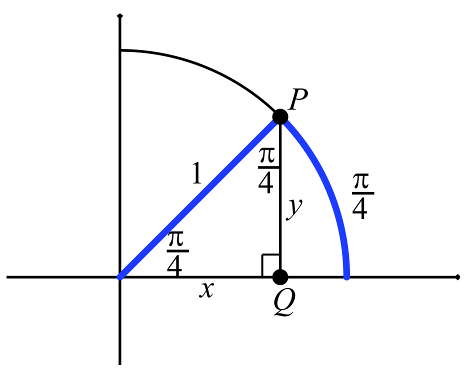


Figure 1.20: The arc $\dfrac{\pi}{4}$ and its associated angle.

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the acute angles are $\dfrac{\pi}{4}$ radians or 45°. We can then use the results shown in the triangle in the middle of Figure 1.18 to conclude that $x = \dfrac{\sqrt{2}}{2}$ and $y = \dfrac{\sqrt{2}}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\boxed{\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.}$$

Progress Check 1.22 (Comparison to the Beginning Activity)
Now that we know that

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

use a method similar to the one used in Progress Check 1.21 to determine the exact values of each of the following:

1. $\cos\left(\dfrac{3\pi}{4}\right)$ and $\sin\left(\dfrac{3\pi}{4}\right)$.
2. $\cos\left(\dfrac{5\pi}{4}\right)$ and $\sin\left(\dfrac{5\pi}{4}\right)$.
3. $\cos\left(\dfrac{7\pi}{4}\right)$ and $\sin\left(\dfrac{7\pi}{4}\right)$.

The Values of Cosine and Sine at $t = \dfrac{\pi}{3}$

Figure 1.21 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{3}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{3}\right), \sin\left(\dfrac{\pi}{3}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{3}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{3}\right).$$

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the angle associated with the arc is $\dfrac{\pi}{3}$ radians or 60°. From what we know about this type of right triangle, the other acute angle in the right triangle is 30° or $\dfrac{\pi}{6}$ radians. We can then use the results shown in the triangle on the right in

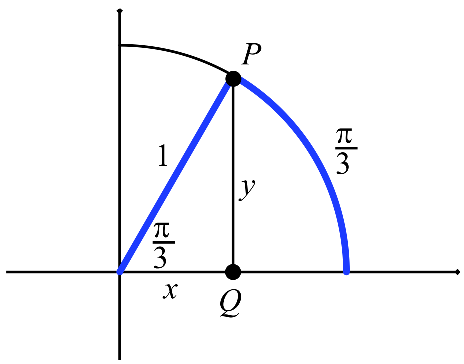


Figure 1.21: The arc $\dfrac{\pi}{3}$ and its associated angle.

Figure 1.18 to conclude that $x = \dfrac{1}{2}$ and $y = \dfrac{\sqrt{3}}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\boxed{\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.}$$

Reference Arcs (Reference Angles)

In the beginning activity for this section and in Progress Checks 1.21 and 1.22, we saw that we could relate the coordinates of the terminal point of an arc of length greater than $\dfrac{\pi}{2}$ on the unit circle to the coordinates of the terminal point of an arc of length between 0 and $\dfrac{\pi}{2}$ on the unit circle. This was intended to show that we can do this for any angle of length greater than $\dfrac{\pi}{2}$, and this means that if we know the values of the cosine and sine for any arc (or angle) between 0 and $\dfrac{\pi}{2}$, then we can find the values of the cosine and sine for any arc at all. The arc between 0 and $\dfrac{\pi}{2}$ to which we relate a given arc of length greater than $\dfrac{\pi}{2}$ is called a *reference arc*.

> **Definition**. The **reference arc** $\hat{t}$ for an arc t is the smallest non-negative arc (always considered non-negative) between the terminal point of the arc t and the closer of the two x-intercepts of the unit circle. Note that the two x-intercepts of the unit circle are $(-1, 0)$ and $(1, 0)$.

The concept of reference arc is illustrated in Figure 1.22. Each of the thicker arcs has length $\hat{t}$ and it can be seen that the coordinates of the points in the second, third, and fourth quadrants are all related to the coordinates of the point in the first quadrant. The signs of the coordinates are all determined by the quadrant in which the point lies.

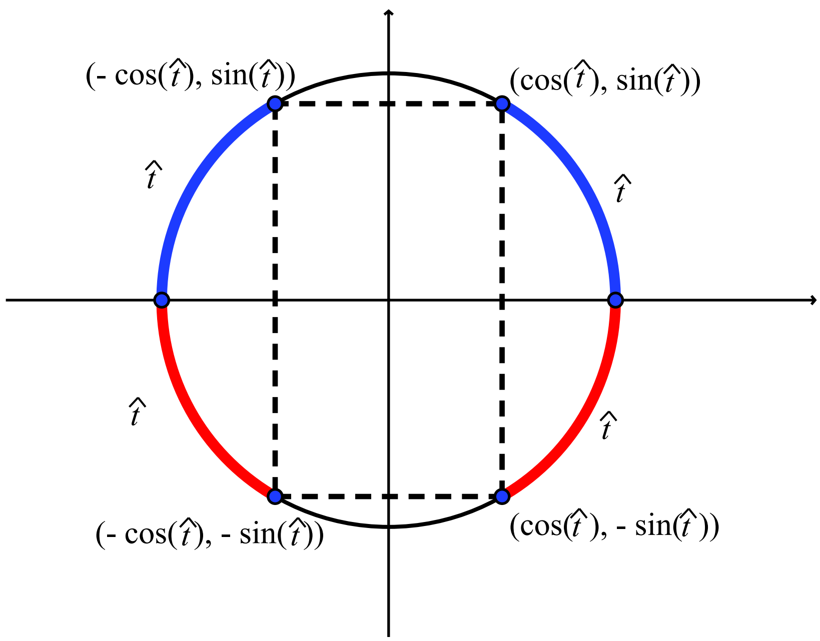


Figure 1.22: Reference arcs.

How we calculate a reference arc for a given arc of length t depends upon the quadrant in which the terminal point of t lies. The diagrams in Figure 1.23 illustrate how to calculate the reference arc for an arc of length t with $0 \leq t \leq 2\pi$.

In Figure 1.23, we see that for an arc of length t with $0 \leq t \leq 2\pi$:

- If $\dfrac{\pi}{2} < t < \pi$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(-1, 0)$. So the reference arc is $\pi - t$. In this case, Figure 1.23 shows that

$$\cos(\pi - t) = -\cos(t) \qquad \text{and} \qquad \sin(\pi - t) = \sin(t).$$

- If $\pi < t < \dfrac{3\pi}{2}$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(-1, 0)$. So the reference arc is $t - \pi$. In this case, Figure 1.23 shows that

$$\cos(t - \pi) = -\cos(t) \qquad \text{and} \qquad \sin(t - \pi) = -\sin(t).$$

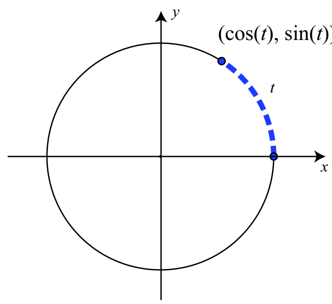
- If $\dfrac{3\pi}{2} < t < 2\pi$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(1, 0)$. So the reference arc is $2\pi - t$. In this case, Figure 1.23 shows that

$$\cos(2\pi - t) = \cos(t) \qquad \text{and} \qquad \sin(2\pi - t) = -\sin(t).$$

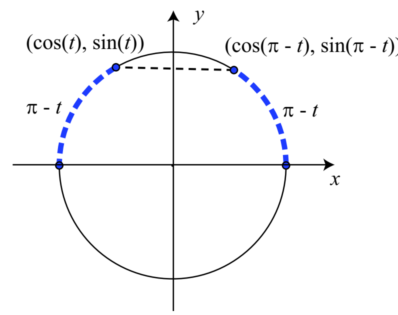
Progress Check 1.23 (Reference Arcs – Part 1)
For each of the following arcs, draw a picture of the arc on the unit circle. Then determine the reference arc for that arc and draw the reference arc in the first quadrant.

1. $t = \dfrac{5\pi}{4}$ **2.** $t = \dfrac{4\pi}{5}$ **3.** $t = \dfrac{5\pi}{3}$

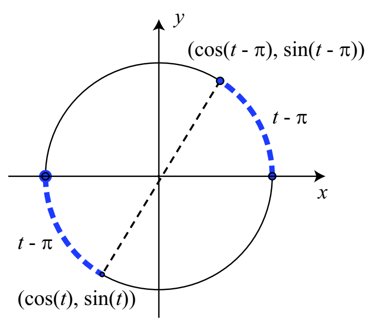
Progress Check 1.24 (Reference Arcs – Part 2)
Although we did not use the term then, in Progress Checks 1.21 and 1.22, we used the facts that $t = \dfrac{\pi}{6}$ and $t = \dfrac{\pi}{4}$ were the reference arcs for other arcs to determine the exact values of the cosine and sine functions for those other arcs. Now use the values of $\cos\left(\dfrac{\pi}{3}\right)$ and $\sin\left(\dfrac{\pi}{3}\right)$ to determine the exact values of the cosine and sine functions for each of the following arcs:



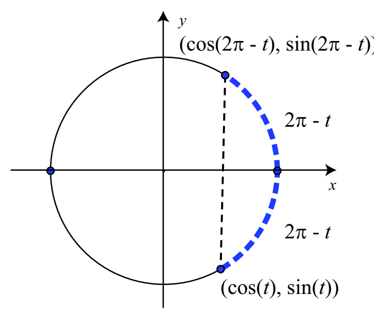
If the arc t is in Quadrant I, then t is its own reference arc.



If the arc t is in Quadrant II, then $\pi - t$ is its reference arc.



If the arc t is in Quadrant III, then $t - \pi$ is its reference arc.



If the arc t is in Quadrant IV, then $2\pi - t$ is its reference arc.

Figure 1.23: Reference arcs

1. $t = \dfrac{2\pi}{3}$ **2.** $t = \dfrac{4\pi}{3}$ **3.** $t = \dfrac{5\pi}{3}$

Reference Arcs for Negative Arcs

Up to now, we have only discussed reference arcs for positive arcs, but the same principles apply when we use negative arcs. Whether the arc t is positive or negative, the reference arc for t is the smallest non-negative arc formed by the terminal point of t and the nearest x-intercept of the unit circle. For example, for the arc $t = -\dfrac{\pi}{4}$ is in the fourth quadrant, and the closer of the two x-intercepts of the unit circle is $(1, 0)$. So the reference arc is $\hat{t} = \dfrac{\pi}{4}$ as shown in Figure 1.24.

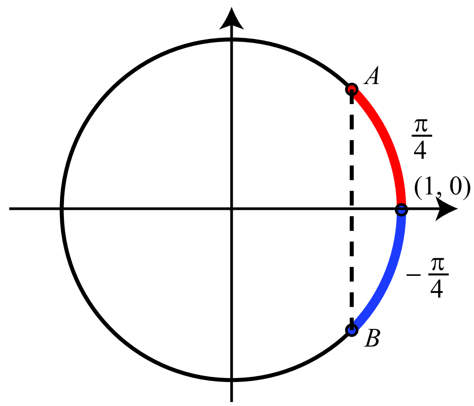


Figure 1.24: Reference Arc for $t = \dfrac{\pi}{4}$.

Since we know that the point A has coordinates $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, we conclude that the point B has coordinates $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$, and so

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Progress Check 1.25 (Reference Arcs for Negative Arcs)
For each of the following arcs, determine the reference arc and the values of the cosine and sine functions.

1. $t = -\dfrac{\pi}{6}$ **2.** $t = -\dfrac{2\pi}{3}$ **3.** $t = -\dfrac{5\pi}{4}$

Example 1.26 (Using Reference Arcs)
Sometimes we can use the concept of a reference arc even if we do not know the length of the arc but do know the value of the cosine or sine function. For example, suppose we know that

$$0 < t < \frac{\pi}{2} \quad \text{and} \quad \sin(t) = \frac{2}{3}.$$

Are there any conclusions we can make with this information? Following are some possibilities.

1. We can use the Pythagorean identity to determine $\cos(t)$ as follows:

$$\begin{aligned}\cos^2(t) + \sin^2(t) &= 1\\ \cos^2(t) &= 1 - \left(\frac{2}{3}\right)^2\\ \cos^2(t) &= \frac{5}{9}\end{aligned}$$

   Since t is in the first quadrant, we know that $\cos(t)$ is positive, and hence

$$\cos(t) = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

2. Since $0 < t < \dfrac{\pi}{2}$, t is in the first quadrant. Hence, $\pi - t$ is in the second quadrant and the reference arc is t. In the second quadrant we know that the sine is positive, so we can conclude that

$$\sin(\pi - t) = \sin(t) = \frac{2}{3}.$$

Progress Check 1.27 (Working with Reference Arcs)
Following is information from Example 1.26:

$$0 < t < \frac{\pi}{2} \quad \text{and} \quad \sin(t) = \frac{2}{3}.$$

Use this information to determine the exact values of each of the following:

1. $\cos(\pi - t)$
2. $\sin(\pi + t)$
3. $\cos(\pi + t)$
4. $\sin(2\pi - t)$

Summary of Section 1.5

In this section, we studied the following important concepts and ideas:

- The values of $\cos(t)$ and $\sin(t)$ for arcs whose terminal points are on one of the coordinate axes are shown in Table 1.3 below.

- Exact values for the cosine and sine functions at $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are known and are shown in Table 1.3 below.

- A **reference arc** for an arc t is the arc (always considered nonnegative) between the terminal point of the arc t and point intersecting the unit circle and the x-axis closest to it.

- If t is an arc that has an arc $\hat{t}$ as a reference arc, then $|\cos(t)|$ and $|\cos(\hat{t})|$ are the same. Whether $\cos(t) = \cos(\hat{t})$ or $\cos(t) = -\cos(\hat{t})$ is determined by the quadrant in which the terminal side of t lies. The same is true for $\sin(t)$.

- We can determine the exact values of the cosine and sine functions at any arc with $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ as reference arc. These arcs between 0 and 2π are shown in Figure 1.17. The results are summarized in Table 1.3 below.

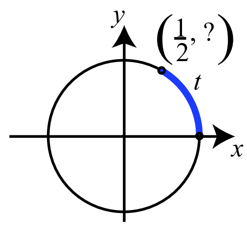
t	$x = \cos(t)$	$y = \sin(t)$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

t	$x = \cos(t)$	$y = \sin(t)$
π	-1	0
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

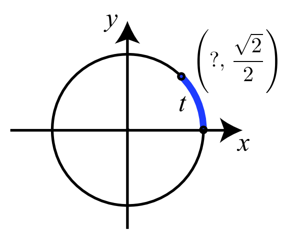
Table 1.3: Exact values of the cosine and sine functions.

Exercises for Section 1.5

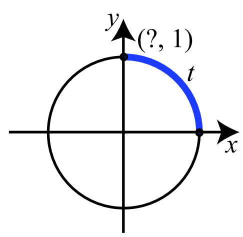
⋆ **1.** A unit circle is shown in each of the following showing information about an arc t. In each case, use the information on the unit circle to determine the values of t, $\cos(t)$, and $\sin(t)$.



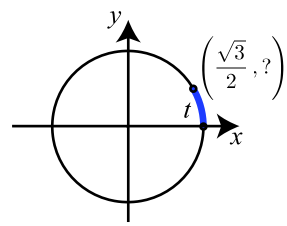
(a)



(c)



(b)



(d)

2. Determine the exact value for each of the following expressions and then use a calculator to check the result. For example,

$$\cos(0) + \sin\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2} \approx 1.8660.$$

⋆ **(a)** $\cos^2\left(\frac{\pi}{6}\right)$

(c) $\dfrac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$

⋆ **(b)** $2\sin^2\left(\frac{\pi}{4}\right) + \cos(\pi)$

(d) $3\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right)$

3. For each of the following, determine the reference arc for the given arc and draw the arc and its reference arc on the unit circle.

⋆ **(a)** $t = \frac{4\pi}{3}$

⋆ **(b)** $t = \frac{13\pi}{8}$

(c) $t = \frac{9\pi}{4}$

⋆ **(d)** $t = -\frac{4\pi}{3}$

(e) $t = -\frac{7\pi}{5}$

(f) $t = 5$

4. For each of the following, draw the given arc t on the unit circle, determine the referenc arc for t, and then determine the exact values for $\cos(t)$ and $\sin(t)$.

- ⋆ **(a)** $t = \dfrac{5\pi}{6}$
- **(b)** $t = \dfrac{5\pi}{4}$
- **(c)** $t = \dfrac{5\pi}{3}$
- ⋆ **(d)** $t = -\dfrac{2\pi}{3}$
- **(e)** $t = -\dfrac{7\pi}{4}$
- **(f)** $t = \dfrac{19\pi}{6}$

5. (**a**) Use a calculator (in radian mode) to determine five-digit approximations for $\cos(4)$ and $\sin(4)$.

(**b**) Use a calculator (in radian mode) to determine five-digit approximations for $\cos(4 - \pi)$ and $\sin(4 - \pi)$.

(**c**) Use the concept of reference arcs to explain the results in parts (a) and (b).

6. Suppose that we have the following information about the arc t.

$$0 < t < \frac{\pi}{2} \quad \text{and} \quad \sin(t) = \frac{1}{5}.$$

Use this information to determine the exact values of each of the following:

- ⋆ **(a)** $\cos(t)$
- **(b)** $\sin(\pi - t)$
- **(c)** $\cos(\pi - t)$
- ⋆ **(d)** $\sin(\pi + t)$
- **(e)** $\cos(\pi + t)$
- **(f)** $\sin(2\pi - t)$

7. Suppose that we have the following information about the arc t.

$$\frac{\pi}{2} < t < \pi \quad \text{and} \quad \cos(t) = -\frac{2}{3}.$$

Use this information to determine the exact values of each of the following:

- (**a**) $\sin(t)$
- (**b**) $\sin(\pi - t)$
- (**c**) $\cos(\pi - t)$
- (**d**) $\sin(\pi + t)$
- (**e**) $\cos(\pi + t)$
- (**f**) $\sin(2\pi - t)$

8. Make sure your calculator is in Radian Mode.

(**a**) Use a calculator to find an eight-digit approximation of $\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) = \sin\left(\dfrac{5\pi}{12}\right)$.

(**b**) Determine the **e**xact value of $\sin\left(\dfrac{\pi}{6}\right)+\sin\left(\dfrac{\pi}{4}\right)$.

(**c**) Use a calculator to find an eight-digit approximation of your result in part (b). Compare this to your result in part (a). Does it seem that

$$\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right).$$

(**d**) Determine the **exact value** of $\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right)+\cos\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right)$.

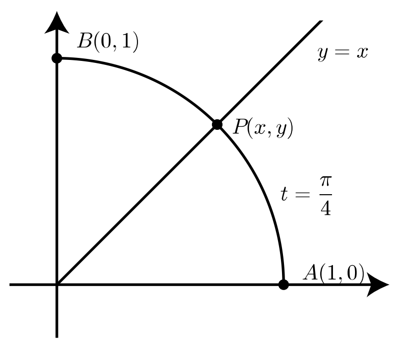
(**e**) Determine an eight-digit approximation of your result in part (d).

(**f**) Compare the results in parts (a) and (e). Does it seem that

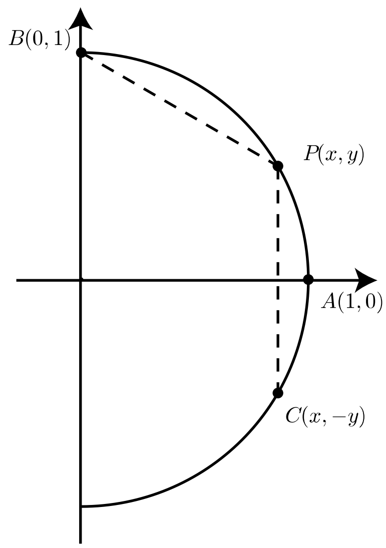
$$\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right).$$

9. This exercise provides an alternate method for determining the exact values of $\cos\left(\dfrac{\pi}{4}\right)$ and $\sin\left(\dfrac{\pi}{4}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \dfrac{\pi}{4}$ on the unit circle. The points $A(1, 0)$ and $B(0, 1)$ are also shown.
Since the point B is the terminal point of the arc of length $\dfrac{\pi}{2}$, we can conclude that the length of the arc from P to B is also $\dfrac{\pi}{4}$. Because of this, we conclude that the point P lies on the line $y = x$ as shown in the diagram. Use this fact to determine the values of x and y. Explain why this proves that

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$



10. This exercise provides an alternate method for determining the exact values of $\cos\left(\frac{\pi}{6}\right)$ and $\sin\left(\frac{\pi}{6}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \frac{\pi}{6}$ on the unit circle. The points $A(1, 0)$, $B(0, 1)$, and $C(x, -y)$ are also shown. Notice that B is the terminal point of the arc $t = \frac{\pi}{2}$, and C is the terminal point of the arc $t = -\frac{\pi}{6}$.



We now notice that the length of the arc from P to B is

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

In addition, the length of the arc from C to P is

$$\frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3}.$$

This means that the distance from P to B is equal to the distance from C to P.

(**a**) Use the distance formula to write a formula (in terms of x and y) for the distance from P to B.

(**b**) Use the distance formula to write a formula (in terms of x and y) for the distance from C to P.

(**c**) Set the distances from (a) and (b) equal to each other and solve the resulting equation for y. To do this, begin by squaring both sides of the equation. In order to solve for y, it may be necessary to use the fact that $x^2 + y^2 = 1$.

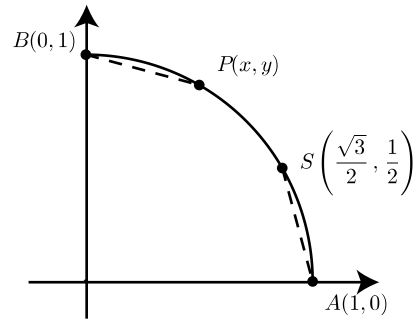
(**d**) Use the value for y in (c) and the fact that $x^2 + y^2 = 1$ to determine the value for x.

Explain why this proves that

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

11. This exercise provides an alternate method for determining the exact values of $\cos\left(\dfrac{\pi}{3}\right)$ and $\sin\left(\dfrac{\pi}{3}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \dfrac{\pi}{3}$ on the unit circle. The points $A(1, 0)$, $B(0, 1)$, and $S\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ are also shown. Notice that B is the terminal point of the arc $t = \dfrac{\pi}{2}$.

From Exercise (10), we know that S is the terminal point of an arc of length $\dfrac{\pi}{6}$.



We now notice that the length of the arc from A to P is $\dfrac{\pi}{3}$. In addition, since the length of the arc from A to B is $\dfrac{\pi}{2}$ and the and the length of the arc from B to P is $\dfrac{\pi}{3}$. This means that the distance from P to B is equal

$$\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

Since both of the arcs have length $\dfrac{\pi}{6}$, the distance from A to S is equal to the distance from P to B.

(**a**) Use the distance formula to determine the distance from A to S.

(**b**) Use the distance formula to write a formula (in terms of x and y) for the distance from P to B.

(**c**) Set the distances from (a) and (b) equal to each other and solve the resulting equation for y. To do this, begin by squaring both sides of the equation. In order to solve for y, it may be necessary to use the fact that $x^2 + y^2 = 1$.

(**d**) Use the value for y in (c) and the fact that $x^2 + y^2 = 1$ to determine the value for x.

Explain why this proves that

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

1.6 Other Trigonometric Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How is the tangent function defined? What is the domain of the tangent function?
- What are the reciprocal functions and how are they defined? What are the domains of each of the reciprocal functions?

We defined the cosine and sine functions as the coordinates of the terminal points of arcs on the unit circle. As we will see later, the sine and cosine give relations for certain sides and angles of right triangles. It will be useful to be able to relate different sides and angles in right triangles, and we need other circular functions to do that. We obtain these other circular functions – tangent, cotangent, secant, and cosecant – by combining the cosine and sine together in various ways.

Beginning Activity

Using radian measure:

1. For what values of t is $\cos(t) = 0$?

2. For what values of t is $\sin(t) = 0$?

3. In what quadrants is $\cos(t) > 0$? In what quadrants is $\sin(t) > 0$?

4. In what quadrants is $\cos(t) < 0$? In what quadrants is $\sin(t) < 0$?

The Tangent Function

Next to the cosine and sine, the most useful circular function is the tangent.[3]

[3]The word tangent was introduced by Thomas Fincke (1561-1656) in his *Flenspurgensis Geometriae rotundi libri* XIIII where he used the word tangens in Latin. From *"Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

Definition. The **tangent function** is the quotient of the sine function divided by the cosine function. So the tangent of a real number t is defined to be $\dfrac{\sin(t)}{\cos(t)}$ for those values t for which $\cos(t) \neq 0$. The common abbreviation for the tangent of t is

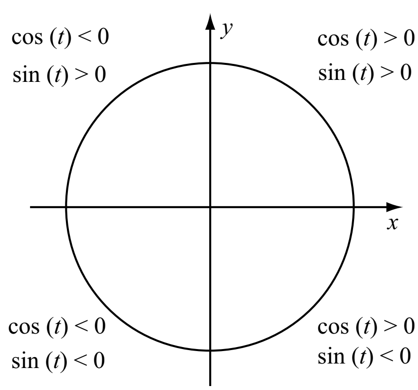
$$\tan(t) = \frac{\sin(t)}{\cos(t)}.$$

In this definition, we need the restriction that $\cos(t) \neq 0$ to make sure the quotient is defined. Since $\cos(t) = 0$ whenever $t = \dfrac{\pi}{2} + k\pi$ for some integer k, we see that $\tan(t)$ is defined when $t \neq \dfrac{\pi}{2} + k\pi$ for all integers k. So

The **domain of the tangent function** is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

Notice that although the domain of the sine and cosine functions is all real numbers, this is not true for the tangent function.

When we worked with the unit circle definitions of cosine and sine, we often used the following diagram to indicate signs of $\cos(t)$ and $\sin(t)$ when the terminal point of the arc t is in a given quadrant.



Progress Check 1.28 (Signs and Values of the Tangent Function)
Considering t to be an arc on the unit circle, for the terminal point of t:

1. In which quadrants is $\tan(t)$ positive?
2. In which quadrants is $\tan(t)$ negative?

3. For what values of t is $\tan(t) = 0$?

4. Complete Table 1.4, which gives the values of cosine, sine, and tangent at the common reference arcs in Quadrant I.

t	$\cos(t)$	$\sin(t)$	$\tan(t)$
0		0	
$\dfrac{\pi}{6}$		$\dfrac{1}{2}$	
$\dfrac{\pi}{4}$		$\dfrac{\sqrt{2}}{2}$	
$\dfrac{\pi}{4}$		$\dfrac{\sqrt{3}}{2}$	
$\dfrac{\pi}{2}$		1	

Table 1.4: Values of the Tangent Function

Just as with the cosine and sine, if we know the values of the tangent function at the reference arcs, we can find its values at any arc related to a reference arc. For example, the reference arc for the arc $t = \dfrac{5\pi}{3}$ is $\dfrac{\pi}{3}$. So

$$\begin{aligned}\tan\left(\frac{5\pi}{3}\right) &= \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} \\ &= \frac{-\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} \\ &= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= -\sqrt{3}\end{aligned}$$

We can shorten this process by just using the fact that $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$ and that $\tan\left(\dfrac{5\pi}{3}\right) < 0$ since the terminal point of the arc $\dfrac{5\pi}{3}$ is in the fourth quadrant.

$$\tan\left(\frac{5\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}.$$

Progress Check 1.29 (Values of the Tangent Function)

1. Determine the exact values of $\tan\left(\dfrac{5\pi}{4}\right)$ and $\tan\left(\dfrac{5\pi}{6}\right)$.

2. Determine the exact values of $\cos(t)$ and $\tan(t)$ if it is known that $\sin(t) = \dfrac{1}{3}$ and $\tan(t) < 0$.

The Reciprocal Functions

The remaining circular or trigonometric functions are reciprocals of the cosine, sine, and tangent functions. Since these functions are reciprocals, their domains will be all real numbers for which the denominator is not equal to zero. The first we will introduce is the secantfunction. [4] function.

Definition. The **secant function** is the reciprocal of the cosine function. So the secant of a real number t is defined to be $\dfrac{1}{\cos(t)}$ for those values t where $\cos(t) \neq 0$. The common abbreviation for the secant of t is

$$\sec(t) = \frac{1}{\cos(t)}.$$

Since the tangent function and the secant function use $\cos(t)$ in a denominator, they have the same domain. So

[4] The term secant was introduced by was by Thomas Fincke (1561-1656) in his Thomae Finkii Flenspurgensis Geometriae rotundi libri XIIII, Basileae: Per Sebastianum Henricpetri, 1583. Vieta (1593) did not approve of the term secant, believing it could be confused with the geometry term. He used Transsinuosa instead. From *"Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

The **domain of the secant function** is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

Next up is the cosecant function. [5]

Definition. The **cosecant function** is the reciprocal of the secant function. So the cosecant of a real number t is defined to be $\dfrac{1}{\sin(t)}$ for those values t where $\sin(t) \neq 0$. The common abbreviation for the cotangent of t is

$$\csc(t) = \frac{1}{\sin(t)}.$$

Since $\sin(t) = 0$ whenever $t = k\pi$ for some integer k, we see that

The **domain of the cosecant function** is the set of all real numbers t for which $t \neq k\pi$ for every integer k.

Finally, we have the cotangent function. [6]

Definition. The **cotangent function** is the reciprocal of the tangent function. So the cotangent of a real number t is defined to be $\dfrac{1}{\tan(t)}$ for those values t where $\tan(t) \neq 0$. The common abbreviation for the cotangent of t is

$$\cot(t) = \frac{1}{\tan(t)}.$$

Since $\tan(t) = 0$ whenever $t = k\pi$ for some integer k, we see that

The **domain of the cotangent function** is the set of all real numbers t for which $t \neq k\pi$ for every integer k.

[5] Georg Joachim von Lauchen Rheticus appears to be the first to use the term cosecant (as cosecans in Latin) in his *Opus Palatinum de triangulis*. From *Earliest Known Uses of Some of the Words of Mathematics* at `http://jeff560.tripod.com/mathword.html`.

[6] The word cotangent was introduced by Edmund Gunter in *Canon Triangulorum* (Table of Artificial Sines and Tangents) where he used the term cotangens in Latin. From *Earliest Known Uses of Some of the Words of Mathematics* at `http://jeff560.tripod.com/mathword.html`.

A Note about Calculators

When it is not possible to determine exact values of a trigonometric function, we use a calculator to determine approximate values. However, please keep in mind that many calculators only have keys for the sine, cosine, and tangent functions. With these calculators, we must use the definitions of cosecant, secant, and cotangent to determine approximate values for these functions.

Progress Check 1.30 (Values of Trignometric Functions

When possible, find the exact value of each of the following functional values. When this is not possible, use a calculator to find a decimal approximation to four decimal places.

1. $\sec\left(\dfrac{7\pi}{4}\right)$
2. $\csc\left(\dfrac{-\pi}{4}\right)$
3. $\tan\left(\dfrac{7\pi}{8}\right)$
4. $\cot\left(\dfrac{4\pi}{3}\right)$
5. $\csc(5)$

Progress Check 1.31 (Working with Trignometric Functions

1. If $\cos(x) = \dfrac{1}{3}$ and $\sin(x) < 0$, determine the exact values of $\sin(x)$, $\tan(x)$, $\csc\,(x)$, and $\cot(x)$.
2. If $\sin(x) = -\dfrac{7}{10}$ and $\tan(x) > 0$, determine the exact values of $\cos(x)$ and $\cot(x)$.
3. What is another way to write $(\tan(x))(\cos(x))$?

Summary of Section 1.6

In this section, we studied the following important concepts and ideas:

- The **tangent function** is the quotient of the sine function divided by the cosine function. So is the quotient of the sine function divided by the cosine function. That is,
$$\tan(t) = \frac{\sin(t)}{\cos(t)},$$
for those values t for which $\cos(t) \neq 0$. The **domain of the tangent function** is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

- The reciprocal functions are the secant, cosecant, and tangent functions.

Reciprocal Function	**Domain**
$\sec(t) = \dfrac{1}{\cos(t)}$	The set of real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.
$\csc(t) = \dfrac{1}{\sin(t)}$	The set of real numbers t for which $t \neq k\pi$ for every integer k.
$\cot(t) = \dfrac{1}{\tan(t)}$	The set of real numbers t for which $t \neq k\pi$ for every integer k.

Exercises for Section 1.6

⋆ **1.** Complete the following table with the exact values of each functional value if it is defined.

t	$\cot(t)$	$\sec(t)$	$\csc(t)$
0			
$\dfrac{\pi}{6}$			
$\dfrac{\pi}{4}$			
$\dfrac{\pi}{3}$			
$\dfrac{\pi}{2}$			

2. Complete the following table with the exact values of each functional value if it is defined.

t	$\cot(t)$	$\sec(t)$	$\csc(t)$
$\dfrac{2\pi}{3}$			
$\dfrac{7\pi}{6}$			
$\dfrac{7\pi}{4}$			
$-\dfrac{\pi}{3}$			
π			

3. Determine the quadrant in which the terminal point of each arc lies based on the given information.

⋆ **(a)** $\cos(x) > 0$ and $\tan(x) < 0$.

⋆ **(b)** $\tan(x) > 0$ and $\csc(x) < 0$.

(c) $\cot(x) > 0$ and $\sec(x) > 0$.

(d) $\sin(x) < 0$ and $\sec(x) > 0$.

(e) $\sec(x) < 0$ and $\csc(x) > 0$.

(f) $\sin(x) < 0$ and $\cot(x) > 0$.

⋆ **4.** If $\sin(t) = \dfrac{1}{3}$ and $\cos(t) < 0$, determine the exact values of $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

5. If $\cos(t) = -\dfrac{3}{5}$ and $\sin(t) < 0$, determine the exact values of $\sin(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

6. If $\sin(t) = -\dfrac{2}{5}$ and $\tan(t) < 0$, determine the exact values of $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

7. If $\sin(t) = 0.273$ and $\cos(t) < 0$, determine the five-digit approximations for $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

8. In each case, determine the arc t that satisfies the given conditions or explain why no such arc exists.

⋆ **(a)** $\tan(t) = 1$, $\cos(t) = -\dfrac{1}{\sqrt{2}}$, and $0 < t < 2\pi$.

⋆ **(b)** $\sin(t) = 1$, $\sec(t)$ is undefined, and $0 < t < \pi$.

(c) $\sin(t) = \dfrac{\sqrt{2}}{2}$, $\sec(t) = -\sqrt{2}$, and $0 < t < \pi$.

(d) $\sec(t) = -\dfrac{2}{\sqrt{3}}$, $\tan(t) = \sqrt{3}$, and $0 < t < 2\pi$.

(e) $\csc(t) = \sqrt{2}$, $\tan(t) = -1$, and $0 < t < 2\pi$.

9. Use a calculator to determine four-digit decimal approximations for each of the following.

(a) $\csc(1)$

(b) $\tan\left(\dfrac{12\pi}{5}\right)$

(c) $\cot(5)$

(d) $\sec\left(\dfrac{13\pi}{8}\right)$

(e) $\sin^2(5.5)$

(f) $1 + \tan^2(2)$

(g) $\sec^2(2)$