

Polynomial functions

Many common functions are polynomial functions. In this unit we describe polynomial functions and look at some of their properties.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise when a rule describes a polynomial function, and write down the degree of the polynomial,
- recognize the typical shapes of the graphs of polynomials, of degree up to 4,
- understand what is meant by the multiplicity of a root of a polynomial,
- sketch the graph of a polynomial, given its expression as a product of linear factors.

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1. Introduction

A polynomial function is a function such as a quadratic, a cubic, a quartic, and so on, involving only non-negative integer powers of x . We can give a general definition of a polynomial, and define its degree.

2. What is a polynomial?

A polynomial of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where the a 's are real numbers (sometimes called the *coefficients* of the polynomial). Although this general formula might look quite complicated, particular examples are much simpler. For example,

$$f(x) = 4x^3 - 3x^2 + 2$$

is a polynomial of degree 3, as 3 is the highest power of x in the formula. This is called a cubic polynomial, or just a *cubic*. And

$$f(x) = x^7 - 4x^5 + 1$$

is a polynomial of degree 7, as 7 is the highest power of x . Notice here that we don't need every power of x up to 7: we need to know only the *highest* power of x to find out the degree. An example of a kind you may be familiar with is

$$f(x) = 4x^2 - 2x - 4$$

which is a polynomial of degree 2, as 2 is the highest power of x . This is called a *quadratic*.

Functions containing other operations, such as square roots, are not polynomials. For example,

$$f(x) = 4x^3 + \sqrt{x} - 1$$

is not a polynomial as it contains a square root. And

$$f(x) = 5x^4 - 2x^2 + 3/x$$

is not a polynomial as it contains a 'divide by x '.



Key Point

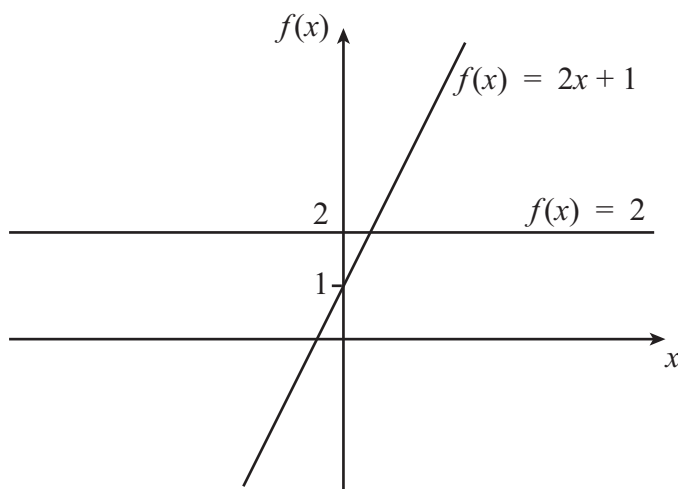
A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0.$$

The *degree* of a polynomial is the highest power of x in its expression. Constant (non-zero) polynomials, linear polynomials, quadratics, cubics and quartics are polynomials of degree 0, 1, 2, 3 and 4 respectively. The function $f(x) = 0$ is also a polynomial, but we say that its degree is 'undefined'.

3. Graphs of polynomial functions

We have met some of the basic polynomials already. For example, $f(x) = 2$ is a constant function and $f(x) = 2x + 1$ is a linear function.

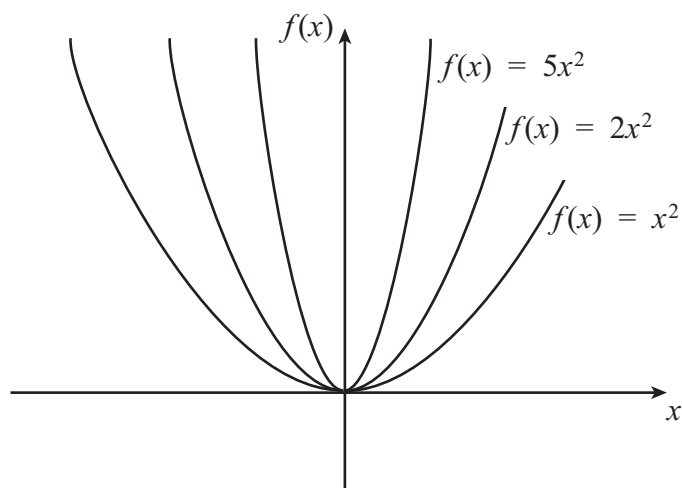


It is important to notice that the graphs of constant functions and linear functions are always straight lines.

We have already said that a quadratic function is a polynomial of degree 2. Here are some examples of quadratic functions:

$$f(x) = x^2, \quad f(x) = 2x^2, \quad f(x) = 5x^2.$$

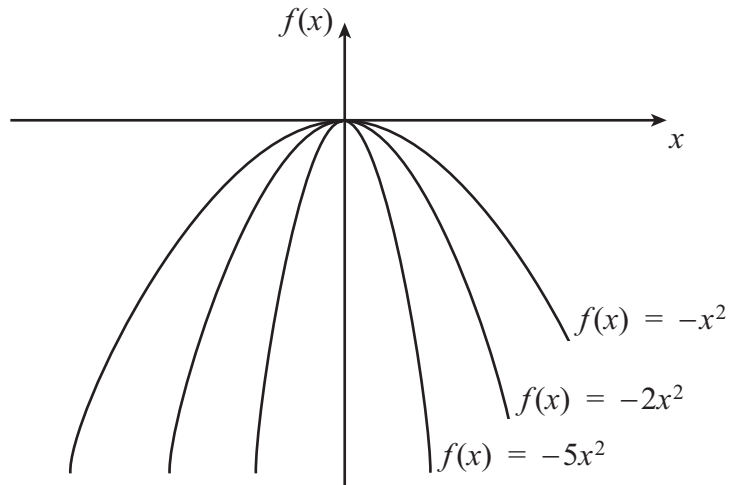
What is the impact of changing the coefficient of x^2 as we have done in these examples? One way to find out is to sketch the graphs of the functions.



You can see from the graph that, as the coefficient of x^2 is increased, the graph is stretched vertically (that is, in the y direction).

What will happen if the coefficient is negative? This will mean that all of the positive $f(x)$ values will now become negative. So what will the graphs of the functions look like? The functions are now

$$f(x) = -x^2, \quad f(x) = -2x^2, \quad f(x) = -5x^2.$$

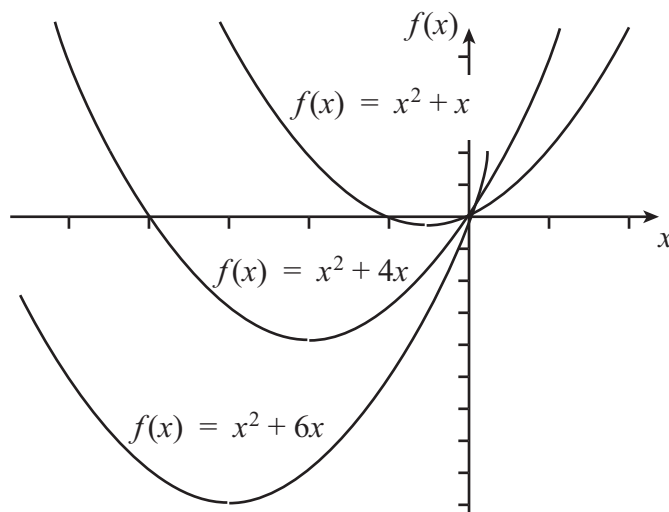


Notice here that all of these graphs have actually been reflected in the x -axis. This will always happen for functions of any degree if they are multiplied by -1 .

Now let us look at some other quadratic functions to see what happens when we vary the coefficient of x , rather than the coefficient of x^2 . We shall use a table of values in order to plot the graphs, but we shall fill in only those values near the turning points of the functions.

x	-5	-4	-3	-2	-1	0	1	2
$x^2 + x$			6	2	0	0	2	6
$x^2 + 4x$		0	-3	-4	-3	0		
$x^2 + 6x$	-5	-8	-9	-8	-5			

You can see the symmetry in each row of the table, demonstrating that we have concentrated on the region around the turning point of each function. We can now use these values to plot the graphs.

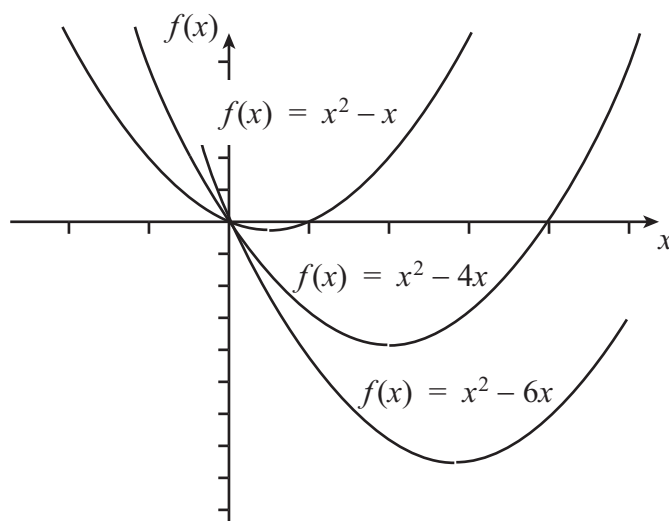


As you can see, increasing the positive coefficient of x in this polynomial moves the graph down and to the left.

What happens if the coefficient of x is negative?

x	-2	-1	0	1	2	3	4	5
$x^2 - x$	6	2	0	0	2	6		
$x^2 - 4x$			0	-3	-4	-3	0	
$x^2 - 6x$				-5	-8	-9	-8	-5

Again we can use these tables of values to plot the graphs of the functions.

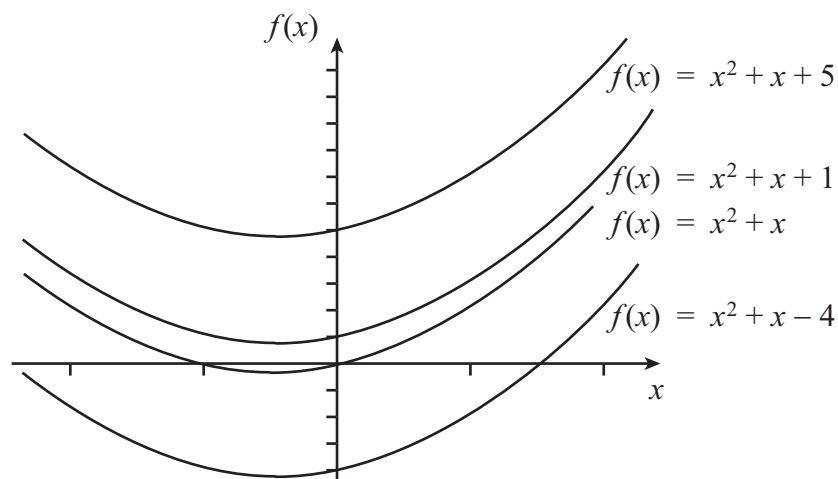


As you can see, increasing the negative coefficient of x (in absolute terms) moves the graph down and to the right.

So now we know what happens when we vary the x^2 coefficient, and what happens when we vary the x coefficient. But what happens when we vary the constant term at the end of our polynomial? We already know what the graph of the function $f(x) = x^2 + x$ looks like, so how does this differ from the graph of the functions $f(x) = x^2 + x + 1$, or $f(x) = x^2 + x + 5$, or $f(x) = x^2 + x - 4$? As usual, a table of values is a good place to start.

x	-2	-1	0	1	2
$x^2 + x$	2	0	0	2	6
$x^2 + x + 1$	3	1	1	3	7
$x^2 + x + 5$	7	5	5	7	11
$x^2 + x - 4$	-2	-4	-4	-2	2

Our table of values is particularly easy to complete since we can use our answers from the $x^2 + x$ column to find everything else. We can use these tables of values to plot the graphs of the functions.

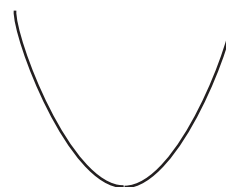


As we can see straight away, varying the constant term translates the $x^2 + x$ curve vertically. Furthermore, the value of the constant is the point at which the graph crosses the $f(x)$ axis.

4. Turning points of polynomial functions

A turning point of a function is a point where the graph of the function changes from sloping downwards to sloping upwards, or vice versa. So the gradient changes from negative to positive, or from positive to negative. Generally speaking, curves of degree n can have up to $(n - 1)$ turning points.

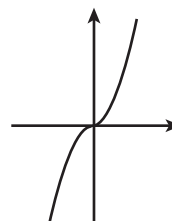
For instance, a quadratic has only one turning point.



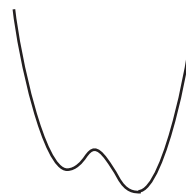
A cubic could have up to two turning points, and so would look something like this.



However, some cubics have fewer turning points: for example $f(x) = x^3$. But no cubic has more than two turning points.



In the same way, a quartic could have up to three turning points, and so would look something like this.



Again, some quartics have fewer turning points, but none has more.



Key Point

A polynomial of degree n can have up to $(n - 1)$ turning points.

5. Roots of polynomial functions

You may recall that when $(x - a)(x - b) = 0$, we know that a and b are roots of the function $f(x) = (x - a)(x - b)$. Now we can use the converse of this, and say that if a and b are roots, then the polynomial function with these roots must be $f(x) = (x - a)(x - b)$, or a multiple of this.

For example, if a quadratic has roots $x = 3$ and $x = -2$, then the function must be $f(x) = (x - 3)(x + 2)$, or a constant multiple of this. This can be extended to polynomials of any degree. For example, if the roots of a polynomial are $x = 1$, $x = 2$, $x = 3$, $x = 4$, then the function must be

$$f(x) = (x - 1)(x - 2)(x - 3)(x - 4),$$

or a constant multiple of this.

Let us also think about the function $f(x) = (x - 2)^2$. We can see straight away that $x - 2 = 0$, so that $x = 2$. For this function we have only one root. This is what we call a repeated root, and a root can be repeated any number of times. For example, $f(x) = (x - 2)^3(x + 4)^4$ has a repeated root $x = 2$, and another repeated root $x = -4$. We say that the root $x = 2$ has multiplicity 3, and that the root $x = -4$ has multiplicity 4.

The useful thing about knowing the multiplicity of a root is that it helps us with sketching the graph of the function. If the multiplicity of a root is odd then the graph cuts through the x -axis at the point $(x, 0)$. But if the multiplicity is even then the graph just touches the x -axis at the point $(x, 0)$.

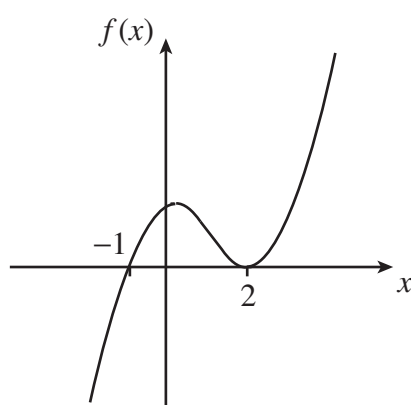
For example, take the function

$$f(x) = (x - 3)^2(x + 1)^5(x - 2)^3(x + 2)^4.$$

- The root $x = 3$ has multiplicity 2, so the graph touches the x -axis at $(3, 0)$.

- The root $x = -1$ has multiplicity 5, so the graph crosses the x -axis at $(-1, 0)$.
- The root $x = 2$ has multiplicity 3, so the graph crosses the x -axis at $(2, 0)$.
- The root $x = -2$ has multiplicity 4, so the graph touches the x -axis at $(-2, 0)$.

To take another example, suppose we have the function $f(x) = (x - 2)^2(x + 1)$. We can see that the largest power of x is 3, and so the function is a cubic. We know the possible general shapes of a cubic, and as the coefficient of x^3 is positive the curve must generally increase to the right and decrease to the left. We can also see that the roots of the function are $x = 2$ and $x = -1$. The root $x = 2$ has even multiplicity and so the curve just touches the x -axis here, whilst $x = -1$ has odd multiplicity and so here the curve crosses the x -axis. This means we can sketch the graph as follows.



Key Point

The number a is a root of the polynomial function $f(x)$ if $f(a) = 0$, and this occurs when $(x - a)$ is a factor of $f(x)$.

If a is a root of $f(x)$, and if $(x - a)^m$ is a factor of $f(x)$ but $(x - a)^{m+1}$ is not a factor, then we say that the root has multiplicity m .

At a root of odd multiplicity the graph of the function crosses the x -axis, whereas at a root of even multiplicity the graph touches the x -axis.

Exercises

1. What is a polynomial function?

2. Which of the following functions are polynomial functions?

- (a) $f(x) = 4x^2 + 2$ (b) $f(x) = 3x^3 - 2x + \sqrt{x}$ (c) $f(x) = 12 - 4x^5 + 3x^2$
 (d) $f(x) = \sin x + 1$ (e) $f(x) = 3x^{12} - 2/x$ (f) $f(x) = 3x^{11} - 2x^{12}$

3. Write down one example of each of the following types of polynomial function:
 (a) cubic (b) linear (c) quartic (d) quadratic
4. Sketch the graphs of the following functions on the same axes:
 (a) $f(x) = x^2$ (b) $f(x) = 4x^2$ (c) $f(x) = -x^2$ (d) $f(x) = -4x^2$
5. Consider a function of the form $f(x) = x^2 + ax$, where a represents a real number. The graph of this function is represented by a parabola.
 (a) When $a > 0$, what happens to the parabola as a increases?
 (b) When $a < 0$, what happens to the parabola as a decreases?
6. Write down the maximum number of turning points on the graph of a polynomial function of degree:
 (a) 2 (b) 3 (c) 12 (d) n
7. Write down a polynomial function with roots:
 (a) 1, 2, 3, 4 (b) 2, -4 (c) 12, -1, -6
8. Write down the roots and identify their multiplicity for each of the following functions:
 (a) $f(x) = (x - 2)^3(x + 4)^4$ (b) $f(x) = (x - 1)(x + 2)^2(x - 4)^3$
9. Sketch the following functions:
 (a) $f(x) = (x - 2)^2(x + 1)$ (b) $f(x) = (x - 1)^2(x + 3)$

Answers

1. A polynomial function is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0,$$

where each a_0, a_1 , etc. represents a real number, and where n is a natural number (including 0).

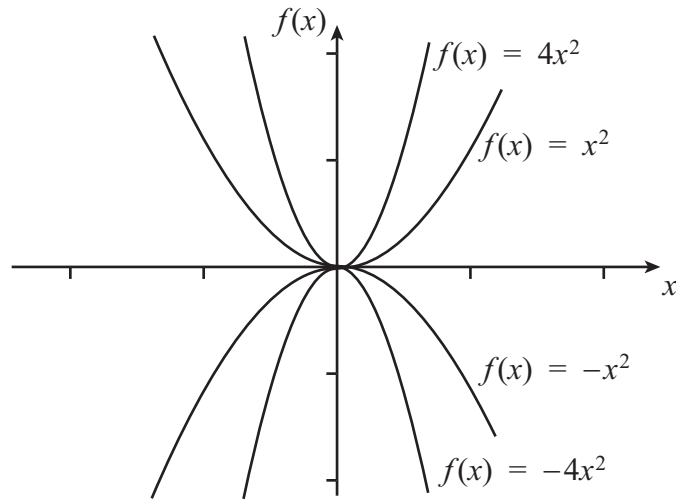
2.

- (a) $f(x) = 4x^2 + 2$ is a polynomial
 (b) $f(x) = 3x^3 - 2x + \sqrt{x}$ is not a polynomial, because of \sqrt{x}
 (c) $f(x) = 12 - 4x^5 + 3x^2$ is a polynomial
 (d) $f(x) = \sin x + 1$ is not a polynomial, because of $\sin x$
 (e) $f(x) = 3x^{12} - 2/x$ is not a polynomial, because of $2/x$
 (f) $f(x) = 3x^{11} - 2x^{12}$ is a polynomial

3.

- (a) The highest power of x must be 3, so examples might be $f(x) = x^3 - 2x + 1$ or $f(x) = x^3 - 2$.
 (b) The highest power of x must be 1, so examples might be $f(x) = x$ or $f(x) = 6x - 5$.
 (c) The highest power of x must be 4, so examples might be $f(x) = x^4 - 3x^3 + 2x$ or $f(x) = x^4 - x - 5$.
 (d) The highest power of x must be 2, so examples might be $f(x) = x^2$ or $f(x) = x^2 - 5$.

4.



5.

- (a) When $a > 0$, the parabola moves down and to the left as a increases.
- (b) When $a < 0$, the parabola moves down and to the right as a decreases.

6.

- (a) 1 turning point
- (b) 2 turning points
- (c) 11 turning points
- (d) $(n - 1)$ turning points.

7.

- (a) $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ or a multiple
- (b) $f(x) = (x - 2)(x + 4)$ or a multiple
- (c) $f(x) = (x - 12)(x + 1)(x + 6)$ or a multiple.

8.

- (a) $x = 2$ odd multiplicity
 $x = -4$ even multiplicity
- (b) $x = 1$ odd multiplicity
 $x = -2$ even multiplicity
 $x = 4$ odd multiplicity

9)

