1.2. TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE
The Unit Circle

Consider the **unit circle** given by

\[ x^2 + y^2 = 1 \]

That is, the unit circle with center at the origin \((0,0)\) and the radius 1.
## The Trigonometric Functions

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<th>Abbreviation (Function symbols)</th>
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<tr>
<td>Cosine</td>
<td>cos</td>
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<tr>
<td>Tangent</td>
<td>tan</td>
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<tr>
<td>Cosecant</td>
<td>csc</td>
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<tr>
<td>Secant</td>
<td>sec</td>
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<tr>
<td>Cotangent</td>
<td>cot</td>
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</table>
Radian measure of $\theta$ is $t$ because $s = r\theta$ (with $r = 1$)
Definitions of Trigonometric Functions

Let \( t \) be a real number and let \((x, y)\) be the point on the unit circle corresponding to \( t \).

\[
\sin t = y \\
\cos t = x \\
\tan t = \frac{y}{x}, \quad x \neq 0
\]

\[
\csc t = \frac{1}{y}, \quad y \neq 0 \\
\sec t = \frac{1}{x}, \quad x \neq 0 \\
\cot t = \frac{x}{y}, \quad y \neq 0
\]
Important Right Triangles
(Please remember)

- General right triangle: $c = \sqrt{a^2 + b^2}$
- Isosceles right triangle: $45^\circ - 45^\circ - 90^\circ$
- 30°-60°-90° triangle:
  - $30^\circ$: 1
  - $60^\circ$: $\sqrt{3}$
  - $90^\circ$: 2

\[
\begin{align*}
\text{general right triangle} & : c = \sqrt{a^2 + b^2} \\
\text{isosceles right triangle} & : 45^\circ - 45^\circ - 90^\circ \\
30^\circ-60^\circ-90^\circ \text{ triangle} & : 30^\circ: 1, 60^\circ: \sqrt{3}, 90^\circ: 2
\end{align*}
\]
0, $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$, and $2\pi$. 
$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$
\[ 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi. \]
Example
Evaluate the six trigonometric functions at each real number.

\( a. \ t = \frac{\pi}{6} \)

\( t = \frac{\pi}{6} \) is defined by the first sector of the unit circle.

\[
\cos \frac{\pi}{6} = x \text{ coordinate} = \frac{\sqrt{3}}{2}
\]

\[
\sin \frac{\pi}{6} = y \text{ coordinate} = \frac{1}{2}
\]
\[
\cos \frac{\pi}{6} = x - \text{coordinate} = \frac{\sqrt{3}}{2}
\]

\[
\sin \frac{\pi}{6} = y - \text{coordinate} = \frac{1}{2}
\]

\[
\tan \frac{\pi}{6} = \frac{y}{x} = y \times \frac{1}{x} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}
\]
\[
\cos \frac{\pi}{6} = x - \text{coordinate} = \frac{\sqrt{3}}{2}
\]

\[
\sec \frac{\pi}{6} =
\]

\[
\sin \frac{\pi}{6} = y - \text{coordinate} = \frac{1}{2}
\]

\[
\csc \frac{\pi}{6} =
\]

\[
\tan \frac{\pi}{6} = \frac{y}{x} = y \times \frac{1}{x} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}
\]

\[
\cot \frac{\pi}{6} =
\]
Rationalizing the denominator means making the denominator a natural number when the denominator includes a square root.

Example:

\[
\frac{1}{\sqrt{3}} = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}
\]
Example
Evaluate the six trigonometric functions at each real number.

b. \( t = \frac{5\pi}{4} \)

\( t = \frac{5\pi}{4} \) is defined by the 5th sector of the unit circle.

\[
\cos \frac{5\pi}{4} = x - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]

\[
\sin \frac{5\pi}{4} = y - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]
\[
\cos \frac{5\pi}{4} = x - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]

\[
\sin \frac{5\pi}{4} = y - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]

\[
\tan \frac{5\pi}{4} = \frac{y}{x} = y \times \frac{1}{x} = -\frac{\sqrt{2}}{2} \times -\frac{2}{\sqrt{2}} = 1
\]
\[
\cos \frac{5\pi}{4} = x - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]

\[
\sec \frac{5\pi}{4} = 
\]

\[
\sin \frac{5\pi}{4} = y - \text{coordinate} = -\frac{\sqrt{2}}{2}
\]

\[
\csc \frac{5\pi}{4} =
\]

\[
\tan \frac{5\pi}{4} = \frac{y}{x} = y \times \frac{1}{x} = -\frac{\sqrt{2}}{2} \times -\frac{2}{\sqrt{2}} = 1
\]

\[
\cot \frac{5\pi}{4} =
\]
Example
Evaluate the six trigonometric functions at each real number.

c. $t = 0$
$t = 0$ coincides the positive x-axis.

$\cos 0 = x - coordinate = 1$

$\sin 0 = y - coordinate = 0$
\[
\cos 0 = x - \text{coordinate}=1
\]

\[
\sin 0 = y - \text{coordinate}=0
\]

\[
\tan 0 = \frac{y}{x} = \frac{0}{1} = 0
\]
\[ \cos 0 = x \ - \ coordinate = 1 \]

\[ \sec 0 = \]

\[ \sin 0 = y \ - \ coordinate = 0 \]

\[ \csc 0 = \]

\[ \tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \]

\[ \cot 0 = \]
Example
Evaluate the six trigonometric functions at each real number.

d. \( t = \pi \)

\( t = \pi \) coincides the positive x-axis.

\[ \cos \pi = x - \text{coordinate} = -1 \]

\[ \sin \pi = y - \text{coordinate} = 0 \]
\[
\cos \pi = x - coordinate = -1
\]

\[
\sin \pi = y - coordinate = 0
\]

\[
\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0
\]
\( \cos \pi = x - coordinate= -1 \)

\( \sec \pi = \)

\( \sin \pi = y - coordinate= 0 \)

\( \csc \pi = \)

\( \tan \pi = \frac{y}{x} = \frac{0}{-1} = 0 \)

\( \cot \pi = \)
The *domain* of the sine and cosine functions is the set of all real numbers.
Range of Sine and Cosine

\[-1 \leq \sin t \leq 1 \quad \text{and} \quad -1 \leq \cos t \leq 1\]
This leads to the general result

\[ \sin(t + 2\pi n) = \sin t \]

and

\[ \cos(t + 2\pi n) = \cos t \]

for any integer \( n \) and real number \( t \). Functions that behave in such a repetitive (or cyclic) manner are called periodic.
Periodic function

A function \( f \) is periodic if there exists a positive real number \( c \) such that
\[
f(t + c) = f(t)
\]
for all \( t \) in the domain of \( f \).

The smallest number \( c \) for which \( f \) is periodic is called period of \( f \).
Even and odd functions

• A function $f$ is **even** if $f(-t) = f(t)$.

• A function $f$ is **odd** if $f(-t) = -f(t)$.
### Even and Odd Trigonometric Functions

<table>
<thead>
<tr>
<th>Even Trigonometric functions</th>
<th>Odd Trigonometric functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(-t) = \cos t$</td>
<td>$\sin(-t) = -\sin t$</td>
</tr>
<tr>
<td>$\sec(-t) = \sec t$</td>
<td>$\csc(-t) = -\csc t$</td>
</tr>
<tr>
<td></td>
<td>$\tan(-t) = -\tan t$</td>
</tr>
<tr>
<td></td>
<td>$\cot(-t) = -\cot t$</td>
</tr>
</tbody>
</table>
Example

Let \( \sin t = \frac{4}{5} \). What is \( \sin(-t) \)?

Solution:

\[
\sin(-t) = -\frac{4}{5}.
\]

**Key point**: \( \sin(-t) = -\sin t \)

Sine is an odd function. Thus, if the input switches the sign, then its output switches the sign.
Example

Use the Period to Evaluate the Sine.

\[ a. \sin \frac{13\pi}{6} = \]

Period of sine=\(2\pi\)

That implies whenever we add or subtract the multiple of \(2\pi\) to the given angle, the values of the sine function are the same.
Example

Use the Period to Evaluate the Cosine.

b. $\cos\left(-\frac{7\pi}{2}\right)$

Period of cosine $= 2\pi$
Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired mode of measurement (degree or radian).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the key with their respective reciprocal functions sine, cosine, and tangent.
Example

To evaluate \( \csc(\pi/8) \), use the fact that

\[
\csc \frac{\pi}{8} = \frac{1}{\sin(\pi/8)}
\]

and enter the following keystroke sequence in \textit{radian} mode.

\[
\left( \text{SIN} \right) \left( \frac{\pi}{8} \right) \left( \times^{-1} \right) \text{ ENTER}
\]

Display 2.6131259
## Example

<table>
<thead>
<tr>
<th>Function</th>
<th>Mode</th>
<th>Calculator Keystrokes</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\sin \frac{2\pi}{3}$</td>
<td>Radian</td>
<td>[SIN] [2] [π] [÷] [3] [ENTER]</td>
<td>0.8660254</td>
</tr>
<tr>
<td>b. cot 1.5</td>
<td>Radian</td>
<td>[TAN] (1.5) [x⁻¹] [ENTER]</td>
<td>0.0709148</td>
</tr>
</tbody>
</table>