Boundary Conditions on

$$\vec{E}, \vec{H}, \vec{D}, \vec{B}$$

General Boundary Conditions: Summary of Results

Vector form	Scalar form	Description
1. $\hat{e}_n \times \left(\vec{E}_1 - \vec{E}_2 \right) = 0$	$E_{t1} - E_{t2} = 0$	Tangential electric field, $\vec{E}$, is continuous.
2. $\hat{e}_n \times \left(\vec{H}_1 - \vec{H}_2 \right) = \vec{j}_s$	$H_{t1} - H_{t2} = j_s$	The discontinuity of the tangential H field equals the surface current.
3. $\hat{e}_n \cdot \left(\vec{D}_1 - \vec{D}_2 \right) = \rho_s$	$D_{n1} - D_{n2} = \rho_s$	The discontinuity of the normal $\vec{D}$ equals the surface charge density.
4. $\hat{e}_n \cdot \left(\vec{B}_1 - \vec{B}_2 \right) = 0$	$B_{n1} - B_{n2} = 0$	The normal component of $\vec{B}$ is continuous.

Derivation of these most general forms is in the text, Chapter 4-9 and 5-7. Study them carefully. In what follows in these notes, we give the derivations of the same four boundary conditions for two important special cases. That is, (a) dielectric to perfect conductor boundary, and (b) general medium to general medium boundary with no surface charges and surface current densities.

There are two ways to solve electromagnetic problems:

1. Find out charge and current distributions everywhere in space and solve Maxwell's Equations everywhere.

2. Solve Maxwell's Equations in a limited region of interest, subject to "boundary conditions" on the boundaries defining this region. Boundary condition means the value of the fields just at the boundary surface.

The second method is used most often. It is especially useful when the boundaries are conductors. Boundary conditions at perfect conductors are fairly simple. We usually approximate good conductors as perfect conductors in this class.

Boundary Conditions at a Surface of Perfect Conductor

Chapter 4-9 Text

Fields inside conductor are zero: $\vec{E}_2 = 0$, $\vec{H}_2 = 0$.

1. **Boundary condition for tangential component of electric field:**

Dielectric (perfect, or lossy)
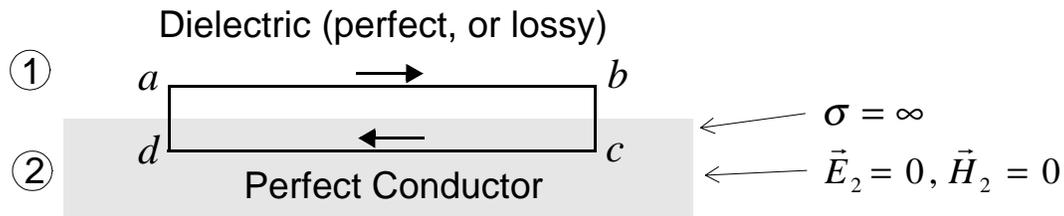
$$\int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{abcd} \vec{B} \cdot d\vec{s}$$

Area enclosed is infinitely small. Therefore RHS $\to 0$.

"ad" and "cb" are infinitely small. Therefore, $\int_a^b$ and $\int_d^a \to 0$.

$\dot{\vec{E}}_2$ (inside conductor) is zero. Therefore, $\int_c^d \to 0$.

This leaves:

$$\int_a^b \vec{E} \cdot d\vec{l} = 0$$

If ab distance is finite, but small, then $\vec{E}$ is constant, and we have:

$$E_t \cdot \Delta = 0 \qquad\qquad E_t = \text{component of } \vec{E} \text{ along } ab;$$
$$\text{i.e., tangential component}$$

Thus, $E_t = 0$.

In vector form, this can be written:

$$\hat{e}_n \times \vec{E} = 0 \text{, where } \hat{e}_n \text{ is unit normal.}$$

A very important corollary of this boundary condition is that, since $E_t = 0$ at the surface of a perfect conductor,

$$\int_a^b \vec{E} \cdot d\vec{l} = 0$$

between any point on or inside the conductor. Therefore, **a perfect conductor is an equipotential surface**.

2. **Boundary condition for tangential component of magnetic field**:

Text, 5-7
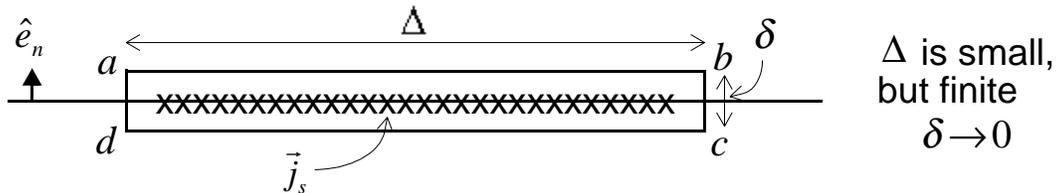
$$\oint \vec{H} \cdot d\vec{l} = \int_{abcd} \vec{j} \cdot d\vec{s} + \frac{d}{dt} \int_{abcd} \vec{D} \cdot d\vec{s}$$

The second term on the right hand side goes to zero, since the area is infinitely small.

However, the first term on the right hand side does **not** go to zero. The reason is that, although the area is infinitely small, $\vec{j}$ can be infinitely large. This means that a surface current $\vec{j}_s$ can exist on the boundary, normal to the area $abcd$. We have already discussed this before.

$$j_s = \vec{j} \cdot \vec{\delta} \qquad \left. \begin{matrix} \vec{\delta} \rightarrow 0 \\ \vec{j} \rightarrow \infty \end{matrix} \right\} \quad \vec{j}_s \text{ finite}$$

Total current enclosed in the area $abcd$ then is $\Delta \cdot j_s$. The direction of j_s must be into the paper if H_t on ab is in the $a \rightarrow b$ direction. Then, the left hand side becomes (by the same reasoning as in 1):

$$\int_a^b \vec{H} \cdot d\vec{l} = H_t \Delta$$

Thus, $H_t \Delta = j_s \Delta$

$$\therefore H_t = j_s, \text{ or } \hat{e}_n \times \vec{H} = \vec{j}_s \qquad \begin{array}{l}\text{This also gives}\\\text{correct direction}\end{array}$$

$$\uparrow$$

$(\hat{e}_n)$ is **outward** normal

($\hat{e}_n$ is frequently written more simply as $\hat{n}$.
This is the notation our text uses.)

3. **Boundary condition on normal component of electric field**:

small, but finite, area ΔA

infinitesimal thickness δ
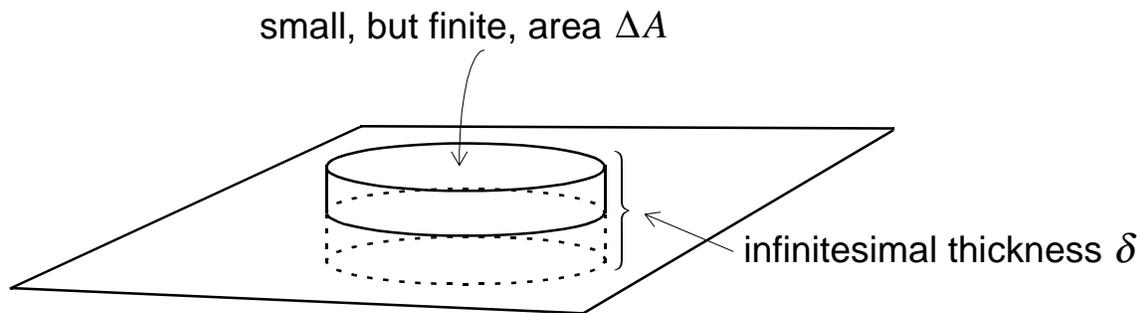
$$\int\limits_{surface\,of\,box} \vec{D} \cdot d\vec{s} = \int\limits_{volume\,of\,box} \rho\,dv$$

If there is only a volume charge density, the right hand side would be zero, since the volume of the box is infinitely small.

However, there can be a surface charge density on the surface of the conductor. Because of that:

right hand side $= \Delta A \cdot \rho_s$

left hand side $= \int\limits_{side\,surface} \vec{D} \cdot d\vec{s} + \int\limits_{top} \vec{D} \cdot d\vec{s} + \int\limits_{bottom} \vec{D} \cdot d\vec{s}$

The integral of $\vec{D} \cdot d\vec{s}$ over the side surface is zero, since its height is infinitesimal. The integral over the bottom surface is zero because the field is zero inside.

$$\therefore \text{LHS} = \int_{top} \vec{D} \cdot d\vec{s} = D_n \, \Delta A$$

positive, if $\vec{D}$ is pointing away from conductor

Thus, $\Delta A \rho_s = \Delta A D_n$

$$D_n = \rho_s, \quad \text{or } \hat{e}_n \cdot \vec{D} = \rho_s$$

Note that ρ_s is positive for $\vec{D}$ pointing outward, as it should be (Field lines originate on positive charges).

4. **Boundary condition on normal component of magnetic field**:

We apply $\int \vec{B} \cdot d\vec{s} = 0$ to the same pillbox as in 3.

Then, obviously, since RHS $= 0$, we get:

$$B_n = 0, \quad \text{or } \hat{e}_n \cdot \vec{B} = 0$$

Summary of B.C. At the surface of a perfect conductor:

$E_t = 0 \quad \hat{e}_n \times \vec{E} = 0$ Tangential component of $E = 0$.

$H_t = j_s \quad \hat{e}_n \times \vec{H} = \vec{j}_s$ Tangential component of $\vec{H} =$ surface current density.

$D_n = \rho_s \quad \hat{e}_n \cdot \vec{D} = \rho_s$ Normal component of $\vec{D} =$ surface charge density.

$B_n = 0 \quad \hat{e}_n \cdot \vec{B} = 0$ Normal component of $\vec{B} = 0$.

Boundary conditions at boundary between two dielectrics (or two general media). **No surface charges or currents:** $\rho_s = 0, j_s = 0$

We use exactly the same methods as we did in the previous sections. The main difference are that:

    a. There are fields on both sides of the boundary, since only inside perfect conductors are the field's zero.

    b. Under normal conditions, there is no surface charge on the boundary.

    c. There is no surface current on boundary, since surface currents can exist only on a perfect conductor surface.

1. Tangential Electric Field Component:

$$\int_a^b \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} = 0$$

$$\vec{E}_{t1}\Delta - \vec{E}_{t2}\Delta = 0$$

$$\therefore \vec{E}_{t1} = \vec{E}_{t2}, \ \text{ or } \ \hat{e}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0$$
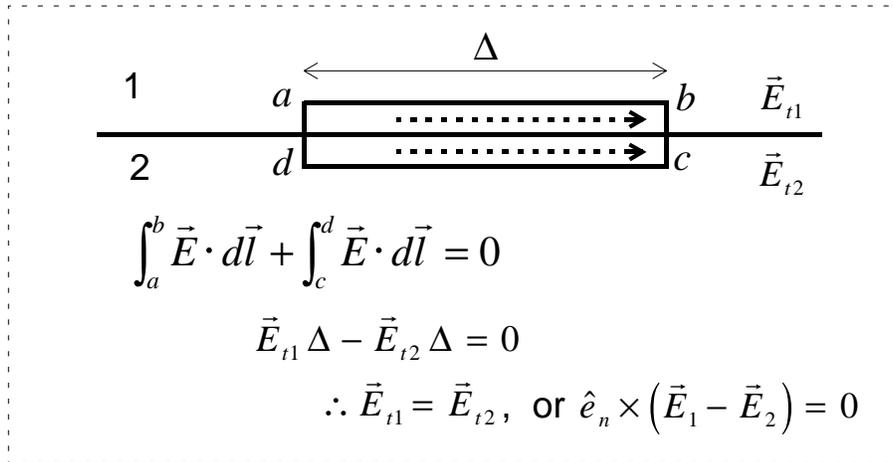
2. Tangential Magnetic Field Component:

Since $j_s = 0$, we get the same result as in 1.

$$H_{t1} = H_{t2}, \ \text{ or } \ \hat{e}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = 0$$

3. Normal Electric Field Component:

Since $\rho_s = 0$, $\displaystyle\int_{top} \vec{D} \cdot d\vec{s} + \int_{bottom} \vec{D} \cdot d\vec{s} = 0$

$$D_{n1} = D_{n2}, \ \text{ or } \ \hat{e}_n \cdot \left(\vec{D}_1 - \vec{D}_2\right) = 0$$

4. Normal Magnetic Field Component:

Same as 3.

$$B_{n1} = B_{n2}, \ \text{ or } \ \hat{e}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0$$

5. **Examples Useful for Remembering 2. and 3.**

$$H = 0$$
$$H = j_s \qquad \text{parallel plate}$$
$$H = 0 \qquad \text{currents}$$

$$E = 0$$
$$E = \rho_s / \varepsilon \qquad \text{parallel plate}$$
$$E = 0 \qquad \text{capacitor}$$

More on Boundary Conditions

A. It can be easily shown that an **equivalent form** of Boundary Condition 1. (that is, tangential $\vec{E}$ continuous: $E_{t1} = E_{t2}$) is:

$V_1 = V_2$ That is, potential is continuous at an interface.

B. Consider $\hat{e}_n \cdot \vec{D} = \rho_s$, which is the boundary condition for the normal component of the electric displacement at the interface between a perfect conductor and a dielectric.

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{E} = -\nabla V$$

Then: $E_n = -\hat{e}_n \cdot \nabla V = -\dfrac{\partial V}{\partial n}$

$$\therefore \rho_s = -\varepsilon \frac{\partial V}{\partial n} \, ,$$

where $\dfrac{\partial V}{\partial n}$ means the normal derivative (that is, derivative in the direction of the normal out of the conducting surface). Example shortly!

C. If the **value** of the potential V is given on a conducting surface, that too is called a boundary condition (to Laplace's Equation).