

VectPlot: A Mathematica Notebook

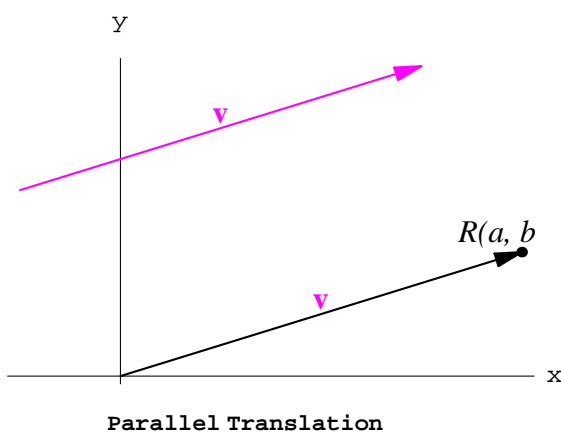
Plotting 2- and 3-Dimensional Vectors

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This notebook demonstrates plotting of 2- and 3-dimensional vectors in *Mathematica*.

1. Basic concepts. Vectors represent mathematical quantities with both *magnitude* (length) and *direction*. The usual practice is to draw a vector \mathbf{v} from a beginning point P to an endpoint Q , with an arrowhead at Q to identify it as the terminal point of the vector. Two vectors \mathbf{v} and \mathbf{w} are considered identical if they have the same length and point in the same direction, even if their initial points are different.

This leads to the algebraic representation of vectors by coordinates. Given a vector \mathbf{v} , translate it parallel to itself (to preserve its direction) until its initial point is the origin. Then the *coordinates* of \mathbf{v} are the coordinates, say (a, b) , of the terminal point R of the translated \mathbf{v} . (See the figure below.) The *length* or *magnitude* $\|\mathbf{v}\|$ of \mathbf{v} is the length of the segment from the origin to the terminal point of \mathbf{v} :



$$(1) \quad \|\mathbf{v}\| = \sqrt{a^2 + b^2}.$$

Note: In the current edition of the text, a single vertical bar denotes the length of a vector. Here, the double-vertical bars — notation of the next edition of the text — appear, to make it easy to distinguish magnitude of a vector from the absolute value of a real number.

Vector arithmetic reduces to carrying out the ordinary arithmetic of real numbers separately on each coordinate:

Definition (Vector Operations). The *sum* of two vectors $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ is the vector

$$\mathbf{v} + \mathbf{w} = (a + c, b + d).$$

The product of the vector $\mathbf{v} = (a, b)$ by the *scalar* (real number) r is

$$r\mathbf{v} = (ra, rb).$$

The *vector difference* of $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ in that order is

$$\mathbf{v} - \mathbf{w} = (a - c, b - d).$$

Note that the vector difference $\mathbf{v} - \mathbf{w}$ is just the sum of the vector \mathbf{v} and the scalar multiple $(-1)\mathbf{w}$ of the vector \mathbf{w} by -1 , which is the *negative* of \mathbf{w} .

The length of the scalar multiple $r\mathbf{v}$ of \mathbf{v} by r is $|r|$ times the length of \mathbf{v} , as the following calculation shows:

$$(2) \quad \|r\mathbf{v}\| = \|(ra, rb)\| = \sqrt{r^2 a^2 + r^2 b^2} = |r| \sqrt{a^2 + b^2} = |r| \|\mathbf{v}\|.$$

Note that multiplication of a vector by a negative number causes the direction of the vector to reverse. Thus, the vector $r\mathbf{v}$ has the same direction as \mathbf{v} if $r > 0$, but has opposite direction if $r < 0$.

Mathematica has a standard graphics package `Arrow` that can plot vectors as directed line segments in the plane. The next short routine illustrates its use on a pair of simple vectors.

Example 1. Let $\mathbf{v} = (-2, 5)$ and $\mathbf{w} = (3, -1)$. Then find and plot (a) $\mathbf{v} + \mathbf{w}$, (b) $\mathbf{v} - \mathbf{w}$, (d) $2\mathbf{v} - 3\mathbf{w}$. (e) Compute $\|2\mathbf{v} - 3\mathbf{w}\|$.

Solution. While *Mathematica* can deal with vector arithmetic, it is certainly easy enough to do these computations without technological aids! Indeed,

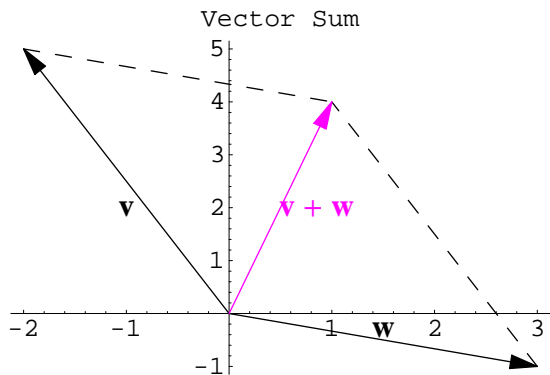
$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (-2 + 3, 5 - 1) = (1, 4), \\ \mathbf{v} - \mathbf{w} &= (-2 - 3, 5 - [-1]) = (-5, 6),\end{aligned}$$

and

$$2\mathbf{v} - 3\mathbf{w} = (-4, 10) - (9, -3) = (-13, 13).$$

The following *Mathematica* program illustrates how to plot the vectors \mathbf{v} , \mathbf{w} , and $\mathbf{v} + \mathbf{w}$. It is easy to modify it to deal with other vectors \mathbf{v} and \mathbf{w} , and other scalars r and s — such as arise in parts (b) and (d). Note that ordered pairs such as $(-2, 5)$ must be entered with braces, *not* parentheses. (The semicolons at the ends of the first four lines suppress printing of the quantities those lines define.) To execute the routine and generate the figure, place the cursor at the end of the blue code, and hit the Enter key (at the lower right edge of the standard Macintosh keyboard).

```
In[11]:= v := {-2, 5};
w := {3, -1};
r := 1;
s := 1;
Needs["Graphics`Arrow`"]
Show[Graphics[{Arrow[{0, 0}, v], Arrow[{0, 0}, w],
{Dashing[{0.03, 0.03}], Line[{v, v + w}], Line[{w, v + w]}},
Text[FontForm["v", {"Times-Bold", 12}], { $\frac{v[[1]]}{2}$ , .4 v[[2]]}],
Text[FontForm["w", {"Times-Bold", 12}],
{ $\frac{w[[1]]}{2}$ , .3 w[[2]]}], RGBColor[1, 0, 1], Arrow[{0, 0}, {1, 4}],
Text[FontForm["v + w", {"Times-Bold", 12}],
{.85 (v + w)[[1], .5 (v + w)[[2]]}],
Axes -> True, PlotLabel -> "Vector Sum"]]
```



Out[15]= - Graphics -

Note: As written, this routine includes two `Lines` commands to connect the endpoints of the vectors \mathbf{v} and \mathbf{w} to the endpoint of $\mathbf{v} + \mathbf{w}$. That illustrates the *parallelogram law*: geometrically, the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram corresponding to the vectors \mathbf{v} and \mathbf{w} , assuming that both \mathbf{v} and \mathbf{w} are nonzero. Since the other linear combinations of this example don't represent the sum of \mathbf{v} and \mathbf{w} , before plotting those delete (or comment out, by putting `(*` before and `*)` at the end of line that has the `Dashing` command.

(e) Since $(-13, 13) = 13(-1, 1)$, by (2) the length of $2\mathbf{v} - 3\mathbf{w}$ is 13 times the length of $(-1, 1)$:

$$\|2\mathbf{v} - 3\mathbf{w}\| = 13 \|(-1, 1)\| = \sqrt{1^2 + 1^2} = 13\sqrt{2} \approx 18.38478.$$

The following routine plots $\mathbf{L} = (1/2)(2\mathbf{v} - 3\mathbf{w})$ and the unit vector $\mathbf{u} = (1/\sqrt{2})(-1, 1)$ in the direction of \mathbf{L} . (See Definition 2.12.) The long length of $2\mathbf{v} - 3\mathbf{w}$ creates scaling problems in plotting both that vector and the unit vector in its direction: all that would be visible of \mathbf{u} would be its arrowhead. This accounts for the program using half $2\mathbf{v} - 3\mathbf{w}$ instead of that vector itself. Execute the program as above.

In[16]:= `Needs["Graphics`Arrow`"]`

$$\mathbf{L} := \left\{ -\frac{13}{2}, \frac{13}{2} \right\}$$

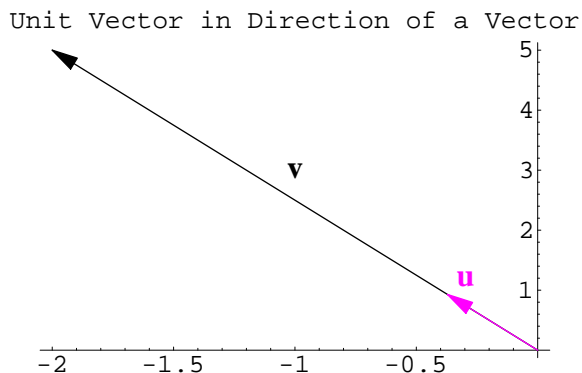
$$\mathbf{u} := \frac{\mathbf{v}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}$$

`Show[Graphics[{Arrow[{0, 0}, v],`

`Text[FontForm["v", {"Times-Bold", 12}], { $\frac{v[[1]]}{2}$, .6 v[[2]]}],`

`RGBColor[1, 0, 1], Arrow[{0, 0}, u],`

```
Text[FontForm["u", {"Times-Bold", 12}],
      {.8 u[[1]], 1.3 u[[2]]}], Axes → True,
PlotLabel → "Unit Vector in Direction of a Vector"]]
```



Out[19]= - Graphics -

The formal definition of two vectors being parallel (Definition 2.13) is that their direction vectors are equal or differ by a factor of -1 . However, the most convenient way to test whether two given vectors are actually parallel is to use the following criterion. Its equivalence to the definition results from the fact that a nonzero multiple of a given vector has the same or opposite direction as the given vector. (See (2) above.)

Two nonzero vectors \mathbf{v} and \mathbf{w} are parallel if and only if $\mathbf{w} = k \mathbf{v}$ for some nonzero real number k .

This makes it easy to see, for example, that the two vectors $(-6, -1)$ and $(2, 1/3) = -(1/3)(-6, -1)$ are parallel.

Another important fact about basic vector arithmetic concerns the geometric interpretation of the difference of two vectors. As the next figure illustrates, it follows from the parallelogram law of addition (see the figure for Example 1(a) above) that the vector difference $\mathbf{v} - \mathbf{w}$ is the vector that when added to \mathbf{w} gives the vector \mathbf{v} . From this it is easy to derive the formula for the *position vector* from a point $A(a_1, a_2)$ to a point $B(b_1, b_2)$. For let $\mathbf{v} = (a_1, a_2) = \mathbf{OA}$ and let $\mathbf{w} = (b_1, b_2) = \mathbf{OB}$. Then (see the figure below) from the parallelogram law

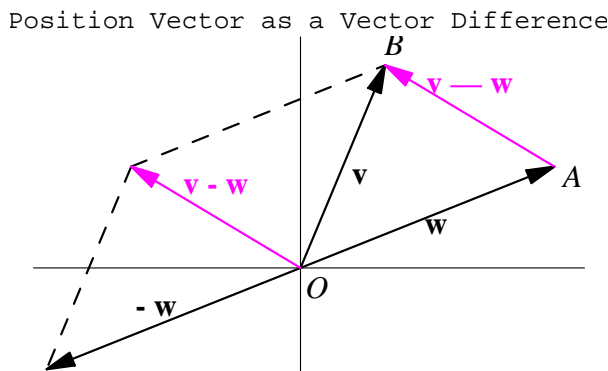
$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB}, \text{ that is, } \mathbf{w} = \mathbf{v} + \mathbf{AB},$$

so that

$$(3) \quad \mathbf{AB} = \mathbf{w} - \mathbf{v} = (b_1 - a_1, b_2 - a_2).$$

To summarize the discussion, Equation (3) says that

The vector $\mathbf{v} - \mathbf{w}$ is representable as the vector drawn from the endpoint of \mathbf{w} to the endpoint of \mathbf{v} .



Another important idea is the representation of vectors in terms of the *standard basis* for \mathbf{R}^2 , which is the ordered set $\{\mathbf{i}, \mathbf{j}\}$, where the vectors \mathbf{i} and \mathbf{j} are

$$(4) \quad \mathbf{i} = (1, 0) \text{ and } \mathbf{j} = (0, 1).$$

It follows immediately from (4) that any vector $\mathbf{v} = (a, b)$ in the plane is a unique linear combination of the standard basis vectors, namely,

$$\mathbf{v} = (a, b) = a \mathbf{i} + b \mathbf{j}.$$

For instance, in Example 1(a) above the vector $\mathbf{v} = (-2, 5)$ is $-2\mathbf{i} + 5\mathbf{j}$ and the vector $\mathbf{w} = (3, -1)$ is $3\mathbf{i} - \mathbf{j}$. (See also Definition 1.16).

2. Three-dimensional vectors. The real plane \mathbf{R}^2 consists of all ordered pairs (x, y) of real numbers. Similarly, *real 3-dimensional Euclidean* or *Cartesian space* is the set of all ordered *triples* (x, y, z) of real numbers. Representing 3-dimensional space on a 2-dimensional surface such as a piece of paper, a chalkboard, or a computer screen is not simple. Indeed, a precise description of that process requires ideas from linear algebra, which you most likely have not studied. The text represents the points of 3-space with

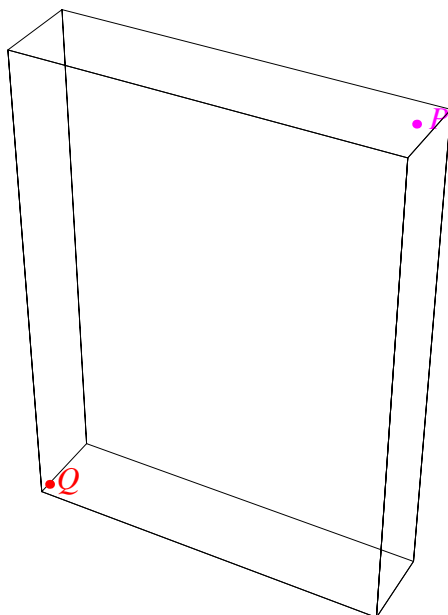
the aid of a *right-handed* coordinate system, like the one shown in the following figures. To locate a point P in \mathbf{R}^3 with coordinates (x, y, z) ,

- first go to the point $Q(x, y)$ in the xy -plane as usual, and then
- go up (if $z > 0$) or down (if $z < 0$) the distance $|z|$ units to reach the point.

The following figure illustrates this with the points $P(2, 2, 3)$ and $Q(-3, 1, -4)$. Note that *Mathematica* plots 3-dimensional points via its built-in `Graphics3D` command. It is also possible to add text, such as labels, at specified places in a figure, and to specify both the size and color of plotted points. As before, execute the following code and watch the picture develop.

```
In[20]:= P = {2, 2, 3};
Q = {-3, 1, -4};
plotpoints =
  {Graphics3D[{PointSize[0.02], RGBColor[1, 0, 1], Point[P],
    Text[FontForm["P", {"Times-Italic", 12}], {2.2, 2.2, 3}]}],
  Graphics3D[{PointSize[0.02], RGBColor[1, 0, 0], Point[Q],
    Text[FontForm["Q", {"Times-Italic", 12}], {-2.8, 1.2, -4}]}]}
Show[plotpoints]
```

```
Out[22]= {- Graphics3D -, - Graphics3D -}
```



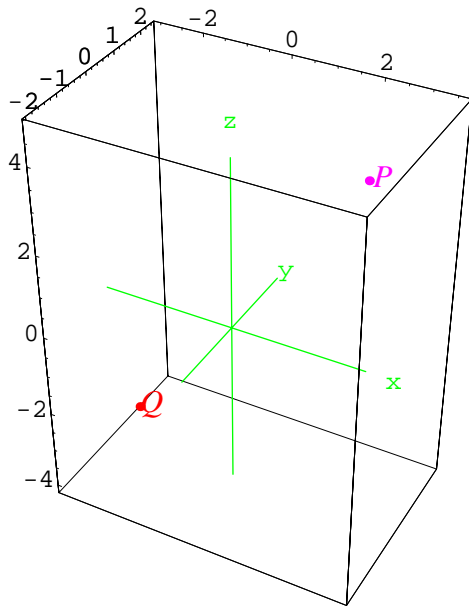
```
Out[23]= - Graphics3D -
```

By default, *Mathematica* draws a box around its three-dimensional diagrams. Its goal is to help your eye better perceive the main features of the diagram. In this case, for instance, the surrounding box emphasizes that the point P lies above, to the right of, and behind the point Q . (Compare Figure 1.9 of the text.) Mathematical diagrams customarily use another means of orienting spatial relationships: the inclusion of coordinate axes and scales. The following modified plot retains the surrounding box, and adds coordinate axes labeled in the customary right-handed fashion: if you point your right index finger in the direction of the positive x axis and your right middle finger in the direction of the positive y axis, then your thumb points in the direction of the positive z axis. The command `Axes -> True` causes *Mathematica* to add ticks, but on the surrounding box rather than the coordinate axes themselves as in the text. As you can see from the resulting figure, *Mathematica's* style makes it easy to perceive the relative size of plotted points' coordinates.

The code introduces variables `xwin`, `yext`, and `zlen` that specify how long a portion of each coordinate axis to plot. (They are not named in a uniform fashion — such as `xwin`, `ywin`, and `zwin` — to forestall *Mathematica's* reporting of possible spelling errors!) In each case, the axis length is twice the absolute value of the largest coordinate that appears in the points P and Q . (The semicolons suppress printing of the values of `xwin`, `yext`, and `zlen`.) To save space, the code from the last program that defines `plotpoints` is not repeated: once that program executes, its information is available in memory as long as your session lasts. That is good to keep in mind.

```
In[24]:= xwin = 3;
         yext = 2;
         zlen = 4;
         coordaxes = Graphics3D[
           {{RGBColor[0, 1, 0], Line[{{-xwin, 0, 0}, {xwin, 0, 0}}]},
            Text["x", {xwin +  $\frac{xwin}{5}$ , 0, 0}]},
           {RGBColor[0, 1, 0], Line[{{0, -yext, 0}, {0, yext, 0}}]},
            Text["y", {0, yext +  $\frac{yext}{5}$ , 0}]},
           {RGBColor[0, 1, 0], Line[{{0, 0, -zlen}, {0, 0, zlen}}]},
            Text["z", {0, 0, zlen +  $\frac{zlen}{5}$ }]}}]
         Show[plotpoints, coordaxes, Axes -> True]
```

```
Out[27]= - Graphics3D -
```

Out[28]= - Graphics3D -

Plotting Points in Three - Space

Just as in two dimensions, vectors in 3-space arise as quantities with both length and direction. Unfortunately, *Mathematica* still does not have an `arrow` command for 3-dimensional plotting, so plotting vectors in 3-space is more challenging. The program below uses the `Polygon` command to put (fairly crude) arrowheads on the end of a vector, cutting the vector about 90% of the way from its initial to terminal point. Without use of advanced concepts of computer-graphics programming, the resulting rendering is much less professional than that above for 2-dimensional vectors. The program below illustrates the vectors \mathbf{v} , \mathbf{w} , and $\mathbf{v} + \mathbf{w}$ from Example 1.12 of the text. Execute it and compare the result with Figure 1.17 of the text.

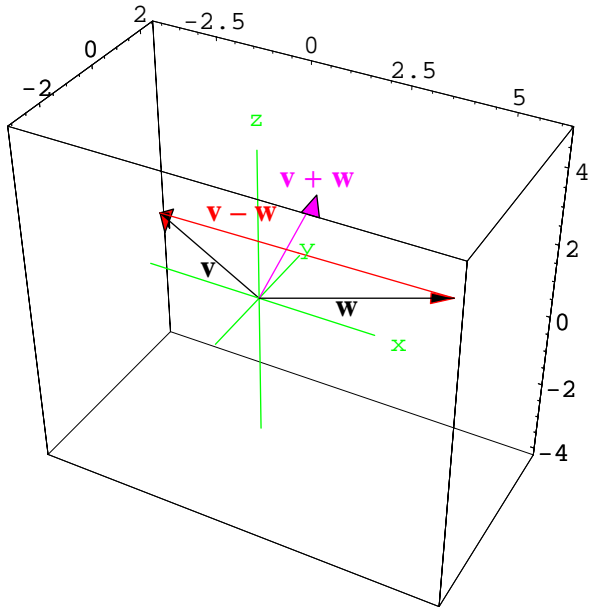
```
In[29]:= v = {-4, 2, 0};
w = {6, -3, 4};
q =  $\sqrt{\mathbf{v} \cdot \mathbf{v}}$ ;
r =  $\sqrt{\mathbf{w} \cdot \mathbf{w}}$ ;
s =  $\sqrt{(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})}$ ;
vplot =
Graphics3D[{{Line[{{0, 0, 0}, v}], Polygon[{{v, {.9 v[[1], .9 v[[2],
.9 v[[3] -  $\frac{1}{q}}$ }, {.9 v[[1], .9 v[[2], .9 v[[3] +  $\frac{1}{q}}$ }}]},
Text[FontForm["v", {"Times-Bold", 12}],
```

```

    {.5 v[[1]], .5 v[[2]], .5 v[[3]] -  $\frac{2}{q}$ }}],
  {Line[{{0, 0, 0}, w}], Polygon[{w, {.9 w[[1]], .9 w[[2]],
    .9 w[[3]] -  $\frac{1}{r}$ }, {.9 w[[1]], .9 w[[2]], .9 w[[3]] +  $\frac{1}{r}$ }]},
  Text[FontForm["w", {"Times-Bold", 12}],
    {.5 w[[1]], .5 w[[2]], .5 w[[3]] -  $\frac{2}{r}$ }}],
  {RGBColor[1, 0, 1], Line[{{0, 0, 0}, v + w}], Polygon[
    {v + w, {.9 (v + w) [[1]] -  $\frac{1}{s}$ , .9 (v + w) [[2]], .9 (v + w) [[3]] -  $\frac{1}{s}$ },
    {.9 (v + w) [[1]] +  $\frac{1}{s}$ , .9 (v + w) [[2]], .9 (v + w) [[3]] -  $\frac{1}{s}$ }]},
  Text[FontForm["v + w", {"Times-Bold", 12}],
    {(v + w) [[1]], (v + w) [[2]], 1.1 (v + w) [[3]]}],
  {RGBColor[1, 0, 0], Line[{v, w}],
  Polygon[{v, {.9 v[[1]], .9 v[[2]] -  $\frac{1}{q}$ , .9 v[[3]] -  $\frac{1}{q}$ },
    {.9 v[[1]], .9 v[[2]] +  $\frac{1}{q}$ , .9 v[[3]] +  $\frac{1}{q}$ }]},
  Text[FontForm["v - w", {"Times-Bold", 12}],
    {.3 (2 v + w) [[1]], .3 (2 v + w) [[2]], .5 (2 v + w) [[3]]}]]]
Show[vplot, coordaxes, Axes → True, Lighting → False]

```

Out[34]= - Graphics3D -



Out[35]= - Graphics3D -