Stable convergence of inner functions

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Abstract

Let $\mathscr{J}$ be the set of inner functions whose derivative lies in the Nevanlinna class. It is natural to record the critical structure of $F \in \mathscr{J}$ by the inner part of its derivative. In this paper, we discuss a natural topology on $\mathscr{J}$ where $F_n \to F$ if the F_n converge uniformly on compact subsets to F and the critical structures of F_n converge to that of F. We show that this occurs precisely when the critical structures of the F_n are uniformly concentrated on Korenblum stars. Building on the works of Korenblum and Roberts, we show that this topology also governs the behaviour of invariant subspaces of a weighted Bergman space which are generated by a single inner function.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. An *inner function* is a holomorphic self-map of the unit disk such that for almost every $\theta \in [0, 2\pi)$, the radial limit $\lim_{r\to 1} F(re^{i\theta})$ exists and has absolute value 1. Let Inn denote the space of all inner functions and $\mathscr{J} \subset \mathrm{Inn}$ be the subspace consisting of inner functions which satisfy

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F'(re^{i\theta})| d\theta < \infty, \tag{1.1}$$

that is, with F' in the Nevanlinna class. The work of Ahern and Clark [1] implies that if $F \in \mathscr{J}$, then F' admits an "inner-outer" decomposition

$$F' = \mathrm{Inn}\, F' \cdot \mathrm{Out}\, F'.$$

Intuitively, $\operatorname{Inn} F' = BS$ describes the "critical structure" of the map F – the Blaschke factor records the locations of the critical points of F in the unit disk, while the singular inner factor describes the "boundary critical structure." In [9], the author proved the following theorem, answering a question posed in [5]:

Theorem 1.1. *Let $\mathscr{J}$ be the set of inner functions whose derivative lies in the Nevanlinna class. The natural map*

$$F \to \operatorname{Inn}(F') \quad : \quad \mathscr{J} / \operatorname{Aut}(\mathbb{D}) \to \operatorname{Inn} / \mathbb{S}^1$$

is injective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and S_μ is the singular factor associated to a measure μ whose support is contained in a countable union of Beurling-Carleson sets.

The above theorem says that an inner function $F \in \mathscr{J}$ is uniquely determined up to a post-composition with a holomorphic automorphism of the disk by its critical structure and describes all possible critical structures of inner functions. We need to quotient Inn by the group of rotations since the inner part is determined up to a unimodular constant. To help remember this, note that *Frostman shifts or post-compositions with elements of* $\operatorname{Aut}(\mathbb{D})$ *do not change the critical set of a function while rotations do not change the zero set.*

By definition, a *Beurling-Carleson set* $E \subset \mathbb{S}^1$ is a closed subset of the unit circle of zero Lebesgue measure whose complement is a union of arcs $\bigcup_k I_k$ with

$$\|E\|_{\mathcal{BC}} = \sum |I_k| \log \frac{1}{|I_k|} < \infty.$$

We say that $E \in \mathcal{BC}(N)$ if $\|E\|_{\mathcal{BC}} \le N$. We denote the collection of all Beurling-Carleson sets by $\mathcal{BC}$.

We will also need the notion of a *Korenblum star* which is the union of Stolz angles emanating from a Beurling-Carleson set $E \subset \mathbb{S}^1$:

$$K_E = B(0, 1/\sqrt{2}) \cup \big\{ z \in \overline{\mathbb{D}} : 1 - |z| \le \operatorname{dist}(\hat{z}, E) \big\}.$$

Here, $\hat{z} = z/|z|$ while dist denotes the Euclidean distance. With the above definition, $K_E \subset \overline{\mathbb{D}}$ is a closed set. We say that the Korenblum star has *norm* $\|E\|_{\mathcal{BC}}$.

We endow $\mathscr{J}$ with the topology of *stable convergence* where $F_n \to F$ if the F_n converge uniformly on compact subsets of the disk to F and the Nevanlinna splitting is preserved in the limit: $\operatorname{Inn} F_n' \to \operatorname{Inn} F'$, $\operatorname{Out} F_n' \to \operatorname{Out} F'$. As observed in [9], in general, one always has semicontinuity in one direction:

$$\operatorname{Inn} F' \geq \limsup(\operatorname{Inn} F_n'). \tag{1.2}$$

Loosely speaking, our main result says that to determine $\operatorname{Inn} F'$, one needs to take the part of the limit of $\operatorname{Inn} F_n'$ that is "uniformly concentrated" on Korenblum stars. A precise statement will be given later in the introduction.

Two examples. If F_n is a finite Blaschke product of degree $n+1$ which has a critical point at $1 - 1/n$ of multiplicity n, and is normalized so that $F_n(0) = 0$, $F_n'(0) > 0$, then the F_n converge to $S_{\delta_1} = \exp\left(\frac{z+1}{z-1}\right)$, the unique inner function with critical structure S_{δ_1}. However, if F_n has n critical points (of multiplicity one) which are sufficiently spread out on the circle $\{z : |z| = 1 - 1/n\}$, then the F_n may converge to the identity mapping even if $\operatorname{Inn} F_n' \to S_{\delta_1}$.

1.1 Measures supported on Beurling-Carleson sets

Let $M_{\mathcal{BC}(N)}(\mathbb{S}^1)$ denote the class of finite positive measures that are supported on a Beurling-Carleson set of norm $\leq N$ and $M_{\mathcal{BC}}(\mathbb{S}^1)$ denote the collection of measures supported on a countable union of Beurling-Carleson sets. For a fixed $N > 0$, the space $M_{\mathcal{BC}(N)}(\mathbb{S}^1)$ comes equipped with the weak topology of measures. A simple "normal families" argument shows that it is a closed subset of $M(\mathbb{S}^1)$, the space all measures that live on the unit circle, see [7, Lemma 7.6].

We endow $M_{\mathcal{BC}}(\mathbb{S}^1)$ with the *inductive limit topology*. Roughly speaking, a sequence of positive measures μ_n converges to μ if up to small error, they converge in $M_{\mathcal{BC}(N)}(\mathbb{S}^1)$. More precisely, for any $\varepsilon > 0$, we want there to exist an $N > 0$ and a "dominated" sequence $\nu_n \to \nu$ such that for all n sufficiently large,

(i) $0 \leq \nu_n \leq \mu_n$,

(ii) $\nu_n \in M_{\mathcal{BC}(N)}(\mathbb{S}^1)$,

(iii) $(\mu_n - \nu_n)(\mathbb{S}^1) < \varepsilon$ and $(\mu - \nu)(\mathbb{S}^1) < \varepsilon$.

1.2 Measures supported on Korenblum stars

We say that a finite positive measure on the closed unit disk μ belongs to the space $M_{\mathcal{BC}(N)}(\overline{\mathbb{D}})$ if its support is contained in a Korenblum star of norm $\le N$, while $\mu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ if its restriction $\mu|_{\mathbb{S}^1} \in M_{\mathcal{BC}}(\mathbb{S}^1)$. We define the *Korenblum topology* on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ by specifying that a sequence of measures $\mu_n \to \mu$ converges in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ if it does so weakly, and up to small error, for all sufficiently large n, most of the mass of μ_n is contained in a Korenblum star K_{E_n} with $\|E_n\|_{\mathcal{BC}} \le N$.

Taking inspiration from the work of Marcus and Ponce [15], we use the word *concentrating* for sequences of measures which converge in this topology. We say that a sequence $\{\mu_n\} \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ is *equidiffuse* if for any $N > 0$,

$$\sup_{E \in \mathcal{BC}(N)} \mu_n(K_E) \to 0, \qquad \text{as } n \to \infty.$$

It is not hard to decompose a convergent sequence $\mu_n \to \mu$ into concentrating and equidiffuse components (that is, to write $\mu_n = \tau_n + \nu_n$ with $\tau_n \to \tau$ and $\nu_n \to \nu$, where τ_n is equidiffuse and ν_n is concentrating). Even though there are infinitely many choices for the sequences $\{\tau_n\}$ and $\{\nu_n\}$, the limits τ and ν are uniquely determined by $\{\mu_n\}$. We leave the verification to the reader.

1.3 Embedding of inner functions

To an inner function I, we associate the measure

$$\mu(I) = \sum (1 - |a_i|)\delta_{a_i} + \sigma(I), \tag{1.3}$$

where the sum ranges over the zeros of I (counted with multiplicity) and $\sigma(I)$ is the singular measure on the unit circle associated with the singular factor of I. This gives an embedding $\operatorname{Inn} / \mathbb{S}^1 \to M(\overline{\mathbb{D}})$. We say that the measure μ records the *zero structure* of I and write $I_\mu := I$. Clearly, the function I_μ is uniquely determined up to a rotation.

We can also embed $\mathscr{J} / \operatorname{Aut}(\mathbb{D}) \to M_{\mathcal{BC}}(\overline{\mathbb{D}})$ by taking $F \to \mu(\operatorname{Inn} F')$. This embedding records the *critical structure* of F. We use the symbol F_μ to denote an inner function with $\operatorname{Inn} F'_\mu = I_\mu$ and $F_\mu(0) = 0$ (again, such a function is unique up to a rotation).

1.4 Main result

We can now state our main theorem:

Theorem 1.2. *The embedding* $\mathscr{J} / \operatorname{Aut}(\mathbb{D}) \to M_{\mathcal{BC}}(\overline{\mathbb{D}})$ *is a homeomorphism onto its image when* $\mathscr{J} / \operatorname{Aut}(\mathbb{D})$ *is equipped with the topology of stable convergence and* $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ *is equipped with the Korenblum topology. More precisely, if* $\mu_n \to \mu$ *in the weak topology of measures and* $F_{\mu_n} \to F_\nu$ *uniformly on compact subsets of the disk, then we can decompose* $\mu_n = \nu_n + \tau_n$ *so that* $\nu_n \to \nu$ *is concentrating and* $\tau_n \to \tau$ *is equidiffuse.*

In fact, our argument gives a slightly stronger result: the sequence $\nu_n \to \nu$ is concentrating if and only if the measures ν_n converge weakly, and up to a small error, the radial projections of ν_n to the unit circle are contained in Beurling-Carleson sets whose norms are uniformly bounded above.

1.5 Connections with the Gauss curvature equation

We now give an alternative (and slightly more general) perspective of our main theorem in terms of conformal metrics and nonlinear differential equations. Given a conformal pseudometric $\lambda(z)|dz|$ on the unit disk with an upper semicontinuous density, its *Gaussian curvature* is given by

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2},$$

where the Laplacian is taken in the sense of distributions. It is well known that the Poincaré metric $\lambda_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$ has constant curvature -4. For a holomorphic self-map of the unit disk $F \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$, consider the pullback

$$\lambda_F := F^* \lambda_{\mathbb{D}} = \frac{|F'|}{1 - |F|^2}.$$

Since curvature is a conformal invariant, e.g. see [13, Theorem 2.5], it follows that

$$k_{\lambda_F} = -4 - 2\pi \sum_{c \in \operatorname{crit}(F)} \lambda_F(c)^{-2} \cdot \delta_c, \tag{1.4}$$

where $\mathrm{crit}(F)$ denotes the critical set of F counted with multiplicity. After the change of variables $u_F = \log \lambda_F$, we naturally arrive at the PDE

$$\Delta u - 4e^{2u} = 2\pi\tilde{\nu}, \qquad \tilde{\nu} \geq 0, \tag{1.5}$$

where $\tilde{\nu} = \sum_{c \in \mathrm{crit}(F)} \delta_c$ is an integral sum of point masses. A theorem of Liouville [13, Theorem 5.1] states that the correspondence $F \to u_F$ is a bijection between

$$\mathrm{Hol}(\mathbb{D}, \mathbb{D}) \,/\, \mathrm{Aut}(\mathbb{D}) \quad \Longleftrightarrow \quad \big\{\text{solutions of (1.5) with } \tilde{\nu} \text{ integral}\big\}.$$

This allows us one to translate questions about critical points of holomorphic self-maps of the disk to problems in PDE.

It turns out that the question of describing inner functions with derivative in the Nevanlinna class is related to studying the Gauss curvature equation with *nearly-maximal* boundary values

$$\begin{cases} \Delta u - 4e^{2u} = 2\pi\tilde{\nu}, & \text{in } \mathbb{D}, \\ u_{\mathbb{D}} - u = \mu, & \text{on } \mathbb{S}^1, \end{cases} \tag{1.6}$$

where $u_{\mathbb{D}} = \log \lambda_{\mathbb{D}}$ is the pointwise *maximal* solution of (1.5) in the sense that it dominates all solutions of (1.5) with any $\tilde{\nu} \geq 0$. In (1.6), we allow $\tilde{\nu} \in M(\mathbb{D})$ to be any positive measure on the unit disk which satisfies the *Blaschke condition*

$$\int_{\mathbb{D}} (1 - |a|) d\tilde{\nu}(a) < \infty, \tag{1.7}$$

and $\mu \in M(\mathbb{S}^1)$ to be any finite positive measure on the unit circle. The first equality in (1.6) is understood weakly in the sense of distributions: we require $u(z)$ and $e^{2u(z)}$ to be in $L^1_{\mathrm{loc}}(\mathbb{D})$, and ask that for any test function $\phi \in C_c^\infty(\mathbb{D})$, compactly supported in the disk,

$$\int_{\mathbb{D}} (u\Delta\phi - 4e^{2u}\phi)|dz|^2 = 2\pi \int_{\mathbb{D}} \phi d\tilde{\nu}, \tag{1.8}$$

while the second equality expresses the fact that the measures $(u_{\mathbb{D}} - u)(d\theta/2\pi)|_{\{|z|=r\}}$ converge to μ as $r \to 1$. If μ and $\tilde{\nu}$ are as above, set

$$\omega(z) = \mu(z) + \nu(z) := \mu(z) + \tilde{\nu}(z)(1 - |z|) \in M(\overline{\mathbb{D}}). \tag{1.9}$$

Theorem 1.3. *Given a measure $\omega = \mu + \nu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, the equation (1.6) admits a unique solution, which we denote $u_{\mu,\nu}$ or u_ω. The solution u_ω is decreasing in ω, that is, $u_{\omega_1} > u_{\omega_2}$ if $\omega_1 < \omega_2$. If $\omega \notin M_{\mathcal{BC}}(\overline{\mathbb{D}})$ then no solution exists.*

We endow the space of solutions of (1.6) with the *stable topology* where $u_{\omega_n} \to u_\omega$ if the u_n converge weakly to u and the ω_n converge weakly to ω. According to [9, Lemma 3.3], if F_ω is an inner function with critical structure ω, then

$$u_\omega = \log \lambda_{F_\omega} = \log \frac{|F'_\omega|}{1 - |F_\omega|^2}.$$

Expressing Theorem 1.2 in this language, we arrive at the following statement:

Theorem 1.4. *The space of integral solutions of the Gauss curvature equation (when $\tilde{\nu}$ is an integral sum of delta masses) with nearly-maximal boundary values naturally embeds in $M_{\mathcal{BC}}(\overline{\mathbb{D}})$ equipped with the Korenblum topology.*

It is likely that the restriction to integral measures $\tilde{\nu}$ is an artifact of the proof and is not truly necessary:

Conjecture 1.5. *The stable topology on the space of solutions with nearly maximal boundary values coincides with the Korenblum topology on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$.*

The difficulty stems from the fact that for general measures $\tilde{\nu} \ge 0$, the Liouville map $F_\omega : \mathbb{D} \to \mathbb{D}$ realizing the conformal metric $\lambda_\omega = \exp u_\omega$ is multi-valued. This only affects our discussion of concentrating sequences (Section 3) as the study of the equidiffuse sequences (Section 4) does not utilize the connection with complex analysis.

1.6 Invariant subspaces of Bergman space

For a fixed $\alpha > -1$ and $1 \le p < \infty$, consider the weighted Bergman space $A^p_\alpha(\mathbb{D})$ which consists of all holomorphic functions on the unit disk satisfying the norm boundedness condition

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \cdot (1 - |z|)^\alpha |dz|^2 \right)^{1/p} < \infty. \tag{1.10}$$

For a function $f \in A^p_\alpha$, let $[f]$ denote the (closed) z-invariant subspace generated by f, that is the closure of the set $\{p(z)f(z)\}$, where $p(z)$ ranges over polynomials. In the work [11], Korenblum equipped subspaces of $A^p_\alpha(\mathbb{D})$ with the *strong topology* where $X_n \to X$ if any $x \in X$ can be obtained as a limit of a converging sequence of $x_n \in X_n$ and visa versa.

We focus our attention on a small but important subclass of invariant subspaces which are generated by a single inner function. Following [6], we say that an invariant subspace is of *κ-Beurling-type* if it is of the form $[BS_\mu]$ where B is a Blaschke product and the measure μ is supported on a countable union of Beurling-Carleson sets. (According to a classical theorem of Korenblum [10] and Roberts [20], if μ does not charge Beurling-Carleson sets, then $[S_\mu] = A^p_\alpha$.) We show:

Theorem 1.6. *For any $\alpha > -1$ and $1 \le p < \infty$, the strong topology on subspaces of κ-Beurling-type agrees with the Korenblum topology on $M_{\mathcal{BC}}(\overline{\mathbb{D}})$.*

In the work [12], Kraus proved that the critical sets of Blaschke products coincide with zero sets of functions in A^2_1. It is therefore plausible that inner functions (modulo Frostman shifts) are in bijection with the collection of z-invariant subspaces of A^2_1 satisfying the codimension one property. The work of Shimorin [21] on the approximate spectral synthesis in Bergman spaces is likely to be of use here.

2 The Gauss curvature equation

In this section, we prove Theorem 1.3 which identifies the nearly-maximal solutions of the Gauss curvature equation with $M_{\mathcal{BC}}(\overline{\mathbb{D}})$. Our main tool is the *Perron method* which we now describe. Consider the Gauss curvature equation

$$-\Delta u = -4e^{2u} - 2\pi\tilde{\nu}, \qquad \tilde{\nu} \ge 0, \tag{2.1}$$

with free boundary (that is, without imposing any restrictions on the behaviour of u near the unit circle). We interpret this in the weak sense: we require that for any non-negative function $\phi \in C^\infty_c(\mathbb{D})$,

$$-\int_{\mathbb{D}} u\Delta\phi\,|dz|^2 = -\int_{\mathbb{D}} 4e^{2u}\phi\,|dz|^2 - 2\pi\int_{\mathbb{D}} \phi\, d\tilde{\nu}. \tag{2.2}$$

Naturally, we say that u is a *subsolution* if one has $\leq$ in (2.2) while the word *supersolution* indicates the sign $\geq$.

Theorem 2.1 (Perron method)**.** *Suppose u is a function on the unit disk which is a subsolution of the Gauss curvature equation (2.1) with free boundary, where $\tilde{\nu} \geq 0$ is a locally finite measure on the unit disk. There exists a unique minimal solution $\Lambda^{\tilde{\nu}}[u]$ which exceeds u. If $\overline{u}$ is a supersolution with $\overline{u} \geq u$ then $\overline{u} \geq \Lambda^{\tilde{\nu}}[u]$.*

The Perron method was first applied to the Gauss curvature equation by Heins [8]; however, since we are dealing with measure-valued singularities, we require more modern machinery [14, 16]. The proof of Theorem 2.1 will be given in Appendix B.

If u is a subsolution of (2.1) defined on $\mathbb{D}_r = \{z : |z| < r\}$, $0 < r < 1$, we use the symbol $\Lambda_r^{\tilde{\nu}}[u]$ to denote the minimal dominating solution on $\mathbb{D}_r$. With this definition, $\Lambda_r^{\tilde{\nu}}[u]$ does not depend on $\tilde{\nu}|_{\mathbb{D}\setminus\mathbb{D}_r}$. One can alternatively describe $\Lambda_r^{\tilde{\nu}}[u]$ as the unique solution of (2.1) on $\mathbb{D}_r$ which agrees with u on $\partial\mathbb{D}_r$.

We record several elementary properties of Perron hulls which follow directly from the definition:

Lemma 2.2. (i) *If $u \geq v$ are two subsolutions of (2.1) then $\Lambda_r^{\tilde{\nu}}[u] \geq \Lambda_r^{\tilde{\nu}}[v]$.*

(ii) *If $\tilde{\nu}_1 \leq \tilde{\nu}_2$ then $\Lambda_r^{\tilde{\nu}_1}[u] \geq \Lambda_r^{\tilde{\nu}_2}[u]$.*

Lemma 2.3. *Suppose u is a subsolution of (2.1) on the unit disk. The family $\Lambda_r^{\tilde{\nu}}[u]$ is non-decreasing in r and*

$$\Lambda^{\tilde{\nu}}[u] \; = \; \lim_{r \to 1} \Lambda_r^{\tilde{\nu}}[u]. \tag{2.3}$$

Lemma 2.4. *Suppose u_n is a sequence of subsolutions of (2.1) with measures $\tilde{\nu}_n$. If $u_n \to u$ and $\tilde{\nu}_n \to \tilde{\nu}$ weakly on the unit disk, then for any $0 < r < 1$,*

$$\liminf_{n \to \infty} \Lambda_r^{\tilde{\nu}_n}[u_n] \geq \Lambda_r^{\tilde{\nu}}[u].$$

The same statement also holds with Λ in place of Λ_r.

The proof of the above lemma uses the following simple observation: since the $u_n \leq u_{\mathbb{D}}$ are locally uniformly bounded, the weak converge of $u_n \to u$ implies that of $e^{2u_n} \to e^{2u}$.

2.1 Generalized Blaschke products

If $\tilde{\nu}$ is a measure on the unit disk satisfying the Blaschke condition

$$\int_{\mathbb{D}} (1 - |a|) d\tilde{\nu}(a) < \infty, \tag{2.4}$$

then $\nu(a) := (1 - |a|)\tilde{\nu}(a)$ is a finite measure. It will be convenient to use the two notations simultaneously. We define the *generalized Blaschke product* with zero structure ν by the formula

$$B_\nu = \exp\biggl(\int_{\mathbb{D}} \log \frac{z - a}{1 - \overline{a}z} \, d\tilde{\nu}(a) \biggr), \tag{2.5}$$

cf. (1.3). While B_ν may not be a single-valued function on the unit disk, its absolute value and hence zero set are well-defined. Multiplying B_ν by a singular inner function S_μ, we obtain the *generalized inner function* $I_\omega = B_\nu S_\mu$ where $\omega = \mu + \nu$.

The following lemma is well known:

Lemma 2.5. (i) *For $\nu \in M(\mathbb{D})$, the measures* $(\log 1/|B_\nu|)(d\theta/2\pi)\big|_{\{|z|=r\}}$ *tend weakly to the zero measure as $r \to 1$.*

(ii) *If $\mu \in M(\mathbb{S}^1)$ is a singular measure, then* $(\log 1/|S_\mu|)(d\theta/2\pi)\big|_{\{|z|=r\}} \to \mu$.

2.2 The space of solutions

We are now ready to prove Theorem 1.3. The heavy-lifting has been done in [9] where Theorem 1.3 was proved in the case when $\tilde{\nu} = 0$. Here, we explain the extension to general measures $\tilde{\nu} \ge 0$ satisfying the Blaschke condition (2.4).

Proof of Theorem 1.3. Suppose that u_ω is a nearly-maximal solution of the Gauss curvature equation with data $\omega = \mu + \nu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$. We claim that

$$u_\omega = \Lambda^{\tilde{\nu}} \biggl[u_{\mathbb{D}} - \log \frac{1}{|I_\omega|} \biggr]. \tag{2.6}$$

Since (2.6) gives a formula for u_ω, it shows that u_ω is the unique nearly-maximal solution with data ω. Combining (2.6) with Lemma 2.2, we see that u_ω is decreasing in ω. Since the function

$$u_{\mathbb{D}} - u_\omega - \log \frac{1}{|I_\omega|}$$

is subharmonic and tends weakly to the zero measure on the unit circle, it is negative in the unit disk. The definition of the Perron hull tells us that

$$u_\omega \ \geq\ \Lambda^{\tilde{\nu}}\left[u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}\right] \ \geq\ u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}.$$

Some rearranging gives

$$u_{\mathbb{D}} - u_\omega \ \leq\ u_{\mathbb{D}} - \Lambda^{\tilde{\nu}}\left[u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}\right] \ \leq\ \log\frac{1}{|I_\omega|}.$$

Taking the weak limit as $r \to 1$ shows that the Perron hull $u_* = \Lambda^{\tilde{\nu}}\big[u_{\mathbb{D}} - \log\frac{1}{|I_\omega|}\big]$ has "deficiency" μ on the unit circle. Since u_* has "singularity" $2\pi\tilde{\nu}$, it is also a nearly-maximal solution of the Gauss curvature equation with data ω. To see that $u_* = u_\omega$, we notice that the difference $u_\omega - u_*$ is a non-negative subharmonic function which tends to the zero measure on the unit circle (and hence must be identically 0). This proves the claim.

Let u_μ be the nearly-maximal solution of the Gauss curvature equation $\Delta u = 4e^{2u}$ with deficiency $\mu \in M_{\mathcal{BC}}(\mathbb{S}^1)$. The existence of u_μ is non-trivial and was proved in [9] using the connection with complex analysis provided by the Liouville correspondence. For any Blaschke measure $\tilde{\nu} \geq 0$ on the unit disk, the Perron method finds the least solution of $\Delta u = 4e^{2u} + 2\pi\tilde{\nu}$ satisfying $u_\mu \ \geq\ u \ \geq\ u_\mu - \log\frac{1}{|B_\nu|}$. By Lemma 2.5, u has the correct boundary behaviour in order to solve (1.6), thereby proving the existence of $u_{\mu,\nu}$.

Conversely, suppose that $\mu \notin M_{\mathcal{BC}}(\mathbb{D})$. It was proved in [9] that u_μ does not exist in this case. To show that $u_{\mu,\nu}$ does not exist for any $\nu \in M(\mathbb{D})$, we argue by contradiction: we use the existence of $u_{\mu,\nu}$ to construct u_μ. To this end, we notice that $\Lambda^0(u_{\mu,\nu})$ is a solution of the Gauss curvature $\Delta u = e^{2u}$ which is squeezed between $u_{\mu,\nu} \leq \Lambda^0(u_{\mu,\nu}) \leq u_{\mu,\nu} + \log\frac{1}{|B_\nu|}$, and so must be u_μ by Lemma 2.5. $\square$

2.3 Fundamental Lemma

The following lemma will play an important role in this work:

Lemma 2.6. *Given two measures $\omega_i = \mu_i + \nu_i \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, $i = 1, 2$, we have*

$$u_{\omega_1+\omega_2} = \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}\right], \tag{2.7}$$

$$= \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[\Lambda^{\tilde{\nu}_1}\left[u_{\mathbb{D}} - \log\frac{1}{|I_{\omega_1}|}\right] - \log\frac{1}{|I_{\omega_2}|}\right]. \tag{2.8}$$

Proof. The proof of (2.7) is very similar to that of (2.6). Since the quantity on the right side (2.7) is a solution of the Gauss curvature equation with "singularity" $2\pi(\tilde{\nu}_1 + \tilde{\nu}_2)$, we simply need to check that it has the correct "deficiency" on the unit circle. To see this, we observe that it is squeezed by quantities with deficiency $\mu_1 + \mu_2$:

$$u_{\omega_1+\omega_2} \geq \Lambda^{\tilde{\nu}_1+\tilde{\nu}_2}\left[u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}\right] \geq u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}.$$

We leave it to the reader to justify the first inequality by checking that $u_{\omega_1+\omega_2} \geq u_{\omega_1} - \log\frac{1}{|I_{\omega_2}|}$ using the argument from the proof of Theorem 1.3. Equation (2.8) can be obtained by substituting (2.6) into (2.7). □

2.4 A change of notation

To make this paper more consistent with [9], we work with conformal metrics rather than with their logarithms. Given a conformal metric λ and a measure $\omega = \mu + \nu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, we will write $\Lambda_r^{\langle\omega\rangle}[\lambda]$ for the composition $\exp\circ\Lambda_r^{\tilde{\nu}}\circ\log$. We adopt this notation for the rest of the paper. We say that ω is *integral* if $\tilde{\nu}$ is an integer sum of delta masses (there is no restriction on μ). Recall that in this case, we use the symbol F_ω to denote an inner function with $F_\omega(0) = 0$ and critical structure ω (which is unique up to multiplication by a unimodular constant).

Lemma 2.7 (Fundamental Lemma)**.** (i) *For any integral measure $\omega \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, we have*

$$\lambda_{F_\omega} = \Lambda^{\langle\omega\rangle}[|I_\omega|\lambda_{\mathbb{D}}].$$

(ii) *If $\omega_1, \omega_2 \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ are integral measures, then*

$$\lambda_{F_{\omega_1+\omega_2}} = \Lambda^{\langle\omega_1+\omega_2\rangle}[|I_{\omega_2}|\lambda_{F_{\omega_1}}] = \Lambda^{\langle\omega_1+\omega_2\rangle}\Big[|I_{\omega_1}|\cdot\Lambda^{\langle\omega_2\rangle}\big[|I_{\omega_2}|\lambda_{\mathbb{D}}\big]\Big].$$

(iii) *More generally, if $\omega_1, \omega_2, \dots, \omega_j \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$, then*

$$\Lambda^{\langle\omega_1+\omega_2+\dots+\omega_j\rangle}\Big[|I_{\omega_1}|\cdot\dots\Lambda^{\langle\omega_{j-1}+\omega_j\rangle}\big[|I_{\omega_{j-1}}|\cdot\Lambda^{\langle\omega_j\rangle}[|I_{\omega_j}|\lambda_{\mathbb{D}}]\big]\dots\Big] =$$

$$= \Lambda^{\langle\omega_1+\omega_2+\dots+\omega_j\rangle}\big[|I_{\omega_1+\omega_2+\dots+\omega_j}|\cdot\lambda_{\mathbb{D}}\big].$$

3 Concentrating sequences

In this section, we study concentrating sequences of inner functions. We show:

Theorem 3.1. *Suppose $\{F_{\mu_n}\}$ is a sequence of inner functions in $\mathscr{J}$ which converge uniformly on compact subsets to $F_\nu \in \mathscr{J}$. If $\mu_n \to \mu$ converge in the Korenblum topology, then $\mu = \nu$.*

To prove the above theorem, we will need some a priori bounds on Blaschke products whose critical structure is supported on a Korenblum star. For a Beurling-Carleson set $E \subset \mathbb{S}^1$ and parameters $\alpha \geq 1$, $0 < \theta \leq 1$, we define the *generalized Korenblum star of order* α as

$$K_E^\alpha(\theta) = \bigl\{z \in \overline{\mathbb{D}} : 1 - |z| \leq \theta \cdot \operatorname{dist}(\hat{z}, E)^\alpha \bigr\}. \tag{3.1}$$

If $\alpha = 1$ and $\theta = 1$, the above definition reduces to the one given earlier: $K_E = K_E^1(1)$. By default we take $\theta = 1$, i.e. we write $K^\alpha(E) = K_E^\alpha(1)$.

Lemma 3.2. *Suppose $F(z)$ is an inner function with $F(0) = 0$ whose "critical structure" $\mu(\operatorname{Inn} F')$ is supported on a Korenblum star K_E of order 1 and "critical mass" $\mu(\operatorname{Inn} F')(\overline{\mathbb{D}}) < M$. Then,*

$$\frac{1 - |F(z)|}{1 - |z|} \leq C(M) \cdot \operatorname{dist}(z, E)^{-4}, \qquad z \in \mathbb{D} \setminus K_E^4, \tag{3.2}$$

where dist *denotes Euclidean distance.*

Under the assumptions of the above lemma, we have:

Corollary 3.3. *The "zero structure" $\mu(F)$ is supported on a higher-order Korenblum star $K_E^4(\theta)$ where θ is a parameter which depends on M. In particular, by Schwarz reflection, F extends to an analytic function on $\mathbb{C} \setminus \mathbf{r}(K_E^4(\theta))$ where $\mathbf{r}(z) = 1/\overline{z}$ denotes the reflection in the unit circle.*

Corollary 3.4. *For a point $\zeta \in \mathbb{S}^1$ on the unit circle,*

$$|F'(\zeta)| \leq C(M) \cdot \operatorname{dist}(\zeta, E)^{-4}.$$

With help of Lemma 3.2 and its corollaries, the proof of Theorem 3.1 runs as follows:

Proof of Theorem 3.1. Special case. We first prove the theorem in the special case when each measure μ_n is supported on a Korenblum star K_{E_n} with norm $\|E_n\|_{\mathcal{BC}} \le N$. In this case, $F_n' \to F'$ converge locally uniformly on $\mathbb{C} \setminus \mathbf{r}(K^4_E(\theta))$, where the domains of definition $\mathbb{C} \setminus \mathbf{r}(K^4_{E_n}(\theta))$ are changing but converge to $\mathbb{C} \setminus \mathbf{r}(K^4_E(\theta))$. According to [9, Section 4], to show that the sequence $F_n = F_{\mu_n}$ is stable, it suffices to check that the outer factors converge at the origin:

$$\lim_{n\to\infty} \int_{\mathbb{S}^1} \log|F_n'| d\theta = \int_{\mathbb{S}^1} \log|F'| d\theta.$$

The proof will be complete if we can argue that the functions $\log|F_n'|$ are uniformly integrable on the unit circle. This means that for any $\delta > 0$, there exists an $\varepsilon > 0$ so that $\int_A \log|F_n'| d\theta < \varepsilon$ for any n, whenever $A \subset \mathbb{S}^1$ is a measurable set with $m(A) < \delta$. This estimate is provided by Corollary 3.4 above and the definition of a Beurling-Carleson set.

General case. The convergence of $\mu_n \to \mu$ in the Korenblum topology means that for any $\varepsilon > 0$, one can find a sequence $\mu_n^N \to \mu^N$ which converges in $M_{\mathcal{BC}(N)}(\overline{\mathbb{D}})$ with $(\mu_n - \mu_n^N)(\overline{\mathbb{D}}) < \varepsilon$ and $(\mu - \mu^N)(\overline{\mathbb{D}}) < \varepsilon$. Furthermore, for any given $0 < r < 1$, we may choose N and $\{\mu_n^N\}$ so that $\operatorname{supp}(\mu_n - \mu_n^N) \subset \mathbb{D} \setminus B(0,r)$. This follows from the rather simple observation that the ball $B(0,r)$ is contained in a Korenblum star.

The lemma for the class $M_{\mathcal{BC}(N)}(\overline{\mathbb{D}})$ gives the convergence of the conformal metrics $\lambda_{F_{\mu_n^N}} \to \lambda_{F_{\mu^N}}$. The fundamental lemma (Lemma 2.7) implies that

$$\frac{1}{|I_{\mu_n - \mu_n^N}|} \cdot \lambda_{F_{\mu_n}} \ge \lambda_{F_{\mu_n^N}} \ge \lambda_{F_{\mu_n}}, \qquad n = 1, 2, \dots$$

For a compact set $K \subset \mathbb{D}$, we first pick an $0 < r < 1$ so that $K \subset B(0,r)$ and then choose $\varepsilon > 0$ sufficiently small to ensure that $1 - \delta < |I_{\mu_n - \mu_n^N}(z)| < 1$ for $z \in K$. Since $\delta > 0$ can be made arbitrarily small, we see that the conformal metrics $\lambda_{F_{\mu_n}} \to \lambda_{F_\mu}$ converge uniformly on compact subsets of the disk. By Liouville's theorem, this is equivalent to convergence of $F_{\mu_n} \to F_\mu$. $\square$

3.1 Blaschke products as approximate isometries

To prove Lemma 3.2, we use the following principle: *away from the critical points, an inner function is close to a hyperbolic isometry.* Our discussion is inspired by the work of McMullen [18, Section 10] which deals with finite Blaschke products of fixed degree. Here, we require "degree independent" estimates. To this end, given an inner function $F(z)$, we consider the quantity

$$\gamma_F(z) = \log \frac{1}{|\operatorname{Inn} F'(z)|} \tag{3.3}$$

which measures how much F deviates from a Möbius transformation near z. Let $G(z, w)$ denote the Green's function on the unit disk. When $w = 0$, $G(z, 0) = \log \frac{1}{|z|}$. If the singular measure $\sigma(F')$ is trivial (e.g. if F is a finite Blaschke product), the above definition reduces to

$$\gamma_F(z) = \sum_{c \in \operatorname{crit}(F)} G(z, c). \tag{3.4}$$

To see that the definition of $\gamma_F(z)$ is Möbius invariant, note that if $M_1, M_2 \in \operatorname{Aut}(\mathbb{D})$ then $\operatorname{Inn}\big[(M_1 \circ F \circ M_2)'\big] = \operatorname{Inn} F' \circ M_2$.

Lemma 3.5 (cf. Proposition 10.9 of [18])**.** *Suppose F is an inner function. At a point $z \in \mathbb{D}$ which is not a critical point of F, the 2-jet of F matches the 2-jet of a hyperbolic isometry with an error of $\mathcal{O}(\gamma(z))$.*

Proof. By Möbius invariance, it suffices to consider the case when $z = F(z) = 0$. Set $\delta = \gamma_F(0)$. To prove the lemma, we need to show that $|F'(0) - 1| = |F''(0)| = \mathcal{O}(\delta)$. The definition of $\gamma_F(0)$ gives $1 - |(\operatorname{Inn} F')(0)| \le \delta$. By the Schwarz lemma applied to $\operatorname{Inn} F'$, we have $1 - |(\operatorname{Inn} F')(z)| = \mathcal{O}(\delta)$ for $z \in B(0, 1/2)$. Applying the fundamental lemma (Lemma 2.7), we arrive at

$$1 - C\delta < \frac{\lambda_F(z)}{\lambda_{\mathbb{D}}(z)} \le 1, \qquad \text{for } z \in B(0, 1/2), \tag{3.5}$$

where C is a universal constant (independent of δ and F). From the above equation, the estimate on the first derivative $|F'(0) - 1| = \mathcal{O}(\delta)$ is immediate. To estimate the second derivative, note that by (3.5), $|F(z) - z| = \mathcal{O}(\delta)$ for all $z \in B(0, 1/2)$, and then use Cauchy's integral formula. □

We write $d_{\mathbb{D}}(z_1, z_2)$ for the hyperbolic distance. A sample application of the above lemma is the following:

Corollary 3.6 (cf. Theorem 10.11 and Corollary 10.7 of [18])**.** *Suppose $F(z)$ is a finite Blaschke product and $[z_1, z_2]$ is a segment of a hyperbolic geodesic. If for each $z \in [z_1, z_2]$, $\gamma_F(z) < \varepsilon_0$ is sufficiently small, then $F(z_1) \neq F(z_2)$. In fact, for any $\delta > 0$, we can choose $\varepsilon_0 > 0$ small enough to guarantee that*

$$(1-\delta) \cdot d_{\mathbb{D}}(z_1, z_2) \;\le\; d_{\mathbb{D}}\big(F(z_1), F(z_2)\big) \;\le\; d_{\mathbb{D}}(z_1, z_2). \tag{3.6}$$

Sketch of proof. If we choose $\varepsilon_0 > 0$ small enough, then $F|_{[z_1,z_2]}$ is so close to an isometry that the geodesic curvature of its image is nearly 0. But a path in hyperbolic space with geodesic curvature less than 1 (the curvature of a horocycle) cannot cross itself, so $F(z_1) \neq F(z_2)$. Similar reasoning gives the second statement. $\square$

Remark. If $\gamma_F(z)$ decays exponentially along $[z_1, z_2]$, i.e. satisfies a bound of the form

$$\gamma_F(z) < M \exp\big(-d_{\mathbb{D}}(z, z_1)\big),$$

for some $M > 0$, then McMullen's argument gives the stronger conclusion

$$d_{\mathbb{D}}(F(z_1), F(z_2)) = d_{\mathbb{D}}(z_1, z_2) + \mathcal{O}(1).$$

See the proof of [18, Theorem 10.11].

Lemma 3.7. *Suppose I is an inner function whose zero structure $\mu(I)$ is contained in a Korenblum star K_E and its critical mass $\mu(I)(\overline{\mathbb{D}}) < M$. Then, $|I(z)| > c(M) > 0$ is bounded from below on $\mathbb{D} \setminus K_E^2$. More precisely,*

$$\log \frac{1}{|I(z)|} \lesssim M \exp\big(-d_{\mathbb{D}}(z, K_E^2)\big), \qquad z \in \mathbb{D} \setminus K_E^2.$$

For a point $z \in \mathbb{D}$, let $[0, z]$ denote the hyperbolic geodesic that joins 0 to z. For $n > 0$, let z_n be the unique point of intersection of $[0, z]$ with ∂K_E^n if it exists.

Proof. We may assume that I is a finite Blaschke product as the general case follows by approximating I by finite Blaschke products whose zero sets are contained in K_E.

Let a be a zero of I. Elementary hyperbolic geometry and the triangle inequality show that the hyperbolic distance

$$\begin{aligned} d_{\mathbb{D}}(z,a) &= d_{\mathbb{D}}(z,z_1) + d_{\mathbb{D}}(z_1,a) - \mathcal{O}(1), \\ &\ge d_{\mathbb{D}}(z,z_1) + d_{\mathbb{D}}(0,a) - d_{\mathbb{D}}(0,z_1) - \mathcal{O}(1), \\ &\ge d_{\mathbb{D}}(z,z_2) + d_{\mathbb{D}}(0,a) - \mathcal{O}(1). \end{aligned}$$

In other words, the Green's function

$$G(z,a) \lesssim G(0,a)\exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr)$$

decays exponentially quickly in the hyperbolic distance $d_{\mathbb{D}}(z,z_2)$. If $a \in B(0,1/2)$, we instead use the "trivial" estimate $G(z,a) \lesssim \exp\bigl(-d_{\mathbb{D}}(z,0)\bigr) \le \exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr)$. Combining the two inequalities, we get

$$G(z,a) \lesssim G^*(0,a)\exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr)$$

where $G^*(z,w) := \min\bigl(G(z,w),1\bigr)$ is the truncated Green's function. Summing over the zeros of I gives

$$\log\frac{1}{|I(z)|} = \sum_{a\in\text{zeros}(I)} G(z,a) \lesssim M\exp\bigl(-d_{\mathbb{D}}(z,z_2)\bigr) \asymp M\exp\bigl(-d_{\mathbb{D}}(z,K_E^2)\bigr),$$

where in the second step we made use of $\sum_{a\in\text{zeros}(I)} G^*(0,a) \asymp \mu(I)(\overline{\mathbb{D}}) \le M$. This proves the lemma. □

Corollary 3.8. *Suppose F is an inner function which satisfies the hypotheses of Lemma 3.2. For $z \in \mathbb{D}\setminus K_E^2$, the characteristic $\gamma_F(z) \lesssim M\exp\bigl(-d_{\mathbb{D}}(z,K_E^2)\bigr)$.*

With these preparations, we can now prove Lemma 3.2:

Proof of Lemma 3.2. Suppose $z \in \mathbb{D}\setminus K_E^4$. Divide $[0,z]$ into two parts: $[0,z_2]$ and $[z_2,z]$. By the Schwarz lemma,

$$d_{\mathbb{D}}(F(0),F(z_2)) \le d_{\mathbb{D}}(0,z_2).$$

However, since F restricted to $[z_2,z]$ is close to a hyperbolic isometry,

$$d_{\mathbb{D}}(F(z_2),F(z)) \ge d_{\mathbb{D}}(z_2,z) - \mathcal{O}(1). \tag{3.7}$$

The triangle inequality gives

$$d_{\mathbb{D}}(F(0), F(z)) \geq d_{\mathbb{D}}(z_4, z) - \mathcal{O}(1),$$

which is equivalent to (3.2). $\square$

4 Equidiffuse sequences

We now turn our attention to equidiffuse sequences. We show:

Theorem 4.1. *Suppose $\{F_n\}$ is a sequence of inner functions in $\mathscr{J}$ which converge uniformly on compact subsets to a function $F \in \mathscr{J}$. If $\mu(\operatorname{Inn} F_n') \to \mu$ is equidiffuse, then $F(z) = z$.*

This proof is similar to the one in [9, Section 6], but requires a slightly more intricate argument since we need to decompose measures supported on the closed unit disk.

Given an arc $I \subset \mathbb{S}^1$ on the unit circle, set

$$\square_{I,r,R} := \{z : z/|z| \in I,\, r \leq |z| \leq R\},$$

with the convention that we include the left edge into $I_{r,R}$ but not the right edge.

4.1 Roberts decompositions

Similarly to the original Roberts decomposition for measures supported on the unit circle [20], our decomposition will depend on two parameters: a real number $c > 0$ and an integer $j_0 \geq 1$. Set $n_j := 2^{2^{(j+j_0)}}$ and $r_j := 1 - 1/n_j$.

Theorem 4.2. *Given a finite measure $\mu \in M(\overline{\mathbb{D}})$ on the closed unit disk, one can write it as*

$$\mu = (\mu_2 + \mu_3 + \mu_4 + \dots) + \nu_{\text{cone}} \tag{4.1}$$

where each measure μ_j, $j \geq 2$, enjoys the following two properties:

$$\operatorname{supp} \mu_j \subset \square_{\mathbb{S}^1, r_{j-1}, 1}, \tag{4.2}$$

and
$$|I_{\mu_j}| > \frac{1}{(1-|z|^2)^{c'}} \quad \textit{for } z \in \square_{\mathbb{S}^1,0,r_{j-2}}, \qquad c' \asymp c; \tag{4.3}$$
while the cone measure ν_{cone} is supported on a Korenblum star $K_{E_{\text{cone}}}$ of norm $\|E_{\text{cone}}\|_{\mathcal{BC}} \le N(c, j_0)$.

Proof. We obtain the decomposition by means of an algorithm which sorts out the mass of μ into various components. For each $j = 2, 3, \dots$, we consider a partition P_j of the unit circle into n_j equal arcs. Since n_j divides n_{j+1}, each next partition can be chosen to be a refinement of the previous one.

As Step 0 of our algorithm, we move $\mu|_{B(0,r_1)}$ into ν_{cone}. (We remove this mass from μ.)

To define μ_j, $j = 2, 3, \dots$, consider all intervals in the partition P_j. Define an interval to be *light* if $\mu(\square_{I,0,1}) \le (c/n_j) \log n_j$ and *heavy* otherwise. We do one of the following three operations:

- L. If I is light, we move the mass $\mu|_{\square_{I,0,1}}$ into μ_j.
- H1. If I is heavy, we look at the box $\square_{I,r_{j-1},r_j}$. If $\mu(\square_{I,r_{j-1},r_j}) \ge (c/n_j) \log n_j$, we move $\mu|_{\square_{I,r_{j-1},r_j}}$ into ν_{cone}.
- H2. If $\mu(\square_{I,r_{j-1},r_j}) < (c/n_j) \log n_j$, we move $\mu|_{\square_{I,r_{j-1},r_j}}$ to μ_j. We also move some mass from $\mu|_{\square_{I,r_j,1}}$ to μ_j so that
$$2(c/n_j) \log n_j \ge \mu_j(\square_{I,0,1}) \ge (c/n_j) \log n_j.$$
  There is some ambiguity in the second step, but the particular choice will not be important for us.

After we followed the above instructions for $j = 2, 3, \dots$, it is possible that the measure μ has not been exhausted completely: some "residual" mass may remain on the unit circle. We move this remaining mass to ν_{cone}.

Define $E_{\text{cone}} := \mathbb{S}^1 \setminus \mathscr{L}$ where $\mathscr{L}$ is the union of the light intervals (of any generation). Since the measure $\nu_{\text{cone}}|_{\mathbb{S}^1}$ is supported on the set of points which lie in heavy intervals at every stage, $\operatorname{supp} \nu_{\text{cone}}|_{\mathbb{S}^1} \subset K_{E_{\text{cone}}}$. Observe that if I is an interval of generation j, the box $\square_{I,r_{j-1},r_j}$ is contained in the union of two Stolz angles emanating from the endpoints of I. If I is heavy, these endpoints are contained in E_{cone}, from which we see that $\operatorname{supp} \nu_{\text{cone}}|_{\mathbb{D}} \subset K_{E_{\text{cone}}}$ as well.

To check that E_{cone} is a Beurling-Carleson set, we follow the computation from Roberts [20]. The relation $\log n_{j+1} = 2 \log n_j$ shows

$$\sum_{\text{light}} |I| \log \frac{1}{|I|} \lesssim \sum_{\text{heavy}} |J| \log \frac{1}{|J|} \lesssim \sum_{j=0}^{\infty} \sum_{J \in P_j \text{ heavy}} \mu_j(J) \le \mu(\mathbb{S}^1), \tag{4.4}$$

where we have used the fact that a light interval is contained in a heavy interval of the previous generation. The estimate (4.3) follows from Lemma 4.3 below. $\square$

Lemma 4.3. *Suppose ν is a finite measure on the closed unit disk with*

$$\operatorname{supp} \nu \subset \{z : 1 - 1/n \le |z| \le 1\}.$$

Assume that for any interval $I \subset \mathbb{S}^1$ of length $1/n$,

$$\nu(\square_{I,1-1/n,1}) \le c \cdot |I| \log \frac{1}{|I|}.$$

Then,

$$|B_\nu| > \frac{1}{(1 - |z|^2)^{c'}}, \qquad |z| < 1 - 2/n,$$

for some $c' \asymp c$.

Sketch of proof. The lemma is well known when $\operatorname{supp} \nu \subseteq \mathbb{S}^1$, e.g. see [20, Lemma 2.2]. For the general case, it suffices to show that $\log \frac{1}{|B_\nu|} \asymp \log \frac{1}{|S_{\hat{\nu}}|}$ where $\hat{\nu}$ is the projection of ν to the unit circle. In turn, we may check that

$$\log \frac{1}{|M_a|} \asymp \log \frac{1}{|S_{(1-|a|)\delta_{\hat{a}}}|}, \qquad z \in B(0, 1 - 2/n), \quad a \in \operatorname{supp} \nu,$$

where $M_a(z) = \frac{z-a}{1-\bar{a}z}$ and $\hat{a} = a/|a|$. To see this, note that the level sets of $\log \frac{1}{|M_a|}$ are circles with hyperbolic center a while the level sets of $\log \frac{1}{|S_{(1-|a|)\delta_{\hat{a}}}|}$ are horocycles which rest on $\hat{a}$. $\square$

4.2 Estimate on the conformal metric

Fix an $\varepsilon > 0$. We now show that if a positive measure $\mu \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ gives $\le \delta(\varepsilon)$ mass to any Korenblum star K_E of norm N with $N = N(\delta, \varepsilon)$ sufficiently large, then F_μ is close to the identity, in the sense that

$$\lambda_{F_\mu}(0) = \Lambda^{\langle\mu\rangle}\big[|I_\mu|\lambda_{\mathbb{D}}\big](0) > \lambda_{\mathbb{D}}(0) - \varepsilon. \tag{4.5}$$

Assuming (4.5), Theorem 4.1 follows from a normal families argument and the Schwarz lemma. Fix a large integer $j_0 \ge 1$ and consider the Roberts decomposition with parameters c and j_0, where c is small enough to ensure that $c' < 1/10$ in (4.3). By asking for N to be sufficiently large, we can guarantee that $\nu_{\mathrm{cone}}(\overline{\mathbb{D}}) \le \delta$. In view of the fundamental lemma, up to small error, we have

$$\Lambda^{\langle\mu\rangle}[|I_\mu|\lambda_{\mathbb{D}}](0) \approx \Lambda^{\langle\mu_2+\mu_3+\cdots+\dots\rangle}\big[|I_{\mu_2+\mu_3+\dots}|\lambda_{\mathbb{D}}\big](0).$$

For an integer $j \ge 2$, consider the conformal metric

$$\lambda_j := \Lambda_{r_0}\Big[|I_{\mu_2}| \cdot \dots \Lambda_{r_{j-3}}\Big[|I_{\mu_{j-1}}| \cdot \Lambda_{r_{j-2}}\big[|I_{\mu_j}| \cdot \lambda_{\mathbb{D}}\big]\Big] \dots\Big]. \tag{4.6}$$

By the monotonicity properties of Λ and Lemma 2.7,

$$\begin{aligned}
\lambda_j &\le \Lambda^{\langle\mu_2+\mu_3+\cdots+\mu_j\rangle}\Big[|I_{\mu_2}| \cdot \dots \Lambda^{\langle\mu_{j-1}+\mu_j\rangle}\Big[|I_{\mu_{j-1}}| \cdot \Lambda^{\langle\mu_j\rangle}\big[|I_{\mu_j}| \cdot \lambda_{\mathbb{D}}\big]\Big] \dots\Big], \\
&= \Lambda^{\langle\mu_2+\mu_3+\cdots+\mu_j\rangle}\big[|I_{\mu_2+\mu_3+\cdots+\mu_j}|\lambda_{\mathbb{D}}\big], \\
&\approx \Lambda^{\langle\mu_2+\mu_3+\cdots+\dots\rangle}\big[|I_{\mu_2+\mu_3+\dots}|\lambda_{\mathbb{D}}\big].
\end{aligned}$$

Therefore, in order to show (4.5), we may show that $\lambda_j(0)$ is close to $\lambda_{\mathbb{D}}(0)$, uniform in $j = 1, 2, \dots$. The advantage of working with the λ_j is that we only apply the operators Λ_r to genuine conformal metrics since the zeros of the inner functions I_{μ_j} are located outside the disks $\mathbb{D}_{r_{j-2}}$. This allows us to estimate the effect of Λ_r by comparing with conformal metrics $\Lambda_r[C]$ which extend constant boundary values on $\partial\mathbb{D}_r$. The proof of (4.5) can now be completed by the argument in [9, Section 6] with help of (4.3) to estimate the $|I_{\mu_j}|$.

4.3 Equidiffuse sequences do not affect the limit

Theorem 4.4. *Suppose $\{F_{\tau_n+\nu_n}\}$ is a sequence of functions in $\mathscr{J}$ which converge uniformly on compact subsets of the disk. Furthermore, suppose that $\nu_n \to \nu$ is concentrating and $\tau_n \to \tau$ is equidiffuse. Then, $F_{\tau_n+\nu_n} \to F_\nu$.*

Proof. In view of the fundamental lemma (Lemma 2.7), we must show that

$$\lim_{n\to\infty} \Lambda^{\langle \tau_n+\nu_n\rangle}\big[|I_{\tau_n+\nu_n}|\lambda_{\mathbb{D}}\big] = \lim_{n\to\infty} \Lambda^{\langle \nu_n\rangle}\big[|I_{\nu_n}|\lambda_{\mathbb{D}}\big].$$

The $\leq$ direction follows from the monotonicity properties of Λ (Lemma 2.2). By Lemmas 2.3 and 2.7, for any $0 < r < 1$, we have

$$\Lambda^{\langle \tau_n+\nu_n\rangle}[|I_{\tau_n+\nu_n}|\lambda_{\mathbb{D}}] \geq \Lambda_r^{\langle \tau_n+\nu_n\rangle}\Big[|I_{\nu_n}| \cdot \Lambda^{\langle \tau_n\rangle}[|I_{\tau_n}|\lambda_{\mathbb{D}}]\Big].$$

Since $\{\tau_n\}$ is equidiffuse,

$$\Lambda^{\langle \tau_n+\nu_n\rangle}[|I_{\tau_n+\nu_n}|\lambda_{\mathbb{D}}] \geq \Lambda_r^{\langle \tau_n+\nu_n\rangle}\Big[(1-o(1))|I_{\nu_n}|\lambda_{\mathbb{D}}\Big].$$

Lemma 2.4 and the fact that $\tau_n|_{\mathbb{D}_r} \to 0$ as $n \to \infty$ show

$$\liminf_{n\to\infty} \Lambda^{\langle \tau_n+\nu_n\rangle}\big[|I_{\tau_n+\nu_n}|\lambda_{\mathbb{D}}\big] \geq \liminf_{n\to\infty} \Lambda_r^{\langle \tau_n+\nu_n\rangle}\big[|I_{\nu_n}|\lambda_{\mathbb{D}}\big] \geq \Lambda_r^{\langle \nu\rangle}\big[|I_\nu|\lambda_{\mathbb{D}}\big].$$

Taking $r \to 1$ completes the proof. □

4.4 An improvement for discrete Blaschke measures

Suppose $\mu \in M(\overline{\mathbb{D}})$ is a measure which records the zero structure of some inner function, that is, a measure of the form (1.3). Consider the Roberts decomposition (4.1) of μ for some choice of parameters c and j_0.

Theorem 4.5. *The measure ν_{cone} is contained in a set of the form*

$$q_E = \{0\} \cup \{z \in \mathbb{D} : \hat{z} \in E\} \tag{4.7}$$

where $E \subset \mathbb{S}^1$ is a Beurling-Carleson set with norm $\|E\|_{\mathcal{BC}} \leq N(c, j_0, \mu(\overline{\mathbb{D}}))$.

As a consequence, when discussing the Korenblum topology on inner functions, we may consider sets of the form (4.7) instead of Korenblum stars.

The proof of Theorem 4.5 requires some elementary lemmas.

Lemma 4.6. *Suppose $x = x_1 + x_2 + \cdots + x_M$ with $0 < x_i < x$. Then,*

$$\sum x_i \log \frac{1}{x_i} \le x \log \frac{M}{x}.$$

Lemma 4.7. *Suppose F_1 and F_2 are two finite sets in the unit circle. Then,*

$$\|F_1 + F_2\|_{\mathcal{BC}} \le \|F_1\|_{\mathcal{BC}} + \|F_2\|_{\mathcal{BC}}.$$

Lemma 4.8. *Suppose $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ is an increasing sequence of finite subsets of the unit circle such that the norms $\|F_n\|_{\mathcal{BC}} \le M$ are bounded. Let F be the closure of their union. Then, F is a Beurling-Carleson set with $\|F\|_{\mathcal{BC}} \le M$.*

For a more general statement, see [7, Lemma 7.6].

Proof of Theorem 4.5. We crudely estimate that the number of point masses of μ (zeros of the inner function I_μ) in a heavy box $\square_{I,r_{j-1},r_j}$ by $n_j \cdot \mu(\overline{\mathbb{D}})$. Radially project these point masses onto the unit circle. These points partition the interval I into several pieces of lengths $x_1, \dots, x_M$, with $x_1 + x_2 + \cdots + x_M = |I|$. By Lemma 4.6 and the fact that $|I| = 1/n_j$,

$$\sum x_i \log \frac{1}{x_i} \;\le\; |I| \log \frac{n_j \cdot \mu(\overline{\mathbb{D}})}{|I|} \;\le\; C(\mu(\overline{\mathbb{D}})) \cdot |I| \log \frac{1}{|I|}.$$

Applying Lemma 4.7, we see that the entropy of the partition of the unit circle by the projections of all point masses in $\mu|_{\mathbb{D}_{r_j}}$ union E_{cone} is bounded by

$$\lesssim \|E_{\text{cone}}\|_{\mathcal{BC}} + \mu(\overline{\mathbb{D}}) \cdot \sum_{\text{heavy}} |I| \log \frac{1}{|I|}$$

with the implicit constant independent of j. However, the latter sum is finite by the Roberts estimate (4.4). The proof is now completed by Lemma 4.8. $\square$

5 Invariant subspaces of Bergman spaces

For a fixed $\alpha > -1$ and $1 \leq p < \infty$, consider the weighted Bergman space $A^p_\alpha(\mathbb{D})$ of holomorphic functions satisfying the norm boundedness condition (1.10). Let $\{I_n\}$ be a sequence of inner functions which converge uniformly on compact subsets to an inner function I. Assume that the zero structures $\mu(I_n) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$ and let $[I_n] \subset A^p_\alpha$ be the z-invariant subspace generated by I_n. In this section, we prove Theorem 1.6 which identifies $\lim_{n\to\infty}[I_n]$ as $[I]$ where I is the "concentrated limit" of the I_n.

In general, one has semicontinuity in one direction:

$$[I] \subseteq \liminf_{n\to\infty} [I_n]. \tag{5.1}$$

To see this, note that if $f \in [I]$, then it may be approximated in norm by $p_k I$ for some polynomials $\{p_k\}_{k=1}^\infty$. Diagonalization allows us to express f as the limit of $p_{k(n)} I_n \in [I_n]$.

5.1 Concentrating sequences: special case

Lemma 5.1. *Suppose $I_n \to I$ if a sequence of inner functions which converges uniformly on compact subsets of the unit disk. If the zero structure of each I_n belongs to a Korenblum star K_{E_n} with $\|E_n\|_{\mathcal{BC}} \leq N$ uniformly bounded above then $[I_n] \to [I]$.*

The above lemma is essentially due to Korenblum [11], albeit with a slightly greater focus on uniformity. For a Beurling-Carleson set E, one can construct an outer function $\Phi_E(z) \in C^\infty(\overline{\mathbb{D}})$ which vanishes precisely on E and does so to infinite order. Examining the construction in [7, Proposition 7.11]), we may assume that Φ_E may enjoys two nice properties:

1. The function $\Phi_E(z)$ varies continuously with the Beurling-Carleson set E, in the sense that $\Phi_{E_n} \to \Phi_E$ uniformly on compact subsets of the disk if $E_n \to E$ and $\|E_n\|_{\mathcal{BC}} \to \|E\|_{\mathcal{BC}}$.

2. For each $N \geq 0$,
$$|\Phi_E(z)| \cdot \mathrm{dist}(z, E)^{-N} \leq C_N$$

is bounded by a constant which depends only on $\|E\|_{\mathcal{BC}}$. It is convenient to take $C_0 = 1$ so that $|\Phi_E(z)| \le 1$ on the disk.

The central idea in Korenblum's vision is the following division principle:

Theorem 5.2 (Korenblum's division principle)**.** *Suppose I is an inner function with $\operatorname{supp}\mu(I) \subset K_E$ and $f \in [I]$. For any $\delta > 0$,*

$$f^\delta(z) := (\Phi_E^\delta / I) f(z) \in A^p_\alpha, \tag{5.2}$$

with the norm estimate $\|f^\delta\|_{A^p_\alpha} \le C\|f\|_{A^p_\alpha}$. Here, the constant C depends on δ, $\|E\|_{\mathcal{BC}}$ and $\mu(I)(\overline{\mathbb{D}})$.

Assuming Theorem 5.2, the proof of Lemma 5.1 runs as follows:

Proof of Lemma 5.1. Suppose that a sequence of functions $f_n \in [I_n]$ converges to f in A^p_α. Norm convergence implies that the f_n converge to f uniformly on compact subsets of $\mathbb{D}$. By Korenblum's division principle, for a fixed $\delta > 0$, the functions $g_n = (\Phi_n^\delta / I_n) \cdot f_n(z)$ have bounded A^p_α norms and converge uniformly on compact subsets to

$$g = (\Phi^\delta / I) \cdot f(z).$$

Fatou's lemma implies that $g \in A^p_\alpha$ and therefore $\Phi^\delta \cdot f = Ig \in [I]$. Taking $\delta \to 0$ shows that $f \in [I]$ and therefore $[I] \supseteq \limsup_{n\to\infty} [I_n]$. By (5.1), the other inclusion is automatic. $\square$

Since the exact statement of Theorem 5.2 is not present in Korenblum's work [11], we give a proof below.

Proof of Korenblum's division principle (Theorem 5.2). We first consider the case when I is a finite Blaschke product and E is a finite set. Afterwards, we will deduce the general case by a limiting argument. If I is a finite Blaschke product, it is clear that $f^\delta \in A^p_\alpha$. We need to give a uniform estimate on its norm.

Recall that K_E^2 denotes the generalized Korenblum star of order 2, see (3.1) for the definition. According to Lemma 3.7, $|1/I(z)| \le C\big(\mu(I)(\overline{\mathbb{D}})\big)$ is uniformly bounded on $\mathbb{D} \setminus K_E^2$ so that $|f^\delta(z)| \le C|f(z)|$ there.

To estimate f^δ on K_E^2, we examine its values on the boundary ∂K_E^2. It is well known that a function in Bergman space does not grow too rapidly:

$$|f(z)| \le C\|f\|_{A^p_\alpha}(1-|z|)^{-\beta}, \qquad z \in \mathbb{D}, \tag{5.3}$$

for some $\beta = \beta(p, \alpha) > 0$. However, the C^∞ decay of the outer function Φ^δ cancels out this grows rate on ∂K_E^2 and we end up with

$$|f^\delta(z)| \le C_2\|f\|_{A^p_\alpha}, \qquad z \in \partial K_E^2.$$

Since $f^\delta \in A^p_\alpha$, we can use the Phragmén-Lindelöf principle to conclude that this bound extends to the interior of K_E^2. Putting the above estimates together completes the proof when I is a finite Blaschke product.

For the general case, we approximate I uniformly on compact subsets by finite Blaschke products I_n whose zeros are contained in $K_E \supset \mu(I)$. Using the semicontinuity property (5.1), we may then approximate $f \in [I]$ by $f_n \in [I_n]$ in the A^p_α-norm. By the finite case of the lemma, $f_n^\delta = (\Phi_E^\delta/I_n)f_n(z) \in A^p_\alpha$ with $\|f_n^\delta\|_{A^p_\alpha}$ bounded above. By Fatou's lemma, $\|f^\delta\|_{A^p_\alpha} \le \liminf_{n\to\infty}\|f_n^\delta\|_{A^p_\alpha}$ as desired. □

5.2 Concentrating sequences: general case

Suppose I is an inner function with $\mu(I) \in M_{\mathcal{BC}}(\overline{\mathbb{D}})$. Let $\{I^N\}$ be a sequence of approximating inner functions with $\mu(I^N) \le \mu(I)$ supported on a Korenblum star of norm $\le N$. We claim that $[I^N] \to [I]$. The inclusion $\liminf_{N\to\infty}[I^N] \supseteq [I]$ is trivial. Conversely, given $f^N \in I^N$ converging to f, the sequence $f^N(I/I_N) \in [I]$ will also converge to f. Since $[I]$ is closed, $f \in [I]$ and $\limsup_{N\to\infty}[I^N] \subseteq [I]$, which proves the claim.

Lemma 5.3. *Suppose $I_n \to I$ is a sequence of inner functions which converges on compact subsets of the disk. If the associated measures $\mu(I_n)$ converge in the Korenblum topology, then $[I_n] \to [I]$.*

Proof. By the definition of the Korenblum topology, there exist 'approximations' $I_n^N \to I^N$ supported on Korenblum stars of norm $\le N$. By Lemma 5.1,

$$\limsup_{n\to\infty}[I_n] \subseteq \limsup_{n\to\infty}[I_n^N] = [I^N] \to_{N\to\infty} [I].$$

The other inclusion follows from (5.1). □

5.3 Equidiffuse sequences

Lemma 5.4. *Suppose $I_n \to I$ is a convergent sequence of inner functions such that the associated measures $\mu(I_n)$ are equidiffuse. Then, $[I_n] \to [1]$.*

To prove the above lemma, we closely follow the work of Roberts [20]. Suppose $\mu \in M(\overline{\mathbb{D}})$ is a measure on the closed unit disk which is very close to being diffuse, that is, gives $\le \varepsilon$ mass to any Korenblum star of order $\le N$. We need to show that the distance $d(1,[I_\mu])$ from the z-invariant subspace $[I_\mu] \subset A^p_\alpha$ to the constant function 1 is small. As in Section 4, we consider the (c, j_0) Roberts decomposition

$$\mu = (\mu_2 + \mu_3 + \mu_4 + \dots) + \nu_{\text{cone}}$$

from Section 4, where the parameter c is small and j_0 is large. If $N(c, j_0)$ is large, then the assumption on μ guarantees that $\nu_{\text{cone}}(\overline{\mathbb{D}}) \le \varepsilon$. Set $\tilde{\mu} = \mu_2 + \mu_3 + \mu_4 + \dots$.

We may instead show that $d(1,[I_{\tilde{\mu}}])$ is small since the triangle inequality would imply that $d(1,[I_\mu])$ is also small. In [20], Roberts proved such an estimate for singular inner functions (in which case, the measures μ_j are supported on the unit circle). Roberts' argument is a clever iterative scheme which is quite similar to the one employed in Section 4. Actually, the techniques of Section 4 are adapted from Roberts' work where the use of the corona theorem is replaced with estimates on conformal metrics. In our setting, the function I_μ might have zeros and therefore μ_j are measures on the closed unit disk. Nevertheless, Roberts' argument ([20, Lemmas 2.3 and 2.4]) extends to this more general case almost verbatim.

Lemma 5.5 (cf. Lemma 2.3 of [20])**.** *Fix $\beta > 0$ so that $\|z^n\|_{A^p_\alpha} \le n^{-\beta}$ for $n \ge 2$. Suppose I is an inner function which enjoys the estimate*

$$|I(z)| \ge n^{-\gamma}, \qquad |z| \le 1 - 1/n. \tag{5.4}$$

If $\gamma > 0$ and $n \ge N(\gamma)$ is sufficiently large, there exists a function $g \in H^\infty(\mathbb{D})$ with

$$\|g\|_\infty \le n^{\beta/3}, \qquad \|1 - gI\|_{A^p_\alpha} \le n^{-2\beta/3}. \tag{5.5}$$

Roberts introduced the function $D[\{n_1, n_2, \dots, n_k\}]$ which is defined recursively by $D[\emptyset] = 0$ and $D[\{n_1, n_2, \dots, n_k\}] = n_1^{\beta/3} D[\{n_2, n_3, \dots, n_k\}] + n_1^{-2\beta/3}$. In view of monotonicity, this definition naturally extends to infinite sequences.

Lemma 5.6 (cf. Lemma 2.4 of [20]). *Suppose $I_{n_1}, I_{n_2}, \dots, I_{n_k}$ are inner functions such that*

$$|I_{n_j}(z)| \ge n^{-\gamma}, \qquad |z| \le 1 - 1/n_j, \qquad j = 1, 2, \dots, k. \tag{5.6}$$

Assume that $\min(n_1, n_2, \dots, n_k) \ge N$. *If* $I = \prod_{j=1}^{k} I_{n_j}$ *then* $d(1, [I]) \le D[\{n_j\}]$.

Roberts noticed that if $j_0 \ge 1$ is large, then the sequence of integers $n_j = 2^{2^{(j+j_0)}}$ in the Roberts decomposition (Theorem 4.2) is sufficiently sparse to ensure that $D[\{n_j\}]$ is small. Of course, the condition (5.6) is easily verified using Lemma 4.3. This completes our sketch of Lemma 5.4. We leave the details to the reader.

Remark. In the special case of the weighted Bergman space A^2_1, we can give an alternative argument based on the methods of this paper. For each I_n, we may form an inner function F_n with $F_n(0) = 0$ and $\operatorname{Inn} F_n' = I_n$. According to Theorem 4.1, $F_n \to z$ uniformly on compact subsets. However, the bound $\|F_n\|_{H^\infty} \le 1$ implies that $\|F_n\|_{H^2} \le \|z\|_{H^2}$ which forces $F_n \to z$ to converge in the H^2-norm. The Littlewood-Paley formula

$$\|F_n\|_{H^2} = \frac{1}{\pi} \int_{\mathbb{D}} |F_n'|^2 \log \frac{1}{|z|^2} \, |dz|^2 \asymp \|F_n'\|_{A^2_1}$$

then shows that $F_n' \to 1$ in the A^2_1-norm. Since $F_n' \in [I_n]$,

$$\lim_{n\to\infty} [I_n] \supset \lim_{n\to\infty} [F_n'] \supset [1] = A^2_1.$$

As in Section 4, equidiffuse sequences cannot change the limiting function: if $I_n = I_n' J_n$ with $J_n \to J$ equidiffuse, then $\lim_{n\to\infty}[I_n] = \lim_{n\to\infty}[I_n']$. In the present setting, the proof is rather trivial.

A Entropy of universal covering maps

Let m be the Lebesgue measure on the unit circle, normalized to have unit mass. It is well known that if F is an inner function with $F(0) = 0$, then m is F-invariant, i.e. $m(E) = m(F^{-1}(E))$ for any measurable set $E \subset \mathbb{S}^1$. In the work [4], M. Craizer showed that if $F \in \mathscr{J}$, then the integral

$$\int_{|z|=1} \log |F'(z)| dm$$

has the dynamical interpretation as the measure-theoretic entropy of m. It is therefore of interest to compute it in special cases. For finite Blaschke products, one may easily compute the entropy using Jensen's formula:

Theorem A.1. *Suppose F is a finite Blaschke product with $F(0) = 0$ and $F'(0) \neq 0$. We have*

$$\frac{1}{2\pi}\int_{|z|=1} \log|F'(z)|d\theta \;=\; \sum_{\text{crit}} \log\frac{1}{|c_i|} - \sum_{\text{zeros}} \log\frac{1}{|z_i|}, \tag{A.1}$$

where in the sum over the zeros of F, we omit the trivial zero at the origin.

In this appendix, we discuss a complementary example:

Theorem A.2 (Pommerenke)**.** *Let P be a relatively closed subset of the unit disk not containing 0. Let $\mathcal{U}_P : \mathbb{D} \to \mathbb{D} \setminus P$ be the universal covering map, normalized so that $\mathcal{U}_P(0) = 0$ and $\mathcal{U}_P'(0) > 0$. Then $\mathcal{U}_P \in \mathscr{J}$ if and only if P is a Blaschke sequence, in which case*

$$\frac{1}{2\pi}\int_{|z|=1} \log|\mathcal{U}_P'(z)|d\theta \;=\; \sum_{p_i\in P} \log\frac{1}{|p_i|} - \sum_{\text{zeros}} \log\frac{1}{|z_i|}. \tag{A.2}$$

A theorem of Frostman says that $\mathcal{U}_P$ is an inner function if and only if the set P has logarithmic capacity 0, see [3, Chapter 2.8]. In particular, $\mathcal{U}_P$ is inner if P is countable.

For brevity, we will write $F = \mathcal{U}_P$. While Pommerenke did not explicitly state (A.2), in the work [19], he proved the equivalent statement

$$\operatorname{Inn} F'(z) \;=\; \prod_{i=1}^{k} F_{p_i}(z) \;=\; \prod_{i=1}^{k} \frac{F(z) - p_i}{1 - \overline{p_i}F(z)}, \tag{A.3}$$

so we feel that it is appropriate to name the above theorem after him. Actually, Pommerenke worked in the significantly greater generality of Green's functions for Fuchsian groups of Widom type, so this is only a special case of his result. Below, we give a more direct proof of Theorem A.2 which may be of independent interest.

A.1 Preliminaries

We first recall a well known property of Nevanlinna averages:

Lemma A.3. *If $f \in \mathcal{N}$ is a function in the Nevanlinna class and is not identically 0, then*

$$\frac{1}{2\pi}\int_{|z|=1} \log|f(z)|d\theta - \lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{|z|=r} \log|f(z)|d\theta\right\} = \sigma(f)(\mathbb{S}^1). \tag{A.4}$$

See [9, Section 3] for a proof. For $x \in \mathbb{D}$, let $F_x = T_x \circ f$ denote the *Frostman shift* of F with respect to x, where $T_x(z) = \frac{z-x}{1-\overline{x}z}$. Frostman showed that if x avoids an exceptional set $\mathscr{E}$ of capacity zero, then F_x is a Blaschke product, in which case $\sigma(F_x) = 0$. We will also need:

Lemma A.4. *Let F be an inner function with $F(0) = 0$. For any $x \in \mathbb{D} \setminus \{0\}$,*

$$\log\frac{1}{|x|} = \sum_{F(y)=x} \log\frac{1}{|y|} + \sigma(F_x). \tag{A.5}$$

Proof. Taking $f = F_x$ in Lemma A.3 gives

$$0 = \lim_{r\to 1}\frac{1}{2\pi}\int_{|z|=r} \log|F_x(z)|d\theta + \sigma(F_x).$$

The lemma follows after applying Jensen's formula and taking $r \to 1$. $\square$

In the case when $F \in \mathscr{J}$, Ahern and Clark [1] observed that the exceptional set $\mathscr{E}$ of F is at most countable and that the singular masses of different Frostman shifts F_x are mutually singular. More precisely, they showed that the measure $\sigma(F_x)$ is supported on the set of points on the unit circle at which the radial limit of F is x. Since the singular inner function $\operatorname{Sing} F_x$ divides F_x', it must also divide its inner part $\operatorname{Inn} F_x' = \operatorname{Inn} F'$. This shows that

$$\sigma(F') \geq \sum_{x\in\mathscr{E}} \sigma(F_x). \tag{A.6}$$

In other words, $\operatorname{Inn} F'$ is divisible by the product $\prod_{x\in\mathscr{E}} \operatorname{Sing} F_x$.

A.2 Proof of Theorem A.2 when P is a finite set

We first prove Theorem A.2 when $P = \{p_1, p_2, \ldots, p_k\}$ is a finite set. In the formula (A.1), one considers the sum $\sum_{\text{crit}} \log \frac{1}{|c_i|}$ over critical points. It appears that the identity (A.5) allows one to sum over the "critical values" $p_1, p_2, \ldots, p_k$ instead. To make this rigorous, we will construct a special approximation $F_n \to F$ by finite Blaschke products with critical values sets $\{p_1, p_2, \ldots, p_k\}$. Assuming the existence of such an approximating sequence, the argument runs as follows: since the entropy can only decrease after taking limits [9, Theorem 4.2],

$$\begin{aligned}
\frac{1}{2\pi}\int_{|z|=1} \log|F'(z)|d\theta &\le \liminf_{n\to\infty} \frac{1}{2\pi}\int_{|z|=1} \log|F_n'(z)|d\theta, \\
&\le \liminf_{n\to\infty}\left\{\log|F_n'(0)| + \sum_{i=1}^{k} \sum_{F_n(q_i)=p_i} \log\frac{1}{|q_i|}\right\}, \\
&= \log|F'(0)| + \sum_{i=1}^{k} \log\frac{1}{|p_i|}.
\end{aligned}$$

However, by (A.6), the other direction is automatic:

$$\begin{aligned}
\frac{1}{2\pi}\int_{|z|=1} \log|F'(z)|d\theta &= \lim_{r\to 1} \frac{1}{2\pi}\int_{|z|=r} \log|F'(z)|d\theta + \sigma(F'), \\
&\ge \log|F'(0)| + \sum_{i=1}^{k} \sigma(F_{p_i}), \\
&= \log|F'(0)| + \sum_{i=1}^{k} \log\frac{1}{|p_i|}.
\end{aligned}$$

Logic dictates that the sequence $F_n \to F$ is stable and the formula (A.2) holds.

A.3 Construction of the approximating sequence

For the construction of the approximating sequence, we employ the gluing technique of Stephenson [22], also see the paper of Bishop [2]. For each puncture p_i, choose a real-analytic arc which joins p_i to a point on the unit circle, so that the arcs are disjoint and do not pass through the origin. Define a *tile* or *sheet* to be the shape

$\mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$. Let $\Gamma = \langle g_1, g_2, \ldots, g_k \rangle$ be the free group on k generators. Consider the countable collection $\{T_g\}_{g \in \Gamma}$ of tiles indexed by elements of Γ. We form a simply-connected Riemann surface S by gluing the lower side of γ_i in T_g to the upper side of γ_i in $T_{g_i g}$. The surface S comes equipped with a natural projection to the disk $\mathbb{D}$ which sends a point in a tile T_g to its representative in the model $\mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$. We may uniformize $\mathbb{D} \cong S$ by taking 0 in the base tile T_e to 0. In this uniformizing coordinate, the projection F becomes a holomorphic self-map of the disk. Since all the slits have been glued up, F is an inner function, and a little thought shows that it is the universal covering map of $\mathbb{D} \setminus \{p_1, p_2, \ldots, p_k\}$.

We now give a slightly different description of the above construction. For this purpose, we need the notion of an ∞-*stack*: a countable collection of tiles $\{T_j\}_{j \in \mathbb{Z}}$, where the lower side of γ_i in T_j is identified with the upper side of γ_i in T_{j+1}. To highlight the dependence on the curve γ_i, we say that the ∞-stack is glued over γ_i. Similarly, by an n-*stack*, we mean a set of n tiles with the above identifications made modulo n. Now, to construct S, we begin with the base tile $T_e \cong \mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$, and at each slit $\gamma_i \subset T_e$, we glue an ∞-stack (i.e. we add the tiles $\{T_j\}_{j \in \mathbb{Z} \setminus \{0\}}$ and treat T_e as T_0). We refer to the tiles that were just added as the tiles of generation 1. To each of the $k-1$ unglued slits in each tile of generation 1, we glue a further ∞-stack of tiles, which we call tiles of generation 2. Repeating this construction infinitely many times gives the Riemann surface S from before.

For the finite approximations, we slightly modify the above procedure. We begin with a base tile $T_e \cong \mathbb{D} \setminus \cup_{i=1}^{k} \gamma_i$ with k slits. At each of these k slits, we glue in an n-stack of sheets (sheets of generation 1). At each of the $k-1$ unresolved slits of sheet of generation 1, we glue in a further n-stack (sheets of generation 2). We repeat for n generations. Finally, at sheets of generation n, we resolve the slits by simply sowing their edges together. This gives us a Riemann surface S_n and a finite Blaschke product F_n with critical values $p_1, p_2, \ldots, p_k$.

Since the Riemann surfaces $S_n \to S$ converge in the Carathéodory topology, the maps $F_n \to F$ converge uniformly on compact sets. With the construction of the special approximating sequence, the proof of Theorem A.2 is complete (when the number of punctures is finite).

A.4 Proof of Theorem A.2 when P is infinite

We handle the infinite case by reducing it to the finite case. This is achieved by the following lemma:

Lemma A.5. *Suppose that* $\mathcal{U}_P$ *is an inner function. Then,*

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_P'(z)|d\theta \geq \frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_Q'(z)|d\theta, \tag{A.7}$$

for any $Q \subseteq P$.

Proof. Topological considerations allow us to factor $\mathcal{U}_P = \mathcal{U}_Q \circ h$, where h is a holomorphic map of the disk. The normalizations $\mathcal{U}_P(0) = \mathcal{U}_Q(0) = 0$ imply that $h(0) = 0$. Since $\mathcal{U}_P$ is inner, h must also be inner. The chain rule and the h-invariance of Lebesgue measure give

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_P'|d\theta = \frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_Q'|d\theta + \frac{1}{2\pi}\int_{|z|=1}\log|h'|d\theta$$

Since h is inner and $h(0) = 0$, $|h'(z)| \geq 1$ for $z \in \mathbb{S}^1$, see e.g. [17, Theorem 4.15]. Dropping second term gives (A.7). $\square$

Proof of Theorem A.2 when P is infinite. The above lemma shows that if P is not a Blaschke sequence, then $\mathcal{U}_P$ cannot be an inner function of finite entropy. Conversely, if $P = \{p_1, p_2, \dots\}$ is a Blaschke sequence, then the integrals

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_{P_k}'(z)|d\theta, \qquad P_k = \{p_1, p_2, \dots, p_k\},$$

are increasing in k and

$$\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_P'(z)|d\theta \geq \lim_{k\to\infty}\frac{1}{2\pi}\int_{|z|=1}\log|\mathcal{U}_{P_k}'(z)|d\theta. \tag{A.8}$$

Since the entropy can only decrease in the limit [9, Theorem 4.2], we must have equality in (A.8). This completes the proof. $\square$

B Existence of Perron hulls

Here, we prove Theorem 2.1 which says that any subsolution of the Gauss curvature equation admits a minimal dominating solution (a Perron hull). We will deduce it from the following the following theorem:

Theorem B.1. *If $\tilde{\nu} \geq 0$ is a Blaschke measure on the unit disk and $h \in L^1(\partial\mathbb{D})$, then the Gauss curvature equation*

$$\begin{cases} \Delta u - 4e^{2u} = 2\pi\tilde{\nu}, & \text{in } \mathbb{D}, \\ u = h, & \text{on } \mathbb{S}^1, \end{cases} \tag{B.1}$$

admits a unique solution.

We can make sense of the boundary condition in (B.1) in two equivalent ways. In the spirit of this paper, we can say that u is a solution if the measures $(u\,d\theta)|_{\{|z|=r\}}$ tend weakly to $h\,d\theta$. To state the second interpretation, note that by Green's formula, if $\phi \in C_0^2(\overline{\mathbb{D}})$ is a C^2 function which vanishes on the unit circle, then

$$\int_{\mathbb{D}} (-u\Delta\phi + \phi\Delta u)|dz|^2 = -\lim_{r\to 1}\int_{\partial\mathbb{D}_r} \partial_{\mathbf{n}}\phi \cdot u\,d\theta,$$

where $\partial_{\mathbf{n}}$ denotes the derivative with respect to the outward unit normal. This suggests that one may can call u a solution of (B.1) if the equality

$$\int_{\mathbb{D}} -u\Delta\phi|dz|^2 + \phi(e^{2u}|dz|^2 + 2\pi\,d\tilde{\nu}) = -\int_{\mathbb{S}^1} \partial_{\mathbf{n}}\phi \cdot h\,d\theta \tag{B.2}$$

holds for every non-negative $\phi \in C_0^2(\overline{\mathbb{D}})$. This definition is used in the book of Marcus and Véron [16] which we use as a reference for this section. The equivalence of the two definitions easily follows from Green's formula. One may define the notions of subsolution and supersolution by changing (B.2) to an inequality.

Proof of Theorem B.1. A well known result of Brezis and Strauss [16, Proposition 2.1.2] says that there exists a unique solution u_h for the equation

$$\begin{cases} \Delta u - 4e^{2u} = 0, & \text{in } \mathbb{D}, \\ u = h, & \text{on } \mathbb{S}^1, \end{cases} \tag{B.3}$$

since we have L^1-data (the choice of non-linearity e^{2u} is not important here). This is a supersolution for the boundary value problem (B.1) since it has boundary measure $h\,d\theta$. Also, $u_h - \log \frac{1}{|I_\nu|}$ is a subsolution for (B.1) since it also has boundary measure $h\,d\theta$. By [16, Theorem 2.2.4], there exists a solution of (B.1). Uniqueness is provided by [16, Proposition 2.2.1]. □

We can now explain the construction of the Perron hull:

Proof of Theorem 2.1. Let u be a subsolution of the Gauss curvature equation $\Delta u - 4e^{2u} = 2\pi\tilde{\nu}$ on the unit disk where $\tilde{\nu}$ is a locally finite measure. For each $0 < r < 1$, we may use Theorem B.1 to produce a solution $\Lambda_r[u]$ on the disk $\mathbb{D}_r = \{z : |z| < r\}$ with boundary values $u|_{\partial\mathbb{D}_r}$, which are guaranteed to be in $L^1(\mathbb{D}_r)$ since u is subharmonic. As $r \to 1$, the $\Lambda_r[u]$ form an increasing family of solutions (defined on an increasing family of domains) which are bounded above by $u_{\mathbb{D}}$, and therefore they must converge to a solution $\Lambda[u]$. This can be easily verified using the definition of a subsolution and the dominated convergence theorem. From the construction, it is clear that $\Lambda_r[u]$ and $\Lambda[u]$ are minimal dominating solutions.

Suppose that $\overline{u} \geq u$ is a dominating supersolution. To show that $\overline{u} \geq \Lambda[u]$, it suffices to show $\overline{u} \geq \Lambda_r[u]$ on $\mathbb{D}_r$ for any $0 < r < 1$. However, this follows from the principle that there is a solution between a subsolution and any supersolution greater than it, see [16, Theorem 2.2.4]. □

References

[1] P. R. Ahern, D. N. Clark, *On inner functions with H_p-derivative*, Michigan Math. J. 21 (1974), no. 2, 115–127.

[2] C. J. Bishop, *An indestructible Blaschke product in the little Bloch space*, Publicacions Matemtiques 37 (1993), no. 1, 95–109.

[3] E. F. Collingwood, A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56. Cambridge University Press, Cambridge, 1966.

[4] M. Craizer, *Entropy of inner functions*, Israel J. Math. 74 (1991), no. 2, 129–168.

[5] K. M. Dyakonov, *Inner functions and inner factors of their derivatives*, Integr. Equ. Oper. Theory 82 (2015), no. 2, 151–155.

[6] H. Hedenmalm, B. Korenblum, K. Zhu, *Beurling type Invariant Subspaces of the Bergman Spaces*, J. London Math. Soc. 53 (1996), no. 2, 601–614.

[7] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics 199, Springer-Verlag, 2000.

[8] M. Heins, *On a class of conformal metrics*, Nagoya Math. J. 21 (1962), 1–60.

[9] O. Ivrii, *Prescribing inner parts of derivatives of inner functions*, J. d'Analyse Math. (to appear).

[10] B. Korenblum, *Cyclic elements in some spaces of analytic functions*, Bull. Amer. Math. Soc. 5 (1981), 317–318.

[11] B. Korenblum, *Outer Functions and Cyclic Elements in Bergman Spaces*, J. Funct. Anal. 151 (1993), no. 1, 104–118.

[12] D. Kraus, *Critical sets of bounded analytic functions, zero sets of Bergman spaces and nonpositive curvature*, Proc. London Math. Soc. 106 (2013), no. 4, 931–956.

[13] D. Kraus, O. Roth, *Conformal metrics*, Lecture Notes Ramanujan Math. Society, Lecture Notes Series 19 (2013), 41–83.

[14] A. C. Ponce, *Elliptic PDEs, Measures and Capacities: From the Poisson Equation to Nonlinear Thomas–Fermi Problems*, EMS Tracts in Mathematics 23, 2016.

[15] M. Marcus, A. C. Ponce, *Reduced limits for nonlinear equations with measures*, J. Funct. Anal 258 (2010), no. 7, 2316–2372.

[16] M. Marcus, L. Véron, *Nonlinear second order elliptic equations involving measures*, De Gruyter Series in Nonlinear Analysis and Applications 21, De Gruyter, Berlin, 2014.

[17] J. Mashreghi, *Derivatives of Inner Functions*, Fields Institute Monographs, 2012.

[18] C. T. McMullen, *Ribbon $\mathbb{R}$-trees and holomorphic dynamics on the unit disk*, J. Topol. 2 (2009), 23–76.

[19] C. Pommerenke, *On the Green's function of Fuchsian groups*, Ann. Acad. Sci. Fenn. Math. 2 (1976), 409–427.

[20] J. W. Roberts, *Cyclic inner functions in the Bergman spaces and weak outer functions in H^p*, $0 < p < 1$, Illinois J. Math. 29 (1985), 25–38.

[21] S. M. Shimorin, *Approximate spectral synthesis in the Bergman space*, Duke Math. J. 101 (2000), no. 1, 1–39.

[22] K. Stephenson, *Construction of an inner function in the little Bloch space*, Trans. Amer. Math. Soc. 308 (1988), no. 2, 713–720.