## Appendix F

## Transforms, Complex Analysis

This appendix discusses Fourier and Laplace transforms as they are used in plasma physics and this book. Also, key properties of complex variable theory that are needed for understanding and inverting these transforms, and to define singular integrals that arise in plasma physics, are summarized here.

Fourier and Laplace transforms are useful in solving differential equations because they convert differentiation in the dependent variable into multiplication by the transform variable. Thus, they convert linear differential equations into algebraic equations in the transformed variables. In addition, Laplace transforms introduce the (temporal) initial conditions and hence causality into the transformed equations and the ultimate (inverse transform) solution.

## F. 1 Fourier Transforms

Fourier transforms are usually used for representing spatial variations because the spatial domain of the response is often localized away from the boundaries. For such situations the spatial domain can be considered infinite: $|\mathbf{x}| \leq \infty$. The Fourier transform $\mathcal{F}$ (transformed functions are indicated by hats over them) and its inverse $\mathcal{F}^{-1}$ are defined in three dimensions by ${ }^{1}$

$$
\begin{array}{ll}
\hat{f}(\mathbf{k})=\mathcal{F}\{f(\mathbf{x})\} \equiv \int d^{3} x e^{-i \mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}), & \text { Fourier transform, } \\
f(\mathbf{x}) \stackrel{\text { ae }}{=} \mathcal{F}^{-1}\{\hat{f}(\mathbf{k})\} \equiv \int \frac{d^{3} k}{(2 \pi)^{3}} e^{i \mathbf{k} \cdot \mathbf{x}} \hat{f}(\mathbf{k}), & \text { inverse Fourier transform. }
\end{array}
$$

[^0]These three dimensional integrals are defined in cartesian coordinates by

$$
\begin{equation*}
\int d^{3} x \equiv \int_{-\infty}^{\infty} d x \int_{-\infty}^{\infty} d y \int_{-\infty}^{\infty} d z, \quad \int d^{3} k \equiv \int_{-\infty}^{\infty} d k_{x} \int_{-\infty}^{\infty} d k_{y} \int_{-\infty}^{\infty} d k_{z} \tag{F.3}
\end{equation*}
$$

Sufficient conditions for the integral in the Fourier transform to converge are that $f(\mathbf{x})$ be piecewise smooth and that the integral of $f(\mathbf{x})$ converges absolutely:

$$
\begin{equation*}
\int d^{3} x|f(\mathbf{x})|<\text { constant, } \quad \text { Fourier transform convergence condition. } \tag{F.4}
\end{equation*}
$$

When these conditions are satisfied, the inverse Fourier transform yields the original function $f(\mathbf{x})$ at all $\mathbf{x}$ except at a discontinuity in the function where it yields the average of the values of $f(\mathbf{x})$ on the two sides of the discontinuity.

Some useful Fourier transforms are (here $k^{2} \equiv \mathbf{k} \cdot \mathbf{k}$ )

$$
\begin{align*}
\mathcal{F}\{1\} & =(2 \pi)^{3} \delta(\mathbf{k}),  \tag{F.5a}\\
\mathcal{F}\left\{\delta\left(\mathbf{x}-\mathbf{x}_{0}\right)\right\} & =e^{-i \mathbf{k} \cdot \mathbf{x}_{0}},  \tag{F.5b}\\
\mathcal{F}\left\{e^{i \mathbf{k}_{0} \cdot \mathbf{x}}\right\} & =(2 \pi)^{3} \delta\left(\mathbf{k}-\mathbf{k}_{0}\right),  \tag{F.5c}\\
\mathcal{F}\left\{e^{-|\mathbf{x}| / \Delta} /|\mathbf{x}|\right\} & =4 \pi /\left(k^{2}+1 / \Delta^{2}\right),  \tag{F.5d}\\
\mathcal{F}\left\{e^{-|\mathbf{x}|^{2} / 2 \Delta^{2}}\right\} & =(\sqrt{2 \pi} \Delta)^{3} e^{-k^{2} \Delta^{2} / 2},  \tag{F.5e}\\
\mathcal{F}\{f(\mathbf{x})\} & =\hat{f}(\mathbf{k}),  \tag{F.5f}\\
\mathcal{F}\{\boldsymbol{\nabla} f(\mathbf{x})\} & =i \mathbf{k} \hat{f}(\mathbf{k}),  \tag{F.5}\\
\mathcal{F}\{\boldsymbol{\nabla} \cdot \mathbf{A}\} & =i \mathbf{k} \cdot \hat{\mathbf{A}}(\mathbf{k}),  \tag{F.5h}\\
\mathcal{F}\{\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})\} & =i \mathbf{k} \times \hat{\mathbf{A}}(\mathbf{k}),  \tag{F.5i}\\
\mathcal{F}\left\{\nabla^{2} f(\mathbf{x})\right\} & =-k^{2} \hat{f}(\mathbf{k}),  \tag{F.5j}\\
\mathcal{F}\left\{\int d^{3} x^{\prime} G\left(\mathbf{x}-\mathbf{x}^{\prime}\right) f\left(\mathbf{x}^{\prime}\right)\right\} & =\hat{G}(\mathbf{k}) \hat{f}(\mathbf{k}) .
\end{align*}
$$

The last relation is called the Fourier convolution relation. Corresponding inverse Fourier transforms can be inferred by taking the inverse Fourier transforms of these relations and using the fact that $\mathcal{F}^{-1} \mathcal{F}\{f(\mathbf{x})\} \stackrel{\text { ae }}{=} f(\mathbf{x})$.

As can be seen from (F.5e), which is indicative of the Fourier transform of the smoothest possible localized function in space, the localization in space $\left(\delta x_{\mathrm{rms}}=\Delta\right)$ times the localization in $\mathbf{k}$-space $\left(\delta k_{\mathrm{rms}}=1 / \Delta\right)$ is subject to the condition:

$$
\begin{equation*}
\delta k \delta x \geq 1, \quad \text { uncertainty relation. }{ }^{2} \tag{F.6}
\end{equation*}
$$

Taking the dot product of the Fourier transform of a vector field with its complex conjugate and integrating over all $\mathbf{k}$-space yields

$$
\begin{equation*}
\int d^{3} x|\mathbf{A}(\mathbf{x})|^{2}=\int \frac{d^{3} k}{(2 \pi)^{3}}|\hat{\mathbf{A}}(\mathbf{k})|^{2}, \quad \text { Parseval's theorem. } \tag{F.7}
\end{equation*}
$$

[^1]
## F. 2 Laplace Transforms

Laplace transforms are often used to analyze the temporal evolution in response to initial conditions from the present time $(t=0)$ forward in time, which defines an infinite half-space time domain $(0<t<\infty)$ problem. The Laplace transform $\mathcal{L}$ and its inverse $\mathcal{L}^{-1}$ are defined by ${ }^{3}$
$\hat{f}(\omega)=\mathcal{L}\{f(t)\} \equiv \int_{0^{-}}^{\infty} d t e^{i \omega t} f(t), \quad$ Laplace transform,
$f(t) \stackrel{\text { ae }}{=} \mathcal{L}^{-1}\{\hat{f}(\omega)\} \equiv \int_{-\infty+i \sigma}^{\infty+i \sigma} \frac{d \omega}{2 \pi} e^{-i \omega t} \hat{f}(\omega)$, inverse Laplace transform. (F.9)
Sufficient conditions for the Laplace transform integral to converge are that $f(t)$ be piecewise smooth and at most of exponential order:
$\lim _{t \rightarrow \infty} f(t)<$ constant $\times e^{\sigma t}, \quad$ Laplace transform convergence condition, (F.10)
which defines the convergence parameter $\sigma$ needed for the path of integration in the inverse Laplace transform (F.9). The function $f(t)$ can grow exponentially in time like $e^{\gamma t}$; then $\sigma>\gamma$ is required for (F.10) to be satisfied. The obtained transform $\hat{f}(\omega)$ is only valid for $\operatorname{Im}\{\omega\}>\sigma$. As indicated by the "ae" (almost everywhere) over the equal sign in (F.9), the inverse Laplace transform yields the original function $f(t)$ for all $t$ except at a discontinuity in the function where it yields the average of the values of $f(t)$ on the two sides of the discontinuity. Because the original function and its inverse Laplace transform are only valid for $t \geq 0$, some people introduce a Heaviside step function $H(t)$ (see Section B.1) into the integral in the definition of the inverse transform in (F.9) to emphasize that fact.

[^2]Some useful Laplace transforms are

$$
\begin{array}{rlr}
\mathcal{L}\left\{e^{-\nu t}\right\} & =\frac{i}{\omega+i \nu}, & \sigma>-\nu, \\
\mathcal{L}\left\{e^{-i \bar{\omega} t}\right\} & =\frac{i}{\omega-\bar{\omega}}, & \sigma>\operatorname{Im} m \bar{\omega}\} \\
\mathcal{L}\left\{e^{\gamma t} \sin \left(\omega_{0} t\right)\right\} & =\frac{-\omega_{0}}{(\omega-i \gamma)^{2}-\omega_{0}^{2}}, & \sigma>\gamma, \\
\mathcal{L}\left\{e^{\gamma t} \cos \left(\omega_{0} t\right)\right\} & =\frac{i(\omega-i \gamma)}{(\omega-i \gamma)^{2}-\omega_{0}^{2}}, & \sigma>\gamma, \\
\mathcal{L}\left\{\frac{e^{-x^{2} / 4 D t}}{\sqrt{\pi t}}\right\} & =\frac{e^{-x \sqrt{-i \omega / D}}}{\sqrt{-i \omega}}, & \\
\mathcal{L}\{H(t)\} & =\frac{i}{\omega}=\frac{1}{-i \omega}, & 1, \\
\mathcal{L}\{\delta(t)\} & =1, \\
\mathcal{L}\left\{\frac{d \delta(t)}{d t}\right\} & =-i \omega, \\
\mathcal{L}\left\{\frac{e^{-i \bar{\omega} t}}{\sqrt{t}}\right\} & =\sqrt{\frac{\pi}{-i(\omega-\bar{\omega})},} \\
\mathcal{L}\{f(t)\} & =\hat{f}(\omega) \\
\mathcal{L}\{\dot{f}(t)\} & =-i \omega \hat{f}(\omega)-f(0), & \\
\mathcal{L}\{\ddot{f}(t)\} & =-\omega^{2} \hat{f}(\omega)+i \omega f(0)-\dot{f}(0), \\
\mathcal{L}\left\{t^{n} f(t)\right\} & =\frac{1}{i^{n}} \frac{d^{n} \hat{f}(\omega)}{d \omega^{n}}, \\
\mathcal{L}\left\{\int_{0}^{t} d t^{\prime} G\left(t-t^{\prime}\right) f\left(t^{\prime}\right)\right\} & =\hat{G}(\omega) \hat{f}(\omega) \tag{F.11n}
\end{array}
$$

In (F.11b) and (F.11i) the frequency $\bar{\omega}$ is in general complex. In (F.11c) and (F.11d) the frequency $\omega_{0}$ and gowth rate $\gamma$ are real. In (F.11g) and (F.11h) the integrals over the delta functions are evaluated by taking account of the lower limit of the Laplace transform integral being $0^{-}$(an infinitesimal negative time near zero) where the delta function vanishes. The last relation is called the Laplace convolution relation. Corresponding inverse Laplace transforms can be inferred by taking the inverse Laplace transforms of these relations and using the fact that $\mathcal{L}^{-1} \mathcal{L}\{f(t)\} \stackrel{\text { ae }}{=} f(t)$. [A Heaviside unit step function $H(t)$ (see Section B.1) is sometimes inserted to remind one that Laplace transforms are only defined for $t>0$, i.e., $\mathcal{L}^{-1} \mathcal{L}\{f(t)\} \stackrel{\text { ae }}{=} H(t) f(t)$.] The simultaneous localization in time and frequency is subject to a condition similar to (F.6):

$$
\begin{equation*}
\delta \omega \delta t \geq 1, \quad \text { uncertainty relation. } \tag{F.12}
\end{equation*}
$$

It is important to be aware of the differences between Fourier and Laplace transforms. The main difference is that Fourier transforms represent functions

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in infinite domains (in space) that have no starting or ending points and no preferred directions of motion in them. In contrast, Laplace transforms represent functions in an infinite half-space of time that begins (with suitable intitial conditions) at $t=0$, increases monotonically, and extends to an infinite time in the future $(t \rightarrow \infty)$. These physical differences are manifested mathematically in their transforms of unity. From (F.5a), the Fourier transform of unity is $\mathcal{F}\{1\}=(2 \pi)^{3} \delta(\mathbf{k})$, which is a function of $\mathbf{k}$ that is singular at $\mathbf{k}=\mathbf{0}$. In contrast, from (F.11a) with $\nu \rightarrow 0$, the corresponding Laplace transfom is $\mathcal{L}\{1\}=i / \omega, \operatorname{Im}\{\omega\}>0$, which is singular for $\omega \rightarrow 0$ but with the nature of the singularity defined (see Sections F. 4 to F.6) by the condition $\operatorname{Im}\{\omega\}>\sigma$. Physically, this condition implies that as time progresses the response grows less rapidly than $e^{\sigma t}$. Thus, Laplace transforms embody the physical property of causality that the response proceeds sequentially in time from its initial conditions whereas Fourier transforms embody no such directionality in the response (or dependence on initial or boundary conditions). This key difference is often highlighted by referring to the relevance of Laplace transforms for initial value problems and for ensuring temporal causality in the solution.

## F. 3 Combined Fourier-Laplace Transforms

Often we will need a combination of a three-dimensional Fourier transform in space and a Laplace transform in time, which is defined by

$$
\begin{equation*}
\hat{f}(\mathbf{k}, \omega)=\mathcal{F} \mathcal{L}\{f(\mathbf{x}, t)\} \equiv \int d^{3} x \int_{0^{-}}^{\infty} d t e^{-i(\mathbf{k} \cdot \mathbf{x}-\omega t)} f(\mathbf{x}, t) . \tag{F.13}
\end{equation*}
$$

The corresponding combined inverse transform is defined by

$$
\begin{equation*}
f(\mathbf{x}, t) \stackrel{\text { ae }}{=} \mathcal{F}^{-1} \mathcal{L}^{-1}\{\hat{f}(\mathbf{k}, \omega)\} \equiv \int \frac{d^{3} k}{(2 \pi)^{3}} \int_{-\infty+i \sigma}^{\infty+i \sigma} \frac{d \omega}{2 \pi} e^{i(\mathbf{k} \cdot \mathbf{x}-\omega t)} \hat{f}(\mathbf{k}, \omega) . \tag{F.14}
\end{equation*}
$$

For a monochromatic wave $\left[\hat{f}(\mathbf{k}, \omega)=f_{\mathbf{k}_{0}, \omega_{0}}(2 \pi)^{4} \delta\left(\mathbf{k}-\mathbf{k}_{0}\right) \delta\left(\omega-\omega_{0}\right)\right]$, we have

$$
\begin{equation*}
f(\mathbf{x}, t)=f_{\mathbf{k}_{0}, \omega_{0}} e^{i\left(\mathbf{k}_{0} \cdot \mathbf{x}-\omega_{0} t\right)}, \quad \text { three-dimensional plane wave. } \tag{F.15}
\end{equation*}
$$

The representation of $f(\mathbf{x}, t)$ in terms of its transform $\hat{f}(\mathbf{k}, \omega)$ in (F.14) is a very useful form that is often used (for both scalar functions and vector fields) and one from which the Fourier and Laplace transforms of spatial and temporal derivatives in (F.5f)-(F.5j) and (F.11j)-(F.111) can be deduced readily.

## F. 4 Properties of Complex Variables, Functions

A complex variable $z=x+i y$ is a two-dimensional variable (vector) that has real $\left[x \equiv \operatorname{Re}\{z\} \equiv z_{R}\right]$ and imaginary $\left[y \equiv \operatorname{Im}\{z\} \equiv z_{I}\right]$ parts. Its cartesian and polar angle representations are

$$
\begin{equation*}
z=x+i y=z_{R}+i z_{I}=r e^{i \theta}, \quad r \equiv|z|=\sqrt{z^{*} z}=\sqrt{x^{2}+y^{2}}, \quad \theta=\arctan y / x . \tag{F.16}
\end{equation*}
$$

The function $e^{i \theta}$ is repesented by

$$
\begin{equation*}
e^{i \theta}=\cos \theta+i \sin \theta, \quad \text { Euler's formula. } \tag{F.17}
\end{equation*}
$$

Thus, the imaginary unit number $i \equiv \sqrt{-1}=e^{i \pi / 2}$. [More generally, one defines $\left.i=e^{i(4 n+1) \pi / 2}, n=0, \pm 1, \pm 2, \ldots ..\right]$ The complex conjugate of $z$ is

$$
\begin{equation*}
z^{*}=x-i y=|z| e^{-i \theta}, \quad \text { complex conjugate. } \tag{F.18}
\end{equation*}
$$

The reciprocal of a complex variable can be written many ways:

$$
\begin{equation*}
\frac{1}{z}=\frac{1}{x+i y}=\frac{x-i y}{(x+i y)(x-i y)}=\frac{x-i y}{x^{2}+y^{2}}=\frac{z^{*}}{|z|^{2}}=\frac{e^{-i \theta}}{|z|} \tag{F.19}
\end{equation*}
$$

A function of a complex variable $w(z) \equiv w_{R}(z)+i w_{I}(z)$ is analytic at a point $z \equiv z_{R}+i z_{I}$ if its derivative $d w / d z$ exists there and is the same irrespective of the direction in the complex $z$-plane along which the derivative is calculated. This criterion for a function to be analytic yields the sufficient conditions

$$
\begin{equation*}
\frac{\partial w_{R}}{\partial z_{R}}=\frac{\partial w_{I}}{\partial z_{I}}, \quad \frac{\partial w_{R}}{\partial z_{I}}=-\frac{\partial w_{I}}{\partial z_{R}}, \quad \text { Cauchy-Riemann conditions for analyticity. } \tag{F.20}
\end{equation*}
$$

A general expansion of a complex function around $z=z_{0}$ is

$$
\begin{equation*}
w(z)=\sum_{n=-\infty}^{\infty} c_{n}\left(z-z_{0}\right)^{n}, \quad \text { Laurent expansion } \tag{F.21}
\end{equation*}
$$

This expansion reduces to a Taylor series expansion if $c_{n}=0$ for all $n<0$; then, $c_{n}=(1 / n!) d^{n} f /\left.d z^{n}\right|_{z=z_{0}}, n=0,1,2, \ldots$.

All functions that are analytic over a region can be expressed in terms of convergent Taylor series, with the radius of convergence bounded by the distance from the expansion point to the nearest singularity. Examples of (entire) functions that are analytic over the entire finite $z$-plane are $z, z^{n}, \sin z, e^{z}$. On the other hand, the function $w_{1}(z)=1+z+z^{2}+\cdots$ has a radius of convergence $|z|<1$. An analytic function can be analytically continued to adjacent regions where the function is analytic through Taylor series expansion about other points in the original analytic region or by other means. For example, the power series in the function $w_{1}(z)$ above can be summed to yield $w_{1}(z)=1 /(1-z)=-1 /(z-1)$, which can be represented by a Laurent series with $c_{-1}=-1$ and $z_{0}=1$ with all other $c_{n}=0$. The function $-1 /(z-1)$ is analytic everywhere except at $z=1$ and represents the analytic continuation of the power series respresentation of $w_{1}(z)$ to all $z \neq 1$.

Nonanalytic functions have singularities ( $z$ values where they are unbounded or about which they are multivalued) and are represented by the Laurent series with $c_{n} \neq 0$ for some $n<0$. Isolated singularities are classified as follows:

- Poles. If the maximum negative power in the Laurent expansion (F.21) is $m$ (i.e., $c_{-m} \neq 0$ and $c_{-n}=0$ for $n>m$ ), then the function $w(z)$ has an $m^{\text {th }}$-order pole at $z=z_{0}$. For example, $w_{1}(z)=-1 /(z-1)$ has a first-order pole at $z=1$ and $1 /(z-1)^{2}$ has a second-order pole at $z=1$.

a)

b)

c)

Figure F.1: Cauchy integral contours $C$ that: a) do not enclose $z_{0}$, b) go "through" $z_{0}$ (really enclose with a small semi-circle), and c) fully enclose $z_{0}$.

- Essential Singularities. If there are an infinite number of negative powers present in the Laurent series (F.21), w(z) has an essential singularity at $z_{0}$. For example, $e^{-1 / z}=1-1 / z+1 / 2 z^{2}-\cdots$ has an essential singularity at $z=0$ and hence is nonanalytic there. The logarithm function $\ln z=$ $\ln |z|+i \theta$ is multivalued (has different values for the same $z$ depending on which $2 \pi$ interval $\theta$ is taken to be in) and has an essential singularity at $z=0$ where it is unbounded. Its "principal value" is usually defined for $0 \leq \theta<2 \pi$ with a branch cut inserted at $\theta=2 \pi$. Additional branches ("Riemann sheets") of $\ln z$ are defined for $2 \pi \leq \theta<4 \pi$, etc. Since the encircling of $z=0$ is the source of the multivaluedness, it is known as a branch point of $\ln z$. Similarly, $\sqrt{z}=|z|^{1 / 2} e^{i \theta / 2}$ has a branch point (essential singularity) at $z=0$ and has two branches that are usually defined for $0 \leq \theta<2 \pi$ and $2 \pi \leq \theta<4 \pi$.


## F. 5 Cauchy Integral

The key properties of integration around a simple, closed contour $C$ in the complex $z$ plane are summarized by a generalized Cauchy integral formula:

$$
\int_{C} d z \frac{f(z)}{z-z_{0}}=\left\{\begin{array}{lll}
0, & \text { if } C \text { does not enclose } z_{0}, & \text { (F.22a) }  \tag{F.22}\\
\pi i f\left(z_{0}\right), & \text { if } C \text { goes through } z_{0}, & \text { (F.22b) } \\
2 \pi i f\left(z_{0}\right), & \text { if } C \text { encloses } z_{0}, & \text { (F.22c) }
\end{array}\right.
$$

Cauchy integral formula.
Here, it is assumed that $f(z)$ is an analytic function of $z$ inside and on the contour $C$, and motion along the contour is in the counterclockwise direction. Also, it is assumed for ( F .22 b ) that the contour $C$ goes through the point $z_{0}$ on a straight path (i.e., $z_{0}$ is not at a square corner or other irregular point on $C$ ) and that $z_{0}$ is on the "inside" edge of the contour $C$ - in a limiting sense. The contours for the three situations in (F.22) are shown in Fig. F.1.

For a general complex function $w(z)$, (F.22c) generalizes to the residue the-
orem for a contour $C$ that encloses isolated pole-type singularities at $z=z_{j}$ :

$$
\begin{equation*}
\int_{C} d z w(z)=2 \pi i \sum_{j} c_{-1}\left(z_{j}\right), \quad \text { Cauchy residue theorem. } \tag{F.23}
\end{equation*}
$$

Here, $c_{-1}\left(z_{j}\right)$ is the residue [coefficient $c_{-1}$ in the Laurent expansion (F.21)] of the function $w(z)$ at the singular point $z=z_{j}$, which is defined by

$$
\begin{array}{ll}
c_{-1}\left(z_{j}\right)=\lim _{z \rightarrow z_{j}}\left[\left(z-z_{j}\right) w(z)\right], & \text { first-order pole, } \quad \text { (F.24a) } \\
c_{-1}\left(z_{j}\right)=\frac{1}{(m-1)!} \lim _{z \rightarrow z_{j}} \frac{d^{m-1}}{d z^{m-1}}\left[\left(z-z_{j}\right)^{m} w(z)\right], & m^{\text {th }} \text {-order pole. } \quad \text { (F.24b) } \tag{F.24}
\end{array}
$$

## F. 6 Inverse Laplace Transform Example

To illustrate the use of these complex variable integration formulas (and develop some inverse transform concepts that are important in plasma physics), consider their use in evaluating the inverse Laplace transform of the weakly damped $\left(\nu \ll \omega_{0}\right)$ oscillator problem given in (??): $\ddot{x}+\nu \dot{x}+\omega_{0}^{2} x=f(x, t)$. For simplicity, assume the initial conditions are $x(0)=x_{0}, \dot{x}(0)=0\left[\theta_{0}=\pi / 2\right.$ in the initial conditions used to derive (??)] and that there is no forcing function $f$. Taking the Laplace transform of the homogeneous damped oscillator equation and solving for the transform of the response, one obtains

$$
\begin{equation*}
\hat{x}(\omega)=\hat{G}(\omega) \hat{S}(\omega)=\frac{x_{0}(\nu-i \omega)}{-\omega^{2}-i \nu \omega+\omega_{0}^{2}}, \quad \hat{S}(\omega) \equiv x_{0} \tag{F.25}
\end{equation*}
$$

The temporal response $x(t)$ is obtained from the inverse Laplace transform:

$$
\begin{align*}
x(t) & =\mathcal{L}^{-1}\{\hat{x}(\omega)\}=\int_{-\infty+i \sigma}^{\infty+i \sigma} \frac{d \omega}{2 \pi} I(\omega)  \tag{F.26}\\
I(\omega) & =-\frac{e^{-i \omega t} x_{0}(\nu-i \omega)}{\left(\omega-\omega_{\nu}+i \nu / 2\right)\left(\omega+\omega_{\nu}+i \nu / 2\right)} \tag{F.27}
\end{align*}
$$

The integrand $I(\omega)$ has first-order poles at $\omega=\omega_{ \pm}$, with residues given by

$$
\begin{equation*}
c_{-1}\left(\omega_{ \pm}\right)= \pm \frac{e^{-i \omega_{ \pm} t}\left(\nu-i \omega_{ \pm}\right)}{2 \omega_{\nu}}, \quad \omega_{ \pm} \equiv \pm \omega_{\nu}-i \nu / 2, \quad \omega_{\nu} \equiv \sqrt{\omega_{0}^{2}-\nu^{2} / 4} \tag{F.28}
\end{equation*}
$$

Figure F.2a illustrates the inverse Laplace transform integration path $(L)$ in (F.26) for an arbitrary $\sigma>0$. As indicated, it is just a line integral from $-\infty+i \sigma$ to $\infty+i \sigma$ along a line that is parallel to the $\omega_{R} \equiv \mathcal{R} e\{\omega\}$ axis, but a distance $\omega_{I} \equiv \operatorname{Im}\{\omega\}=\sigma$ above it. While for this problem we could convert this line integral into a closed contour by adding the (vanishing, for $t>0$ ) integral along the infinite semi-circle in the lower half $\omega$-plane $\left[C_{\mathrm{sc}}\right.$ with $|\omega| \rightarrow \infty$ as shown in Fig. F.2a], we will use a more generally useful procedure. [The vanishing of

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Figure F.2: Illustration of: a) inverse Laplace transform integration path $L$ and infinite semi-circle $C_{\mathrm{sc}}$ in the lower half $\omega$-plane which can be used as a closing contour for $t>0$, and b ) inverse Laplace transform contour $C_{L}$ and dotted contour $C_{0}$ which when added together yield the original integration path $L$.
the inverse Laplace transform for $t<0$ can be shown by closing the contour on an infinite semi-circle in the upper half plane by observing that because of the convergence condition (F.10) there are no singularities within this contour.]

For a general Laplace transform inversion procedure, we analytically continue the Laplace integration contour downward, being careful to deform the contour around the singular points of the integrand, as indicated in Fig. F.2b. The integral along the original Laplace integration path $(L)$ is equal to the sum of the Laplace contour $C_{L}$ and the dotted contour $C_{0}$ between it and the original line integration path $(L)$. However, since there are no singularities of $I(\omega)$ inside the $C_{0}$ contour, this integral vanishes by (F.22a). Thus, the integral in (F.26) becomes

$$
\begin{equation*}
\int_{-\infty+i \sigma}^{\infty+i \sigma} \frac{d \omega}{2 \pi} I(\omega)=\int_{C_{0}} \frac{d \omega}{2 \pi} I(\omega)+\int_{C_{L}} \frac{d \omega}{2 \pi} I(\omega) \Longrightarrow \int_{C_{L}} \frac{d \omega}{2 \pi} I(\omega) \tag{F.29}
\end{equation*}
$$

The $C_{L}$ contour integral includes the two first-order poles at $\omega=\omega_{ \pm}$which are evaluated ${ }^{4}$ with (F.24a) using (F.28) for the residues, plus a line integral along the path $-\infty-i \Sigma$ to $\infty-i \Sigma$ which yields a contribution of order $e^{-\Sigma t}$ :

$$
\begin{align*}
x(t) & =\int_{C_{L}} \frac{d \omega}{2 \pi} I(\omega)=i\left[c_{-1}\left(\omega_{+}\right)+c_{-1}\left(\omega_{-}\right)\right]+\mathcal{O}\left\{e^{-\Sigma t}\right\} \\
& =x_{0} e^{-\nu t / 2}\left[\cos \omega_{\nu} t+\frac{\nu}{2 \omega_{\nu}} \sin \omega_{\nu} t\right]+\mathcal{O}\left\{e^{-\Sigma t}\right\}, \quad t \geq 0 \tag{F.30}
\end{align*}
$$

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The first term is the desired response and is the same as the result (??) obtained via other means in Section E. 2 for the present $\theta_{0}=\pi / 2$ case.

The $\mathcal{O}\left\{e^{-\Sigma t}\right\}$ term in (F.30) represents initial transient responses that decay exponentially in time for $t>1 / \Sigma$. For the present problem since there are no other singularities in the lower half complex $\omega$-plane, we can take $\Sigma \rightarrow \infty$ and this term vanishes. However, for plasma physics responses there are often many (sometimes a denumerable infinity of) singularities in the lower half complex $\omega$ plane and we are usually only interested in the time-asymptotic response. Then, we usually only calculate the responses from the singularities that are highest in the complex $\omega$-plane, and estimate the time scale on which this time-asymptotic response will obtain from the maximum $\Sigma$ for a contour $C_{L}$ that lies just above the next highest singularities. Note that the resultant responses may be growing exponentially in time (if the highest singularities are in the upper half $\omega$-plane), and that the "transients" may also be growing (more slowly) in time (if $\Sigma<0$ ).

The generic physical points evident from this inverse Laplace transform analysis procedure are that: 1) responses are determined by the singularities of the integrand of the inverse Laplace transform, which in turn are usually determined by the singularities of the Laplace transform of the system transfer (Green) function $\hat{G}(\omega) ; 2)$ the singularities that are highest in the complex $\omega$-plane dominate the time-asymptotic response; and 3) the next highest singularities determine the time scale on which this asymptotic response becomes dominant.

## F. 7 Ballistic Propagation Example

As another example, we use Fourier-Laplace transforms and complex variable theory to define the singular responses to "ballistic" propagation of particles along straight-line particle trajectories (??): $\mathbf{x}=\mathbf{x}(t=0)+\mathbf{v} t$. Consider a simple kinetic equation for a distribution $f(\mathbf{x}, \mathbf{v}, t)$ with a kinetic source $S_{f}(\mathbf{x}, \mathbf{v}, t)$ :

$$
\begin{equation*}
\frac{\partial f}{\partial t}+\mathbf{v} \cdot \boldsymbol{\nabla} f=S_{f} \tag{F.31}
\end{equation*}
$$

Taking the Fourier-Laplace transform of this equation using (F.13), (F.5g), and (F.11k), we obtain

$$
\begin{equation*}
-i \omega \hat{f}-f(0)+i \mathbf{k} \cdot \mathbf{v} \hat{f}=\hat{S}_{f} \quad \Longrightarrow \quad \hat{f}(\mathbf{k}, \mathbf{v}, \omega)=\hat{G}(\mathbf{k}, \omega) \hat{S}(\mathbf{k}, \mathbf{v}, \omega), \tag{F.32}
\end{equation*}
$$

with transformed source $\hat{S} \equiv \hat{S}_{f}(\mathbf{k}, \mathbf{v}, \omega)+\check{f}(\mathbf{k}, \mathbf{v}, t=0)$ in which $\check{f}$ represents just a Fourier transform in space rather than a full Fourier-Laplace transform. The full transform $\hat{G}(\mathbf{k}, \omega)$ is in general called a transfer function. Here, it is

$$
\begin{equation*}
\hat{G}(\mathbf{k}, \omega)=\frac{i}{\omega-\mathbf{k} \cdot \mathbf{v}}, \quad \operatorname{Im}\{\omega\}>\sigma, \quad \text { ballistic propagator. } \tag{F.33}
\end{equation*}
$$

This Fourier-Laplace transfer function has a singularity at $\omega=\mathbf{k} \cdot \mathbf{v}$ that is defined (resolved) by the Laplace transform convergence condition (F.10) and hence by the initial-value problem (causality) characteristics of the Laplace
transform. It is called the ballistic propagator in plasma physics because it represents [in $\omega, \mathbf{k}$ transform space - see (F.35) below] motion along straightline particle trajectories.

The kinetic distribution $f$ is obtained from the full inverse transform:

$$
\begin{align*}
f(\mathbf{x}, \mathbf{v}, t) & =\mathcal{F}^{-1} \mathcal{L}^{-1}\{\hat{f}\}=\int \frac{d^{3} k}{(2 \pi)^{3}} \int_{-\infty+i \sigma}^{\infty+i \sigma} \frac{d \omega}{2 \pi} e^{i(\mathbf{k} \cdot \mathbf{x}-\omega t)} \hat{G}(\mathbf{k}, \omega) \hat{S}(\mathbf{k}, \mathbf{v}, \omega) \\
& =\int_{0^{-}}^{t} d t^{\prime} \int d^{3} x^{\prime} G\left(\mathbf{x}-\mathbf{x}^{\prime}, t-t^{\prime}\right) S\left(\mathbf{x}^{\prime}, \mathbf{v}, t^{\prime}\right), \quad t \geq 0 \tag{F.34}
\end{align*}
$$

in which the second line follows from combining the convolution integrals (F.5k) and (F.11n) that result from the inverse Fourier and Laplace transforms of the products of the two transforms $\hat{G}(\mathbf{k}, \omega)$ and $\hat{S}(\mathbf{k}, \mathbf{v}, \omega)$. The Green function $G(\mathbf{x}, t)$ is obtained by first using the same inverse Laplace transform procedure of deforming the Laplace integration contour (see Fig. F.2) downward around the singularity (in this case at $\omega=\mathbf{k} \cdot \mathbf{v}$ ) as was used in the preceding analysis of the damped oscillator. Then, taking account of the first-order pole, evaluating the residue via (F.24a), and using the delta function definition in (??) with $\mathbf{x} \rightarrow \mathbf{x}-\mathbf{v} t$ to evaluate the inverse Fourier transform, we obtain

$$
G(\mathbf{x}, t)=\mathcal{F}^{-1} \mathcal{L}^{-1}\left\{\frac{i}{\omega-\mathbf{k} \cdot \mathbf{v}}\right\}=\int \frac{d^{3} k}{(2 \pi)^{3}} e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{v} t)}=\delta(\mathbf{x}-\mathbf{v} t)
$$

Green function. (F.35)
The inverse Fourier-Laplace transform of $\hat{S}$ is obtained using (F.2), (F.9) and $\mathcal{L}^{-1}$ of (F.11g):

$$
\begin{equation*}
S(\mathbf{x}, \mathbf{v}, t)=S_{f}(\mathbf{x}, \mathbf{v}, t)+f(\mathbf{x}, \mathbf{v}, t=0) \delta(t) \tag{F.36}
\end{equation*}
$$

Substituting (F.35) and (F.36) into (F.34), we obtain for $t \geq 0$

$$
\begin{align*}
f(\mathbf{x}, \mathbf{v}, t) & =\int_{0^{-}}^{t} d t^{\prime} \int d^{3} x^{\prime} \delta\left[\mathbf{x}-\mathbf{x}^{\prime}-\mathbf{v}\left(t-t^{\prime}\right)\right] S\left(\mathbf{x}^{\prime}, \mathbf{v}, t^{\prime}\right) \\
& =f(\mathbf{x}-\mathbf{v} t, \mathbf{v}, t=0)+\int_{0^{-}}^{t} d t^{\prime} S_{f}\left[\mathbf{x}-\mathbf{v}\left(t-t^{\prime}\right), \mathbf{v}, t^{\prime}\right] \tag{F.37}
\end{align*}
$$

which is the "ballistic" response we have been seeking. The first term represents propagation of the initial distribution function along the ballistic straight-line particle trajectories $\mathbf{x}=\mathbf{x}(t=0)+\mathbf{v} t$, while the second represents the time integral of the effect of the propagation of the source function along the same trajectories. Since the solutions propagate (move along) the ballistic motion of the particles, these are called ballistic solutions. Hence, the transform of the Green function that caused this response, which is given in (F.33), is called the ballistic propagator.


Figure F.3: Deformation of $u$ integration contour around the singularity (firstorder pole) at $u=\omega / k$ as $\operatorname{Im}\{\omega\}$ decreases from: a) the original definition region $\operatorname{Im}\{\omega\}>\sigma>0$, b) to the real $\omega$ axis, and c) to the lower half $\omega$-plane.

## F. 8 Singular Integrals In Plasma Physics

Next, we use complex variable theory to define the types of singular integrals that arise in plasma physics from integrating the ballistic propagator over distribution functions. Defining $\mathbf{k} \cdot \mathbf{v}=k u$, the types of integrals that arise are of the form

$$
\begin{equation*}
I(\omega / k) \equiv \int_{-\infty}^{\infty} d u \frac{g(u)}{u-\omega / k}, \quad \operatorname{Im}\{\omega\}>\sigma>0 \tag{F.38}
\end{equation*}
$$

A sufficient condition for this integral to converge is that the integral of $g(u)$ be bounded (i.e., $\left|\int_{-\infty}^{\infty} d u g(u)\right|<$ constant). This integral is analytically continued to lower values of $\operatorname{Im}\{\omega\}$ by deforming the contour around the singularity at $u=\omega / k$ as $\operatorname{Im}\{\omega\}$ moves from the upper to the lower half $\omega$-plane, as indicated in Fig. F. 3 for the usual case of $k>0$. (An integral in the complex plane is analytically continued by deforming its integration contour so it is always on the same side of any pole-type singularities.) Since the integration contour passes under the singularity for $\operatorname{Im}\{\omega\}>0$, "through" it (but actually on a small semi-circle below it) for $\operatorname{Im}\{\omega\}=0$, and encloses it for $\operatorname{Im}\{\omega\}<0$, using (F.22) we see that $I(\omega / k)$ is defined (for ${ }^{5} k>0$ ) by

$$
\int_{-\infty}^{\infty} d u \frac{g(u)}{u-\omega / k} \equiv\left\{\begin{array}{cl}
\int_{-\infty}^{\infty} d u \frac{g(u)}{u-\omega / k}, & \operatorname{Im}\{\omega / k\}>0,(\mathrm{~F} .39 \mathrm{a})  \tag{F.39}\\
\mathcal{P}\left\{\int_{-\infty}^{\infty} d u \frac{g(u)}{u-\omega / k}\right\}+\pi i g(\omega / k), & \operatorname{Im}\{\omega / k\}=0,(\mathrm{~F} .39 \mathrm{~b}) \\
\int_{-\infty}^{\infty} d u \frac{g(u)}{u-\omega / k}+2 \pi i g(\omega / k), & \operatorname{I} m\{\omega / k\}<0 .(\mathrm{F} .39 \mathrm{c})
\end{array}\right.
$$

[^4]

Figure F.4: Areas that cancel in the Cauchy principal value limit process as $\epsilon \rightarrow 0$ to produce a convergent integral are shown cross-hatched.

For $\operatorname{Im}\{\omega\}=0$ the integration over the singularity in the real integral's integrand at $u=\mathcal{R} e\{\omega / k\} \equiv u_{0}$ is defined (i.e., made convergent) by the prescription

$$
\mathcal{P}\left\{\int_{-\infty}^{\infty} d u \frac{g(u)}{u-u_{0}}\right\} \equiv \lim _{\epsilon \rightarrow 0}\left[\int_{-\infty}^{u_{0}-\epsilon} d u \frac{g(u)}{u-u_{0}}+\int_{u_{0}+\epsilon}^{\infty} d u \frac{g(u)}{u-u_{0}}\right]
$$

Cauchy principal value operator $\mathcal{P}$. (F.40)
As shown in Fig. F.4, the Cauchy principal value limit process causes the nearly equal areas on the two sides of the singularity to cancel as $\epsilon \rightarrow 0$; it thereby yields a finite integral as long as $g(u)$ is a continuous function of $u$ at $u=u_{0}$.

The definition of $I(\omega / k)$ in (F.39) appears to be discontinuous as $\operatorname{Im}\{\omega\}$ approaches zero from above and below, but is in fact continuous there. In the limit of $\operatorname{Im}\{\omega\} \sim \epsilon \rightarrow 0$, the singular part of the integrand becomes

$$
\lim _{\epsilon \rightarrow 0} \frac{1}{u-\left(u_{0} \pm i \epsilon\right)}=\lim _{\epsilon \rightarrow 0} \frac{\left(u-u_{0}\right) \pm i \epsilon}{\left(u-u_{0}\right)^{2}+\epsilon^{2}}=\mathcal{P}\left\{\frac{1}{u-u_{0}}\right\} \pm \pi i \delta\left(u-u_{0}\right)
$$

Plemelj formulas. (F.41)
In obtaining the last, imaginary term, we used the definition of the delta function from (??) and (??) in Section B.2. Using the Plemelj formulas, it can be shown that the $\mathcal{I} m\{\omega\} \rightarrow 0$ limits of both (F.39a) and (F.39c) yield (F.39b). Thus, the definition in (F.39) is just what is needed to make $I(\omega / k)$ a continuous function of $\operatorname{Im}\{\omega\}$; hence, (F.39) represents the proper analytic continuation of the function $I(\omega / k)$ defined in (F.38) - from the upper half $\omega$-plane, where it is initially defined, to the entire $\omega$-plane. Note also that since the representations in the various $\operatorname{Im}\{\omega\}$ regions are continuous in the vicinity of $\operatorname{Im}\{\omega\} \simeq 0$, we can use any of the representations there. In plasma physics the representation of $I\{\omega / k\}$ for $\operatorname{Im}\{\omega\}=0$ given in (F.39b) is often used for all $\operatorname{Im}\{\omega\} \simeq 0$.

## REFERENCES

Discussions of transforms and complex variable theory are provided in most advanced engineering mathematics and mathematical physics textbooks, for example:

Greenberg, Advanced Engineering Mathematics (1988,1998), Chapts. 5, 21-24 [?]
Greenberg, Foundations of Applied Mathematics (1978), Chapts. 6, 11-16 [?]
Morse and Feshbach, Methods of Theoretical Physics (1953), Vol. I, Chapt. 4 [?]

Arfken, Mathematical Methods for Physicists (1970) [?]
Kusse and Westwig, Mathematical Physics (1998), Chapts. 6-9 [?]
Classic treatises on the theory of complex variables are
Whittaker and Watson, A Course of Modern Analysis $(1902,1963)$ [?]
Copson, Theory of Functions of a Complex Variable (1935) [?]
Carrier, Crook, Pearson, Functions of a Complex Variable (1966) [?]
An extensive table of Fourier and Laplace (and other) transforms is provided in
Erdèlyi, Tables of Integral Transforms, Vol. 1 (1954) [?]


[^0]:    ${ }^{1}$ The "ae" above the equal sign in the second equation is there to remind us that the inverse transform is equal to the original function "almost everywhere" - namely, everywhere the function $f$ is continuous. At a jump discontinuity the inverse transform is equal to the average of the function across the discontinuity: $[f(\mathbf{x}+\mathbf{0})+f(\mathbf{x}-\mathbf{0})] / 2$.

[^1]:    ${ }^{2}$ This uncertainty relation indicates the degree of localization in $k$-space for a given localization of a function in $x$-space. For the energy density in wave-packets and the probability density in quantum mechanics, the corresponding uncertainty principle is determined using the square of the fluctuating field or wave function; then the uncertainty principle is $\delta k \delta x \geq 1 / 2$.

[^2]:    ${ }^{3}$ In plasma physics it is convenient to use $e^{i \omega t}$ as the integrating factor in the definition of the Laplace transform so that when $\omega$ is real it will represent a (radian) frequency. Many mathematics texts use $e^{-s t}$ or $e^{-p t}(i \omega \Longleftrightarrow-s$ or $-p)$ as the integrating factor to emphasize exponential growth or damping. Most electrical engineering texts use $e^{-j \omega t}(i \omega \Longleftrightarrow-j \omega)$.

[^3]:    ${ }^{4}$ The residue integrals are the negative of (F.23) because the small circular contours around the poles are in the clockwise direction rather than being in the counterclockwise direction for which (F.22) and (F.23) are defined.

[^4]:    ${ }^{5}$ For $k<0$ the integral $I(\omega / k)$ is originally defined for $\operatorname{Im}\{\omega / k\}<0$ and analytically continued to $\operatorname{Im}\{\omega / k\} \geq 0$, which results in $-\pi i g(\omega / k)$ and $-2 \pi i g(\omega / k)$ terms (because of the then clockwise rotation of the integration contour around the pole) on the second and third lines of this definition which are then applicable for $\operatorname{Im}\{\omega / k\}=0$ and $\operatorname{Im}\{\omega / k\}>0$.

