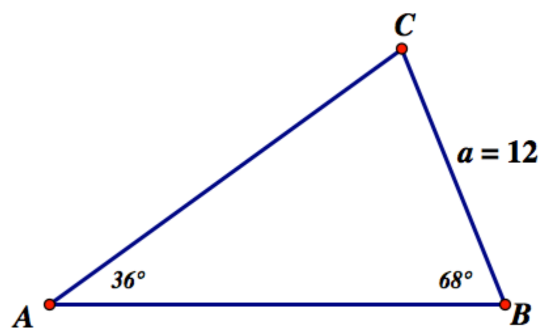
5.8 Triangle Areas by Trig

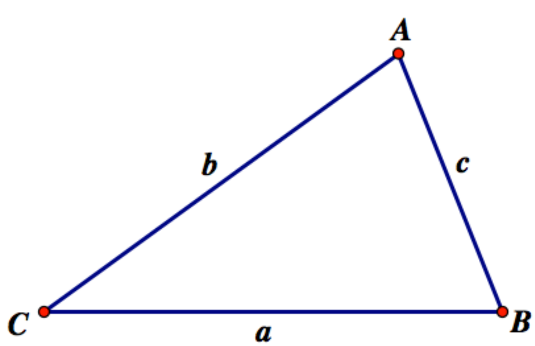
A Practice Understanding Task

Find the area of the following two triangles using the strategies and procedures you have developed in the past few tasks. For example, draw an altitude as an auxiliary line, use right triangle trigonometry, use the Pythagorean theorem, or use the Law of Sines or the Law of Cosines to find needed information.

1. Find the area of this triangle.



2. Find the area of this triangle.



Jumal and Jabari are helping Jumal's father with a construction project. He needs to build a triangular frame as a component of the project, but he has not been given all the information he needs to cut and assemble the pieces of the frame. He is even having a hard time envisioning the shape of the triangle from the information he has been given.

Here is the information about the triangle that Jumal's father has been given.

- Side a = 10.00 meters
- Side b = 15.00 meters
- Angle A = 40.0°

Jumal's father has asked Jumal and Jabari to help him find the measure of the other two angles and the missing side of this triangle.

Carry out each student's strategy as described below. Then draw a diagram showing the shape and dimensions of the triangle that Jumal's father should construct. (Note: To provide as accurate information as possible, Jumal and Jarbari decide to round all calculated sides to the nearest cm—that is, to the nearest hundredth of a meter—and all angle measures to the nearest tenth of a degree.)

<u>Jumal's Approach</u>

- Find the measure of angle B using the Law of Sines
- Find the measure of the third angle C
- Find the measure of side c using the Law of Sines
- Draw the triangle

Jabari's Approach

- Solve for c using the Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos(C)$

  (Jabari is surprised that this approach leads to a quadratic equation, which he solves with the quadratic formula. He is even more surprised when he finds two reasonable solutions for the length of side *c*.)

- Draw both possible triangles and find the two missing angles of each using the Law of Sines

5.8 Triangle Areas by Trig – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice the strategies that have been used in the previous tasks for finding missing sides and angles of non-right triangles. These strategies include drawing an altitude of a triangle as an auxiliary line, using right triangle trigonometry to label sides of the right triangles formed by the altitude, using the Pythagorean theorem to relate sides of the right triangles formed by the altitude, or choosing to use the Law of Sines or the Law of Cosines as appropriate for the given information. Problem 2 leads to an alternative formula for the area of a triangle: $A = \frac{1}{2} ab \sin(C)$. Problem 3 introduces the ambiguous case of the Law of Sines.

Core Standards Focus:

G.SRT.9 (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G.SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G.SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

SMP 8 – Look for and express regularities in repeated reasoning

The Teaching Cycle:

Launch (Whole Class):

Remind students of the strategies they have used in the previous tasks to find missing sides and angles in non-right triangles as described in the introductory paragraph of this task. Students should be able to go right to work on this task following this brief introduction.

Explore (Small Group):

Problem 1 is intended to set up the mathematical thinking that will be used in problem 2 to derive an alternative formula for the area of a triangle based on the length of two sides and an included angle. Students should begin with their familiar formula for finding the area of a triangle, $A = \frac{1}{2}bh$. In problem 1, students can use the Law of Sines to find the lengths of the missing sides of the triangle—one of which can be treated as the base. They will also need to find the length of the altitude associated with this base using a trig ratio. They can then apply their known area formula to find the area of the triangle.

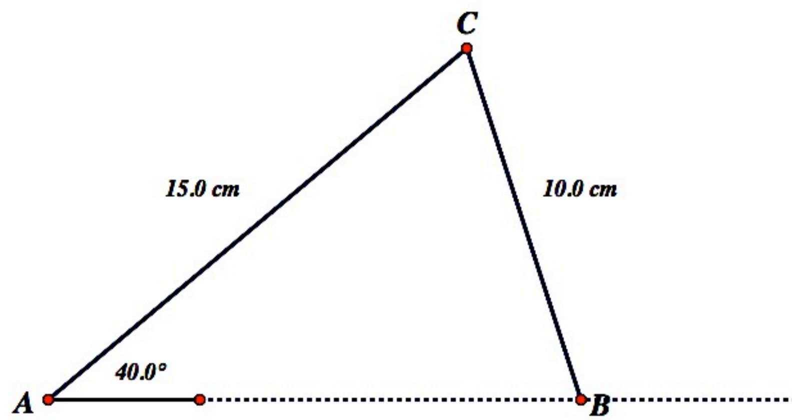
Problem 2 follows the same pattern as problem 1, only the length of the base and corresponding altitude will be a variable and an expression. It is assumed that most students will use side *a* as the base, but the corresponding altitude could be written as $h = b\sin C$ or $h = c\sin B$, leading to two different expressions for the area. Watch for both expressions to emerge.

Problem 3 introduces the ambiguous case of the Law of Sines. Jumal's approach leads to a unique triangle, whereas Jabari's approach leads to two triangles that satisfy the given conditions. Here is an example as to why *SSA* cannot be guaranteed to give us congruent triangles. Identify students who can present both approaches (see notes below).

Discuss (Whole Class):

Only have a presentation on problem 1 if there are students who have not successfully found the area of that triangle. Rather, begin by have two students present their work on problem 2—one who got $A = \frac{1}{2}ab\sin C$ and one who got $A = \frac{1}{2}ac\sin B$ for the area formula. Note that for this *labeled* triangle, both formulas are correct. More important than memorizing either formula is to recognize what information is used in each case to determine the area of the triangle. Ask students what an area "rule" might be for an *unlabeled* triangle. Students should recognize that in each case we are taking *one-half of the product of the length of two sides and the sine of the included angle*.

For question 3, begin by having a student draw a diagram that would represent the given information, such as the diagram below. (Note that this is a *SSA* situation.) Then have one student present Jumal's approach, and one student present Jabari's.



Jumal's approach:

- Find the measure of angle *B* using the Law of Sines:

$$\frac{\sin 40^\circ}{10} = \frac{\sin B}{15} \quad \Rightarrow \quad \frac{15 \sin 40^\circ}{10} = \sin B \quad \Rightarrow \quad \sin^{-1}\left(\tfrac{15}{10} \sin 40^\circ\right) = B = 74.6^\circ$$

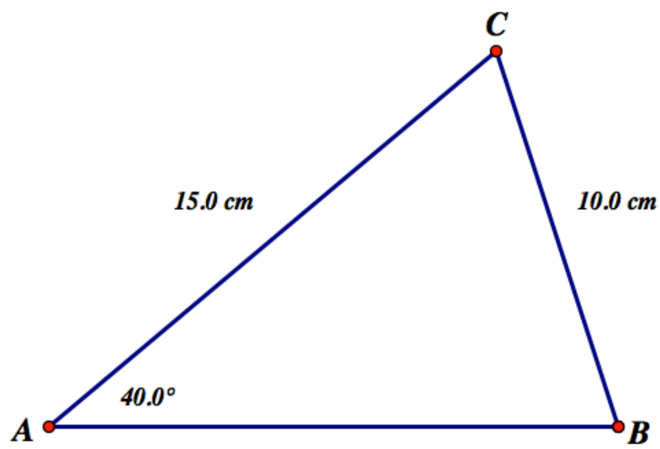
- Find the measure of the third angle *C*:

$$180^\circ - 40^\circ - 74.6^\circ = 65.4^\circ$$

- Find the measure of side *c* using the Law of Sines:

$$\frac{\sin 65.4^\circ}{c} = \frac{\sin 40}{10} \quad \Rightarrow \quad c = \frac{10 \sin 65.4^\circ}{\sin 40^\circ} \approx 14.14\ cm$$

- Draw the triangle:



Jabari's Approach

- Solve for c using the Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos(C)$

$$10^2 = 15^2 + c^2 - 2(15)(c)(\cos 40^\circ)$$

$$\cos 40^\circ = 0.766$$

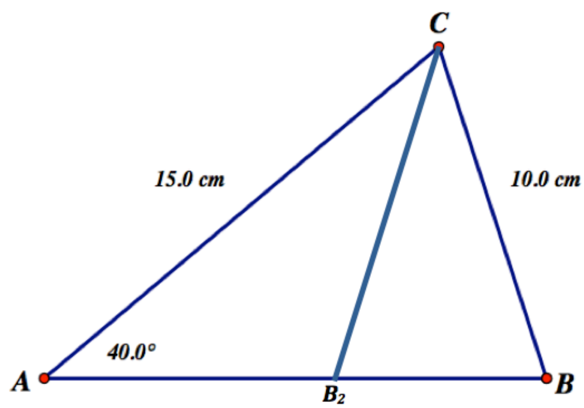
$$100 = 225 + c^2 - 22.98c$$

[Use the quadratic formula to solve for c]

$$0 = c^2 - 22.98c + 125$$

$$\frac{22.98 \pm \sqrt{22.98^2 - 4\cdot 1\cdot 125}}{2\cdot 1} \approx 14.14\ cm \quad or \quad 8.84\ cm$$

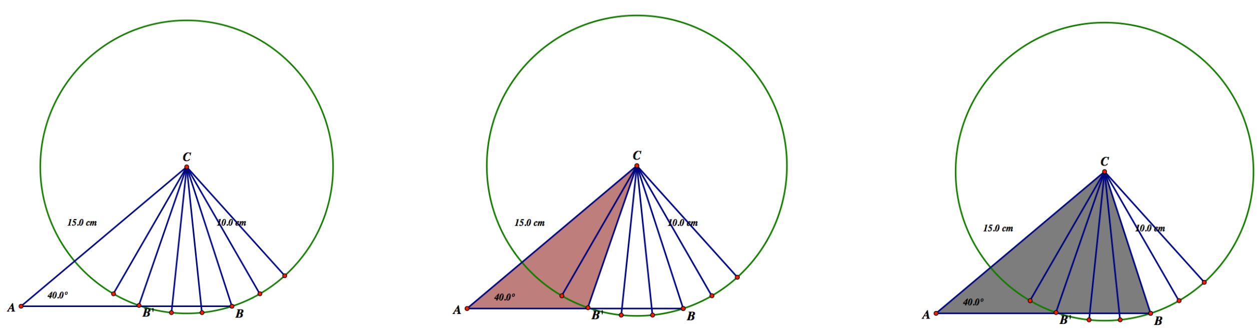
- Draw both possible triangles and find the two missing angles of each using the Law of Sines



$$\frac{\sin 40^\circ}{10} = \frac{\sin C}{8.84} \quad \Rightarrow \quad C = \sin^{-1}\left(\frac{8.84 \sin 40^\circ}{10}\right) = 34.6^\circ$$

$B_2 = 180° – 40° – 34.6° = 105.4°$

It might be helpful for students to think of side *CB* as being free to "swing" around a circle, as in the following diagram. Note that there are two points of intersection between the collection of possible radii of the circle and the line that contains side *AB* of the triangle. Consequently, two triangles can be defined using the same *SSA* criteria.



Aligned Ready, Set, Go: Modeling with Geometry 5.8

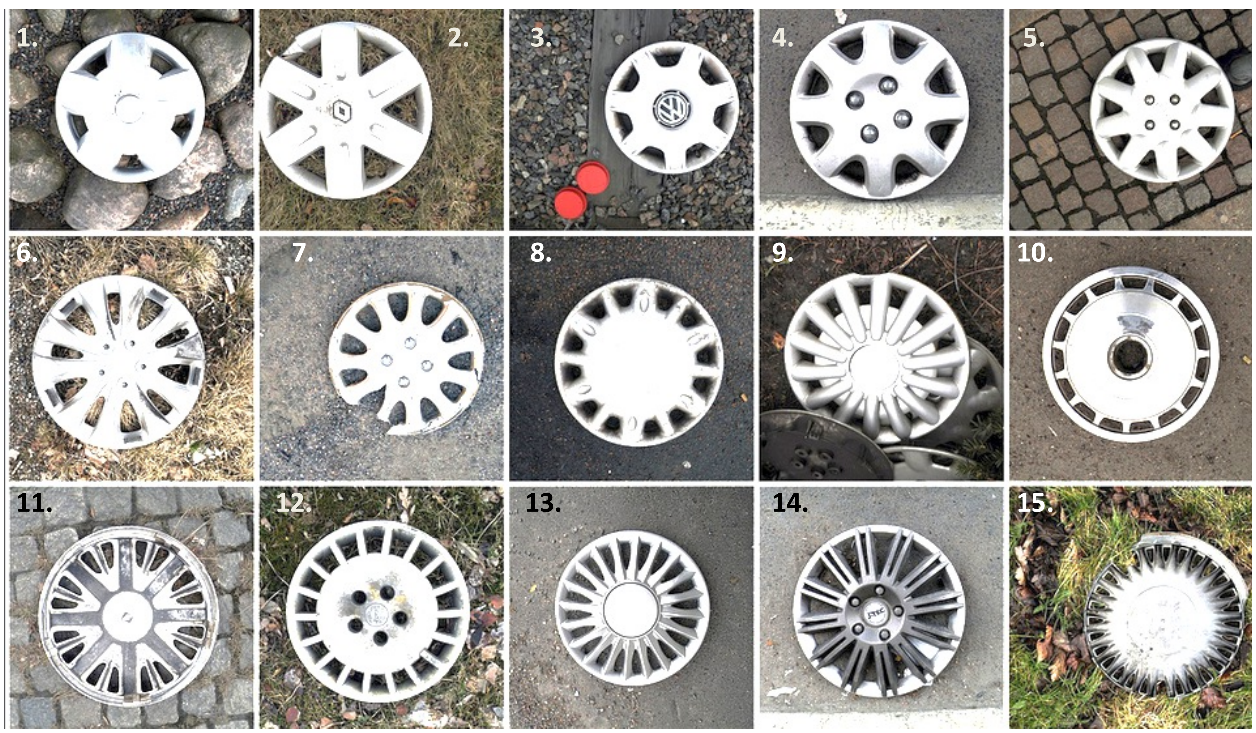
READY, SET, GO! Name Period Date

READY

Topic: Rotational symmetry

Hubcaps have *rotational symmetry*. That means that a hubcap does not have to turn a full circle to appear the same. For instance, a hubcap with this pattern,  will look the same every ¼ turn. It is said to have 90º *rotational symmetry* because for each quarter turn it rotates 90º.

State the *rotational symmetry* for the following hubcaps. Focus your answer on just the spokes, not the center design.(Answers will be in degrees.



SET

Topic: Area formulas for triangles

Area of an Oblique Triangle: The area of **any** triangle is one-half the product of the lengths of two sides times the sine of their included angle. $Area = \frac{1}{2}bc\ \sin A = \frac{1}{2}ab\ \sin C = \frac{1}{2}ab \sin B$

Find the area of the triangle having the indicated sides and angle.

16. $C = 84.5º$, $a = 32$, $b = 40$

17. $A = 29º$, $b = 49$, $c = 50$

18. $B = 72.5º$, $a = 105$, $c = 64$

19. $C = 31º$, $a = 15$, $b = 14$

20. $A = 42º$, $b = 25$, $c = 12$

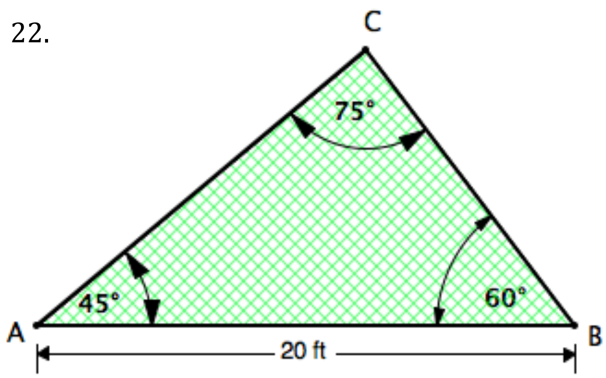
21. $B = 85º$, $a = 15$, $c = 12$

Another formula for the area of a triangle can be derived from the *Law of Sines*.

$$Area = \frac{c^2 \sin A \sin B}{2 \sin C}$$

Use this formula to find the area of the triangles.

22.



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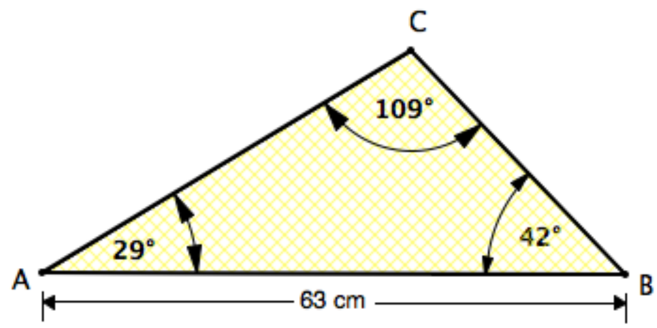


Perhaps you used the *Law of Cosines* to establish the following formula for the area of a triangle. The formula was known as early as circa 100 B.C. and is attributed to the Greek mathematician, Heron.

Heron's Area Formula: Given any triangle with sides of lengths *a, b,* and *c,* the area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{(a+b+c)}{2}.$$

Find the area of the triangle having the indicated sides.

24. $a = 11,\ b = 14,\ c = 20$

25. $a = 12,\ b = 5,\ c = 9$

26. $a = 12.32,\ b = 8.46,\ c = 15.05$

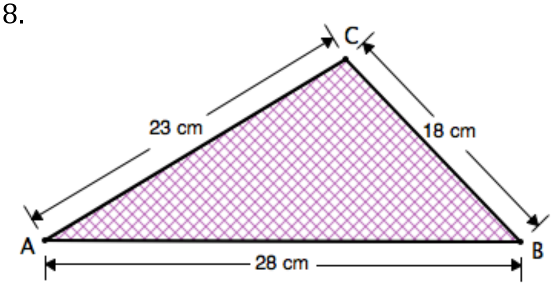
27. $a = 5,\ b = 7,\ c = 10$

GO

Topic: Distinguishing between the *law of sines* and the *law of cosines*

Indicate whether you would use the *Law of Sines* or the *Law of Cosines* to solve the triangles. Do not solve.

28.



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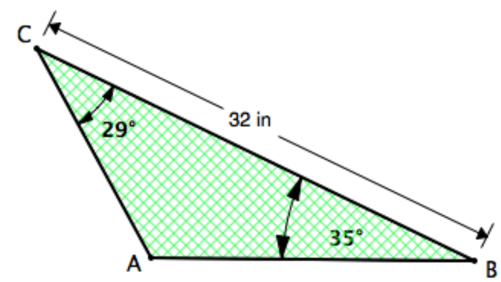

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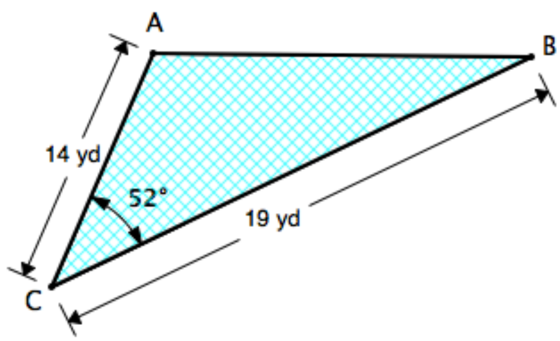

31.