

Pre Calculus: Multiple Angle & Half Angle Formulas

Double Angle Formulas

$$\sin 2u = 2 \sin u \cos u \quad \cos 2u = \cos^2 u - \sin^2 u$$

$$\begin{aligned}\tan 2u &= \frac{2 \tan u}{1 - \tan^2 u} \\ &= \frac{\sin 2u}{\cos 2u} \\ &= 2 \cos^2 u - 1 \\ &= 1 - 2 \sin^2 u\end{aligned}$$

Examples:

- (a) Use a double angle formula to rewrite the expression,

$$\begin{aligned}(\cos x + \sin x)(\cos x - \sin x) \\ = \cos^2 x - \sin^2 x = \cos 2x\end{aligned}$$

- (b) If $\cos \theta = -\frac{2}{3}$ and $\theta \in II$, then find $\cos 2\theta$, $\sin 2\theta$, and $\tan 2\theta$.

Using Pythagorean Theorem, we see the third side is equal to $\sqrt{5}$

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 2 \left(\frac{-2}{3} \right)^2 - 1 = 2 \left(\frac{4}{9} \right) - 1 = \frac{8}{9} - 1 = \frac{-1}{9}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{\sqrt{5}}{3} \right) \left(\frac{-2}{3} \right) = \frac{-4\sqrt{5}}{9}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{-4\sqrt{5}}{9}}{\frac{-1}{9}} = 4\sqrt{5}$$

- (c) If $\tan \theta = \frac{12}{5}$ and $\theta \in III$, then find $\cos 2\theta$, $\sin 2\theta$ and $\tan 2\theta$.

Again, using Pythagorean Theorem, we see the third side is 13

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{-5}{13}\right)^2 - \left(\frac{-12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = \frac{-119}{169}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{-12}{13}\right) \left(\frac{-5}{13}\right) = \frac{120}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{120}{169}}{\frac{-119}{169}} = \frac{120}{-119}$$

(d) Solve algebraically in $[0, 2\pi]$: $\sin 2x = \cos x$

$$2 \sin x \cos x = \cos x$$

$$2 \sin x \cos x - \cos x = 0$$

$$\cos x (2 \sin x - 1) = 0$$

$$\cos x = 0 \quad 2 \sin x - 1 = 0$$

$$\cos x = 0 \quad \sin x = \frac{1}{2}$$

$$x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \right\}$$

(e) Prove the identity: $\cos^4 x - \sin^4 x = \cos 2x$

$$(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos 2x$$

$$(1)(\cos 2x) = \cos 2x$$

$$\cos 2x = \cos 2x$$

(f) Prove the identity: $\cos 3x = 4 \cos^3 x - 3 \cos x$

$$\cos(2x + x) = 4 \cos^3 x - 3 \cos x$$

$$\cos 2x \cos x - \sin 2x \sin x = 4 \cos^3 x - 3 \cos x$$

$$(2 \cos^2 x - 1) \cos x - (2 \sin x \cos x) \sin x = 4 \cos^3 x - 3 \cos x$$

$$2 \cos^3 x - \cos x - 2 \sin^2 x \cos x = 4 \cos^3 x - 3 \cos x$$

$$2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x) = 4 \cos^3 x - 3 \cos x$$

$$2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x = 4 \cos^3 x - 3 \cos x$$

$$4 \cos^3 x - 3 \cos x = 4 \cos^3 x - 3 \cos x$$

Half Angle formulas:

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1-\cos u}{2}}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1+\cos u}{2}}$$

$$\begin{aligned}\tan \frac{u}{2} &= \frac{\sin \frac{u}{2}}{\cos \frac{u}{2}} = \frac{1 - \cos u}{\sin u} \\ &= \frac{\sin u}{1 + \cos u} \quad \text{use whichever makes more sense}\end{aligned}$$

- (a) Find the exact value of $\sin 22.5^\circ$ *Thought:* $22.5^\circ = \frac{45^\circ}{2}$.

$$\begin{aligned}\sin \frac{45^\circ}{2} &= \pm \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}\end{aligned}$$

- (b) If $\tan \theta = \frac{5}{12}$, then find the exact value of $\sin \frac{\theta}{2}$.

$$\begin{aligned}\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} = \pm \sqrt{\frac{1 - \frac{12}{13}}{2}} = \pm \sqrt{\frac{\frac{13}{13} - \frac{12}{13}}{2}} \\ &= \pm \sqrt{\frac{\frac{1}{13}}{2}} = \pm \sqrt{\frac{1}{26}} = \frac{\pm \sqrt{26}}{26}\end{aligned}$$

(c) If $\tan \theta = \frac{5}{12}$, then find the exact value of $\cot \frac{\theta}{2}$.

$$\cot \frac{\theta}{2} = \frac{\sin \theta}{1 - \cos \theta} = \frac{\frac{5}{\sqrt{13}}}{1 - \frac{12}{\sqrt{13}}} = \frac{\frac{5}{\sqrt{13}}}{\frac{1}{\sqrt{13}}} = 5$$

(d) Find $\tan \left(\frac{\theta}{2}\right)$ if $\sin \theta = \frac{2}{5}$ and $\theta \in II$.

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - \frac{-\sqrt{21}}{5}}{\frac{2}{5}} = \frac{\frac{5}{5} + \frac{\sqrt{21}}{5}}{\frac{2}{5}} = \frac{5 + \sqrt{21}}{2}$$

(e) Solve : $\sin^2 x = 2 \sin^2 \left(\frac{x}{2}\right)$

$$\sin^2 x = 2 \left(\pm \sqrt{\frac{1 - \cos x}{2}} \right)^2$$

$$\sin^2 x = 2 \left(\frac{1 - \cos x}{2} \right)$$

$$1 - \cos^2 x = 1 - \cos x$$

$$0 = \cos^2 x - \cos x$$

$$0 = \cos x (\cos x - 1)$$

$$0 = \cos x \quad 0 = \cos x - 1$$

$$0 = \cos x \quad 1 = \cos x$$

$$x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, 0 \right\}$$

(f) Use the half angle formulas to simplify the expression:

$$\sqrt{\frac{1 + \cos 4x}{2}} = \cos \frac{4x}{2} = \cos 2x$$

recall: $\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}}$ so let $u = 4x$

(g) Use the half angle formulas to simplify the expression:

$$\sqrt{\frac{1 - \cos(x - 1)}{2}} = \sin \frac{x - 1}{2}$$

recall: $\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}}$ so let $u = (x - 1)$