Tau: The True Circle Constant

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The Sydney University Mathematics Society

18th May 2017

The Circle Constant

- Circles are everywhere!




Historical Importance

- First approximated by the Ancient Babylonians in $\sim$1700BCE
- Independently calculated by ancient Egyptians $\sim$ 1600 BCE, Greeks $\sim$500 BCE, Indians $\sim$400BCE and Chinese $\sim$500ACE.

Social Importance

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- We have calculated about 10 trillion digits of π.
- But only need 39 digits to measure the size of the observable universe within the width of a hydrogen atom
- Countless books, films, songs, etc dedicated to π.

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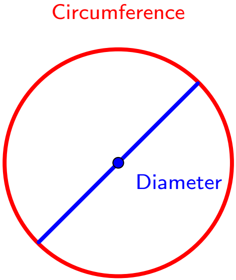
- The length of the *n*th word in the 10000-word novel *"Not a Wake"* by Michael Keith corresponds to the *n*th digit of pi.

Pi (π)

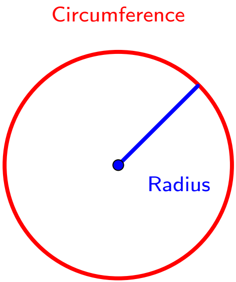
- π is a mathematical constant approximately equal to $3.14159265\ldots$
- Defined to be the ratio between the circumference and diameter of a circle



$$\pi = \frac{C}{D} \approx 3.1415\ldots$$

Tau (τ)

- Defined to be the ratio between the circumference of a circle and its radius



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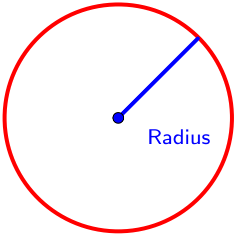

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- τ is mathematically equivalent to 2π.

Why does this matter?

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- Much of today's notation was popularised by Euler in the 18th Century, including π.
- τ has been historically used as the circle constant - Al Kashi.

Why choose $\tau = 2\pi$?

- Why do we choose the golden ratio to be $\varphi = \dfrac{1+\sqrt{5}}{2}$ instead of 2φ or φ^2 or $\varphi - 1$?

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- Why do we choose the golden ratio to be $\varphi = \dfrac{1+\sqrt{5}}{2}$ instead of 2φ or φ^2 or $\varphi - 1$?
- Why do we choose Euler's constant $e = 2.7182818284$ instead of e^2 or $\frac{1}{e}$?

Definition of a Circle

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- Circles are clearly defined by a centre and radius.
- Recall

$$\pi = \frac{C}{d} \qquad\qquad \tau = \frac{C}{r}$$

Definition of a Circle

"But the diameter is just double the radius!"

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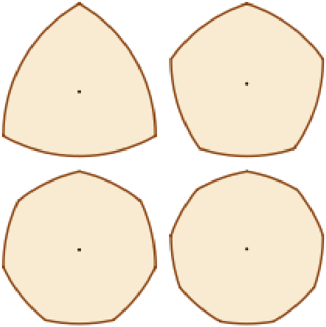
- There are shapes with constant diameter but not a constant radius



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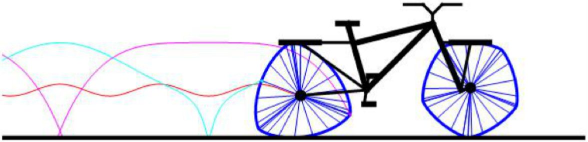
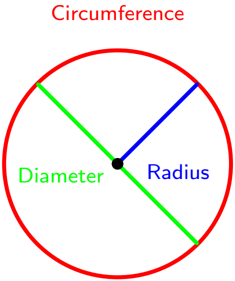


Circumference of a circle



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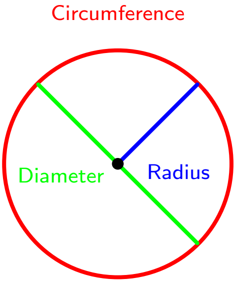
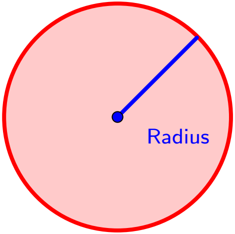


- Circumference is represented by $C = 2\pi r = \tau r$.
- But it's also πD.

Area of a circle



- Can be represented by $A = \pi r^2 = \frac{1}{2}\tau r^2$.

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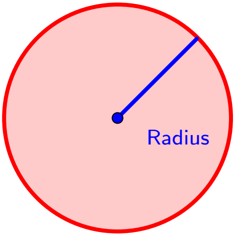

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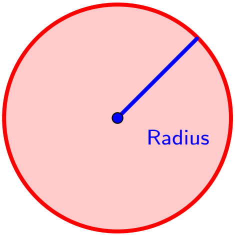

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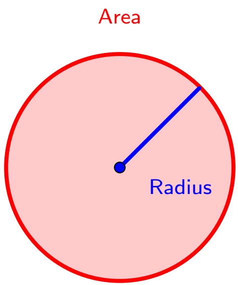

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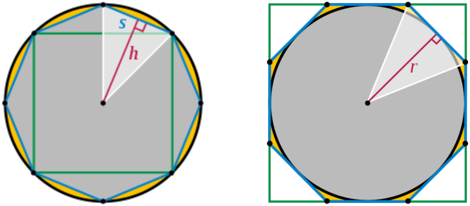
- Or $\frac{1}{4}\pi D^2$
- Recall: π is defined by the diameter and τ is defined by the radius.
- But what if the $\frac{1}{2}$ is actually telling us something?

Area of a circle

- Archimedes calculated the area of a circle by showing that it has the same area as a triangle with base length C and height r.



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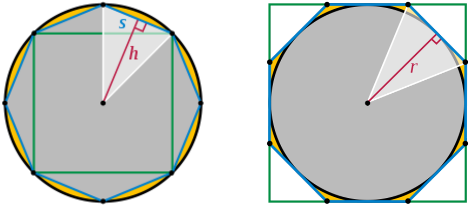
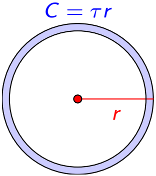


- So that

$$A = \frac{1}{2}Cr = \frac{1}{2}\tau r^2.$$

Area of a circle

- The formula $A = \frac{1}{2}\tau r^2$ also comes from integrating the circumference of a circle over the radius:



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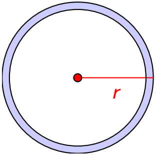

- So that

$$A = \int_0^r \tau r = \frac{1}{2}\tau r^2$$

Area of a circle

- This form is seen all over mathematics and physics, if we have that $y \propto x$, then $y = \lambda x$ so that

$$\int y \; dx = \int \lambda x \; dx = \frac{1}{2}\lambda x^2 + c$$

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- Displacement of an object falling under gravity

$$y = \int v \ dt = \int_0^t gt \ dt = \frac{1}{2}gt^2$$

Area of a circle

- Kinetic Energy

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- Potential energy in a spring

$$U = \int F \ dx = \int_0^x kx \ dx = \frac{1}{2}kx^2$$

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- Area of a sector given by radius r and angle θ:

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- taking $\theta = \tau$:

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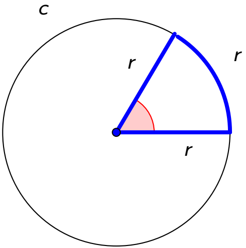
Radian Measure

- Radians defined by the angle subtended by a circular arc of length equal to the *radius* of the circle.



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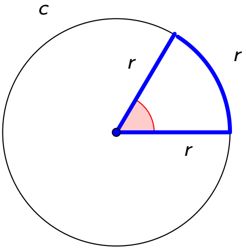
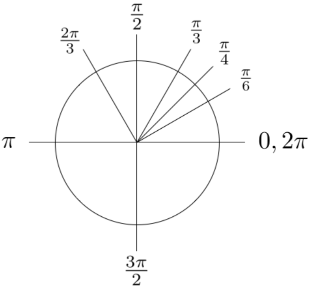
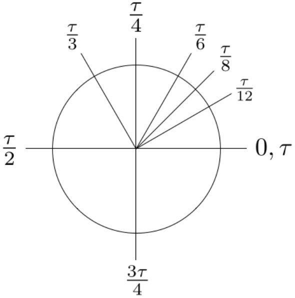


- This is a natural and important measure for angles, eg $\dfrac{d}{d\theta}\sin\theta = \cos\theta$ and $\dfrac{d}{d\theta}\cos\theta = -\sin\theta$.

Radian Measure

- But with this definition, there are 2π radians in a revolution, and τ radians in a revolution.





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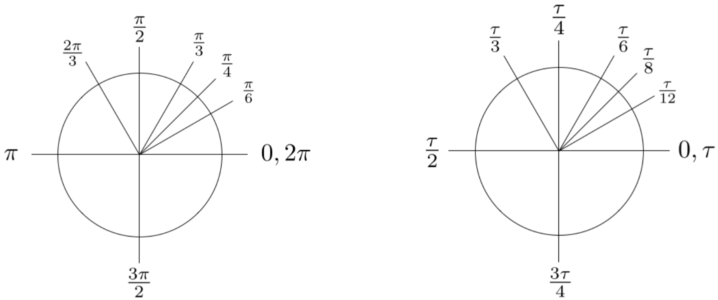
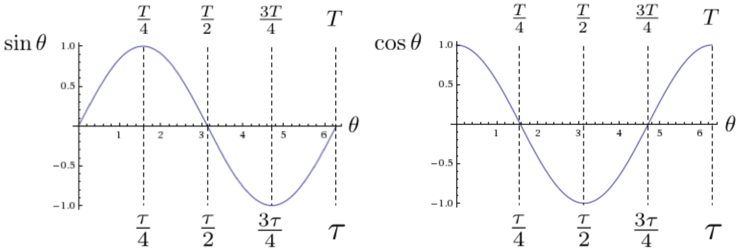


- In this way, τ radians represents τ of a revolution.

Radian Measure

- Leads to many notational simplifications:



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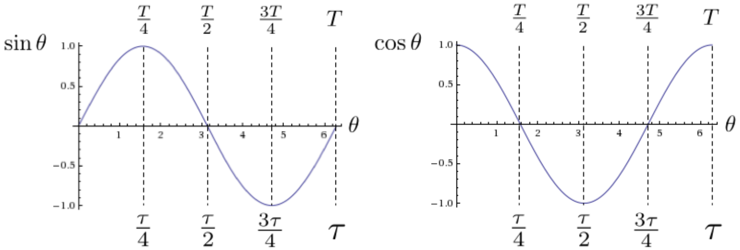

- Angular component of polar coordinates now range from $[0, \tau]$ instead of $[0, 2\pi]$.

Examples of Tau

- Gaussian distribution (normal distribution)

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- Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikx}dk$$

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Examples of Tau

- Riemann Zeta Function for positive even integers:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{B_n}{2(2n)!}(2\pi)^{2n}$$

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- Stirling's Approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Examples of Tau

- Angular Frequency:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

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- Kepler's Third Law of Planetary Motion:

$$T^2 = \frac{4\pi^2}{GM} a^3$$

Euler's Identity

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- or

$$e^{i\frac{\tau}{2}} = -1$$

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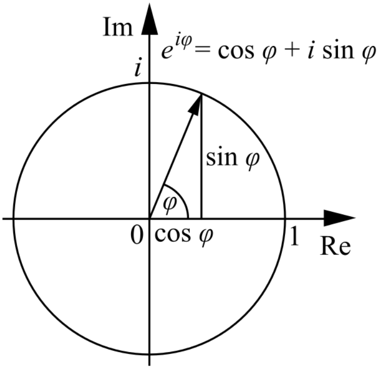

- $e^{i\pi}$ is equivalent to saying that a rotation by a half-turn is equivalent to -1

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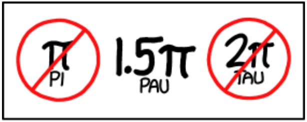
- The circle constant means so much to our society, making this a relevant and important point for discussion.
- τ shouldn't be "2 times π".
- π should be "Half τ".

References

- Michael Hartl, *The Tau Manifesto* (2015)
- Bob Palais, "π is wrong!" (2001)



A COMPROMISE SOLUTION TO THE PI/TAU DISPUTE