The Circle Constant

Tau, the better circle constant

Circle Definition

Radian Measure

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Tau: The True Circle Constant

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The Circle Constant

- Circles are everywhere!
Historical Importance

- First approximated by the Ancient Babylonians in ~1700BCE
- Independently calculated by ancient Egyptians ~ 1600 BCE, Greeks ~500 BCE, Indians ~400BCE and Chinese ~500ACE.
Social Importance

“Of all known mathematical constants, pi continues to attract the most attention” - Ivars Peterson
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- But only need 39 digits to measure the size of the observable universe within the width of a hydrogen atom.
Social Importance

“Of all known mathematical constants, pi continues to attract the most attention” - Ivars Peterson

- We have calculated about 10 trillion digits of π.
- But only need 39 digits to measure the size of the observable universe within the width of a hydrogen atom
- Countless books, films, songs, etc dedicated to π.
Piphiology

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Piphiology

- Piphiology - The memorization of the digits of pi as a hobby.
- Rajveer Meena recited 70,000 digits of pi in India taking 9 hours and 27 minutes on 21 March 2015.
- Akira Haraguchi claims to have recited 100,000 digits of pi on October 3, 2006.
- The length of the $n$th word in the 10000-word novel “Not a Wake” by Michael Keith corresponds to the $n$th digit of pi.
Pi ($\pi$)

- $\pi$ is a mathematical constant approximately equal to 3.14159265\ldots
- Defined to be the ratio between the circumference and diameter of a circle

$$\pi = \frac{C}{D} \approx 3.1415\ldots$$
Defined to be the ratio between the circumference of a circle and its radius

$$\tau = \frac{C}{r} \approx 6.2832\ldots$$
Tau ($\tau$)

- Defined to be the ratio between the circumference of a circle and its radius

\[
\tau = \frac{C}{r} \approx 6.2832 \ldots
\]

- $\tau$ is mathematically equivalent to $2\pi$. 
Why does this matter?

“Ideally, notation should emphasize the most important parameters and features of a mathematical expression or statement, while downplaying the routine or uninteresting parameters and features.” - Terence Tao (2008)
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- Much of today’s notation was popularised by Euler in the 18th Century, including $\pi$. 
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“Ideally, notation should emphasize the most important parameters and features of a mathematical expression or statement, while downplaying the routine or uninteresting parameters and features.” - Terence Tao (2008)

- Much of today’s notation was popularised by Euler in the 18th Century, including $\pi$.
- $\tau$ has been historically used as the circle constant - Al Kashi.
Why choose $\tau = 2\pi$?

- Why do we choose the golden ratio to be $\varphi = \frac{1 + \sqrt{5}}{2}$ instead of $2\varphi$ or $\varphi^2$ or $\varphi - 1$?
Why choose $\tau = 2\pi$?

- Why do we choose the golden ratio to be $\varphi = \frac{1 + \sqrt{5}}{2}$ instead of $2\varphi$ or $\varphi^2$ or $\varphi - 1$?
- Why do we choose Euler’s constant $e = 2.7182818284$ instead of $e^2$ or $\frac{1}{e}$?
Definition of a Circle

“A circle is the set of all points in a plane that are at a given distance from a given point” - Wikipedia
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- Circles are clearly defined by a centre and radius.
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- Circles are clearly defined by a centre and radius.
- Recall

\[ \pi = \frac{C}{d} \]

\[ \tau = \frac{C}{r} \]
Definition of a Circle

“But the diameter is just double the radius!”
Definition of a Circle

“But the diameter is just double the radius!”

- There are shapes with constant diameter but not a constant radius
Definition of a Circle
Circumference of a circle

- Circumference is represented by \( C = 2\pi r = \tau r \).
Circumference of a circle

- Circumference is represented by $C = 2\pi r = \tau r$.
- But it’s also $\pi D$. 
Can be represented by \( A = \pi r^2 = \frac{1}{2} \tau r^2 \).
Area of a circle

- Can be represented by $A = \pi r^2 = \frac{1}{2} \tau r^2$.
- Or $\frac{1}{4} \pi D^2$
Area of a circle

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- Recall: $\pi$ is defined by the diameter and $\tau$ is defined by the radius.
Area of a circle

- Can be represented by $A = \pi r^2 = \frac{1}{2} \tau r^2$.
- Or $\frac{1}{4} \pi D^2$
- Recall: $\pi$ is defined by the diameter and $\tau$ is defined by the radius.
- But what if the $\frac{1}{2}$ is actually telling us something?
Area of a circle

- Archimedes calculated the area of a circle by showing that it has the same area as a triangle with base length $C$ and height $r$. 

$$A = \frac{1}{2} Cr = \frac{1}{2} \tau r^2.$$
Area of a circle

- Archimedes calculated the area of a circle by showing that it has the same area as a triangle with base length $C$ and height $r$.

- So that

$$A = \frac{1}{2} Cr = \frac{1}{2} \tau r^2.$$
Area of a circle

- The formula $A = \frac{1}{2} \tau r^2$ also comes from integrating the circumference of a circle over the radius:
The formula \( A = \frac{1}{2} \tau r^2 \) also comes from integrating the circumference of a circle over the radius:

So that

\[
A = \int_0^r \tau r = \frac{1}{2} \tau r^2
\]
Area of a circle

This form is seen all over mathematics and physics, if we have that $y \propto x$, then $y = \lambda x$ so that

$$\int y \, dx = \int \lambda x \, dx = \frac{1}{2} \lambda x^2 + c$$
Area of a circle

- This form is seen all over mathematics and physics, if we have that $y \propto x$, then $y = \lambda x$ so that
  \[
  \int y \, dx = \int \lambda x \, dx = \frac{1}{2} \lambda x^2 + c
  \]
- Displacement of an object falling under gravity
  \[
  y = \int v \, dt = \int_0^t gt \, dt = \frac{1}{2} gt^2
  \]
Area of a circle

- Kinetic Energy

\[ K = \int p \, dv = \int mv \, dv = \frac{1}{2}mv^2 \]
Area of a circle

- **Kinetic Energy**

  \[ K = \int p \, dv = \int mv \, dv = \frac{1}{2}mv^2 \]

- **Potential energy in a spring**

  \[ U = \int F \, dx = \int_0^x kx \, dx = \frac{1}{2}kx^2 \]
Area of a circle

- Area of a sector given by radius $r$ and angle $\theta$:

$$A = \frac{1}{2} r^2 \theta$$
Area of a circle

- Area of a sector given by radius $r$ and angle $\theta$:

$$A = \frac{1}{2} r^2 \theta$$

- taking $\theta = \tau$:

$$A = \frac{1}{2} r^2 \tau$$
Radian Measure

- Radians defined by the angle subtended by a circular arc of length equal to the *radius* of the circle.
Radian Measure

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![Diagram of a circle with a central angle and a triangle formed by the radius and the arc]

- This is a natural and important measure for angles, eg

\[
\frac{d}{d\theta} \sin \theta = \cos \theta \quad \text{and} \quad \frac{d}{d\theta} \cos \theta = -\sin \theta.
\]
But with this definition, there are $2\pi$ radians in a revolution, and $\tau$ radians in a revolution.
Radian Measure

- But with this definition, there are $2\pi$ radians in a revolution, and $\tau$ radians in a revolution.

- In this way, $\tau$ radians represents $\tau$ of a revolution.
Radian Measure

- Leads to many notational simplifications:
Radian Measure

- Leads to many notational simplifications:

- Angular component of polar coordinates now range from \([0, \tau]\) instead of \([0, 2\pi]\).
Examples of Tau

- Gaussian distribution (normal distribution)

\[ \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]
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\[
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\]

- Fourier Transform

\[
f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x} \, dk
\]
\[
F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx
\]
Examples of Tau

- Riemann Zeta Function for positive even integers:

\[
\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{B_n}{2(2n)!}(2\pi)^{2n}
\]
Examples of Tau

- Riemann Zeta Function for positive even integers:

\[ \zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{B_n}{2(2n)!}(2\pi)^{2n} \]

- Stirling’s Approximation:

\[ n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]
Examples of Tau

Angular Frequency:

\[ \omega = \frac{2\pi}{T} = 2\pi f \]
Examples of Tau

- **Angular Frequency:**
  \[ \omega = \frac{2\pi}{T} = 2\pi f \]

- **Kepler’s Third Law of Planetary Motion:**
  \[ T^2 = \frac{4\pi^2}{GM} a^3 \]
Euler’s Identity

- The most beautiful formula in mathematics

\[ e^{i\pi} = -1 \]
Euler’s Identity

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- Also rearranges to give
  \[e^{i\pi} + 1 = 0\]
Euler’s Identity

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- Also rearranges to give
  \[ e^{i\pi} + 1 = 0 \]

- or
  \[ e^{i\frac{\pi}{2}} = -1 \]
Euler’s Identity

- $e^{i\pi}$ is equivalent to saying that a rotation by a half-turn is equivalent to -1.
Euler’s Identity

- So if we use tau, we actually have

\[ e^{i\tau} = 1 \]
Euler’s Identity

- So if we use tau, we actually have
  \[ e^{i\tau} = 1 \]

- Or
  \[ e^{i\tau} = 1 + 0. \]
Conclusion

- $\tau$ appears naturally all over mathematics and physics and is notationally superior to $\pi$
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- The circle constant means so much to our society, making this a relevant and important point for discussion.
- $\tau$ shouldn’t be “2 times $\pi$”.
Conclusion

- \( \tau \) appears naturally all over mathematics and physics and is notationally superior to \( \pi \).
- The circle constant means so much to our society, making this a relevant and important point for discussion.
- \( \tau \) shouldn’t be “2 times \( \pi \)”.
- \( \pi \) should be “Half \( \tau \)).
References

- Bob Palais, “π is wrong!” (2001)