

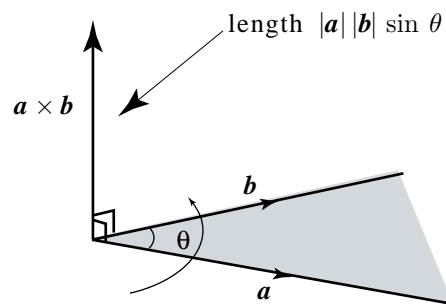
# The vector product

## Introduction

On this leaflet we describe how to find the **vector product** of two vectors.

## 1. Definition of the vector product

The result of finding the vector product of two vectors,  $\mathbf{a}$  and  $\mathbf{b}$ , is a vector of modulus  $|\mathbf{a}| |\mathbf{b}| \sin \theta$  in the direction of  $\hat{\mathbf{e}}$ , where  $\hat{\mathbf{e}}$  is a unit vector perpendicular to the plane containing  $\mathbf{a}$  and  $\mathbf{b}$  in a sense defined by the right-handed screw rule as shown below. The symbol used for the vector product is the times sign,  $\times$ . Do not use a dot,  $\cdot$ , because this is the symbol used for a scalar product.



$$\text{vector product: } \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{e}}$$

## 2. A formula for finding the vector product

A formula exists for finding the vector product of two vectors given in cartesian form:

If  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

### Example

Evaluate the vector product  $\mathbf{a} \times \mathbf{b}$  if  $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$  and  $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ .

### Solution

By inspection  $a_1 = 3$ ,  $a_2 = -2$ ,  $a_3 = 5$ ,  $b_1 = 7$ ,  $b_2 = 4$ ,  $b_3 = -8$ , and so

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= ((-2)(-8) - (5)(4))\mathbf{i} - ((3)(-8) - (5)(7))\mathbf{j} + ((3)(4) - (-2)(7))\mathbf{k} \\ &= -4\mathbf{i} + 59\mathbf{j} + 26\mathbf{k} \end{aligned}$$

### 3. Using determinants to evaluate a vector product

Evaluation of a vector product using the previous formula is very cumbersome. There is a more convenient and easily remembered method for those of you who are familiar with determinants. The vector product of two vectors  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  can be found by evaluating the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

To find the  $\mathbf{i}$  component of the vector product, imagine crossing out the row and column containing  $\mathbf{i}$  and finding the determinant of what is left, that is

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

The resulting number is the  $\mathbf{i}$  component of the vector product. The  $\mathbf{j}$  component is found by crossing out the row and column containing  $\mathbf{j}$  and evaluating

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1$$

and then changing the sign of the result. Finally the  $\mathbf{k}$  component is found by crossing out the row and column containing  $\mathbf{k}$  and evaluating

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

If  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

#### Example

Find the vector product of  $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{b} = 9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ .

#### Solution

The two given vectors are represented in the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$\mathbf{a} \times \mathbf{b} = (-8 - (-12))\mathbf{i} - (6 - 18)\mathbf{j} + (-18 - (-36))\mathbf{k} = 4\mathbf{i} + 12\mathbf{j} + 18\mathbf{k}$$

#### Exercises

1. If  $\mathbf{a} = 8\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  and  $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$  show that  $\mathbf{a} \times \mathbf{b} = -5\mathbf{i} - 18\mathbf{j} - 29\mathbf{k}$ . Show also that  $\mathbf{b} \times \mathbf{a}$  is not equal to  $\mathbf{a} \times \mathbf{b}$ , but rather that  $\mathbf{b} \times \mathbf{a} = 5\mathbf{i} + 18\mathbf{j} + 29\mathbf{k}$ .