

# **Performance of Nonlinear Approximate Adaptive Controllers**

# **Performance of Nonlinear Approximate Adaptive Controllers**

**Mark French**

*University of Southampton, UK*

**Csaba Szepesvári**

*Mindmaker Ltd., Budapest, Hungary*

**Eric Rogers**

*University of Southampton, UK*



**WILEY**

Copyright © 2003      John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester,  
West Sussex PO19 8SQ, England  
Telephone   (+44) 1243 779777

Email (for orders and customer service enquiries): [cs-books@wiley.co.uk](mailto:cs-books@wiley.co.uk)  
Visit our Home Page on [www.wileyeurope.com](http://www.wileyeurope.com) or [www.wiley.com](http://www.wiley.com)

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except under the terms of the Copyright, Designs and Patents Act 1988 or under the terms of a licence issued by the Copyright Licensing Agency Ltd, 90 Tottenham Court Road, London W1T 4LP, UK, without the permission in writing of the Publisher. Requests to the Publisher should be addressed to the Permissions Department, John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex PO19 8SQ, England, or emailed to [permreq@wiley.co.uk](mailto:permreq@wiley.co.uk), or faxed to (+44) 1243 770620.

This publication is designed to provide accurate and authoritative information in regard to the subject matter covered. It is sold on the understanding that the Publisher is not engaged in rendering professional services. If professional advice or other expert assistance is required, the services of a competent professional should be sought.

#### ***Other Wiley Editorial Offices***

John Wiley & Sons Inc., 111 River Street, Hoboken, NJ 07030, USA

Jossey-Bass, 989 Market Street, San Francisco, CA 94103-1741, USA

Wiley-VCH Verlag GmbH, Boschstr. 12, D-69469 Weinheim, Germany

John Wiley & Sons Australia Ltd, 33 Park Road, Milton, Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01, Jin Xing Distripark, Singapore 129809

John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada M9W 1L1

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

#### ***British Library Cataloguing in Publication Data***

A catalogue record for this book is available from the British Library

ISBN 0-471-49809-2

Produced from PDF files supplied by the authors

Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham, Wiltshire

This book is printed on acid-free paper responsibly manufactured from sustainable forestry in which at least two trees are planted for each one used for paper production.

*To Beata and Nic*

# Contents

<b>Preface</b> . . . . .	xiii
<b>1 Introduction</b> . . . . .	1
1.1 Overview . . . . .	1
1.2 Control problems . . . . .	6
1.3 Uncertain systems . . . . .	14
1.4 Some control designs . . . . .	19
1.4.1 A robust damping design . . . . .	20
1.4.2 An adaptive design for parametric uncertainties . . . . .	22
1.4.3 An adaptive design for non-parametric uncertainties: adaptive damping . . . . .	25
1.4.4 A further adaptive design for non-parametric uncertainties . . . . .	28
1.4.5 Uncertainty descriptions for approximate adaptive designs . . . . .	39
1.5 Performance . . . . .	45
1.5.1 Non-singular cost functionals . . . . .	47
1.6 Robust or adaptive? . . . . .	49
1.6.1 Control designs and their classification . . . . .	49
1.6.2 Smoothness and resolution scaling . . . . .	53
1.6.3 Limiting behaviour of the model – resolution divergence and scaling . . . . .	55
1.6.4 Multi-resolution function approximator designs can outperform robust designs . . . . .	58
1.7 When is the complexity of the function approximator designs justified? . . . . .	58
1.7.1 The adaptive damping design can be inferior . . . . .	58
1.7.2 A concluding justification for function approximator designs . . . . .	59
<b>2 Approximation Theory</b> . . . . .	61
2.1 Introduction . . . . .	61
2.2 Approximation theory . . . . .	61
2.2.1 The generic setup . . . . .	62
2.2.2 Some approximation models . . . . .	62
2.2.3 Approximation spaces . . . . .	65
2.2.4 Smoothness spaces . . . . .	66
2.2.5 Relating approximation and smoothness spaces . . . . .	69

2.2.6	Some properties of linear approximations . . . . .	71
2.2.7	Nonlinear approximations . . . . .	76
2.2.8	The curse of dimensionality . . . . .	80
2.2.9	Approximation over unbounded domains . . . . .	83
2.3	Some constructions: multi-variable, vector-valued and multi-resolution approximations . . . . .	84
2.3.1	Multi-variable approximation via a tensor product construction . . . . .	84
2.3.2	Approximation of vector-valued functions . . . . .	86
2.3.3	Multi-resolution models . . . . .	87
2.4	Best approximations . . . . .	87
<b>3</b>	<b>Uncertainty Modelling, Control Design and System Performance . . . . .</b>	<b>91</b>
3.1	Introduction . . . . .	91
3.2	Systems . . . . .	91
3.2.1	Plants . . . . .	91
3.2.2	Controllers . . . . .	92
3.2.3	Closed loops . . . . .	93
3.2.4	Normal forms . . . . .	94
3.3	Uncertainty modelling . . . . .	96
3.3.1	Structural spaces . . . . .	98
3.3.2	Parametric uncertainty . . . . .	99
3.3.3	$L^\infty$ uncertainty . . . . .	100
3.3.4	$L^2$ uncertainty . . . . .	102
3.3.5	Motivation for the $L^2$ model . . . . .	103
3.3.6	Mixed $L^2/L^\infty$ uncertainty . . . . .	106
3.3.7	An important remark on the uncertainty models . . . . .	106
3.3.8	Uncertainty shaping . . . . .	107
3.4	Worst-case performance . . . . .	110
3.4.1	Asymptotic performance . . . . .	110
3.4.2	Transient performance: cost functionals . . . . .	111
3.4.3	Worst-case transient performance . . . . .	114
3.4.4	Optimisation and constructive controller design . . . . .	114
3.5	Average-case performance . . . . .	117
3.5.1	Identification in $L^2$ vs $L^\infty$ . . . . .	118
3.6	Control design . . . . .	118
<b>4</b>	<b>The Chain of Integrators . . . . .</b>	<b>121</b>
4.1	Introduction . . . . .	121
4.1.1	State feedback . . . . .	122
4.1.2	Output feedback . . . . .	123
4.2	An adaptive controller . . . . .	124
4.2.1	The effect of small disturbances . . . . .	127
4.3	Robust modifications . . . . .	133
4.3.1	Projection modification . . . . .	133
4.3.2	$\sigma$ modification . . . . .	138
4.3.3	Dead-zone modification . . . . .	140

4.4	A transient cost comparison between two robust modifications . . . . .	141
4.4.1	Systems, control designs and performance cost functional . . .	142
4.4.2	Conservative uncertainty level: dead-zone modification out- performs projection modification . . . . .	142
4.4.3	Conservative disturbance level: projection modification out- performs dead-zone modification . . . . .	145
4.5	Summary and justification for dead-zone . . . . .	147
<b>5</b>	<b>Function Approximator Designs for the Integrator Chain . . . . .</b>	<b>149</b>
5.1	Introduction . . . . .	149
5.2	Disturbances and approximation error . . . . .	151
5.3	Stability of the approximate adaptive controller . . . . .	151
5.3.1	System stability . . . . .	151
5.4	Performance cost functionals . . . . .	161
5.4.1	System uncertainty and the state domain constraint . . . . .	165
5.4.2	Gram matrices: bounding the size of model parameters . . . . .	165
5.4.3	System uncertainty models and state performance bounds . . .	169
5.4.4	System uncertainty models and control effort performance bounds . . . . .	170
5.5	Worst-case semi-global results . . . . .	171
5.5.1	Statement of semi-global result . . . . .	171
5.6	Global stability . . . . .	175
5.6.1	An unstable example . . . . .	178
5.6.2	Achieving global stability by adjusting the adaptive structure matrix . . . . .	181
5.6.3	Achieving global stability by adding a robust damping term . .	189
5.7	Augmentations of the control law . . . . .	191
<b>6</b>	<b>Resolution Divergence . . . . .</b>	<b>193</b>
6.1	Introduction . . . . .	193
6.2	Some notational conventions . . . . .	195
6.3	Resolution divergence when the rate of change of the control signal is penalised . . . . .	196
6.3.1	System and control task . . . . .	196
6.3.2	Controller: basic dead-zone modified Lyapunov design . . . . .	196
6.3.3	Stability . . . . .	197
6.3.4	Resolution divergence when the adaption gain diverges with model complexity . . . . .	199
6.3.5	Scaling of the adaption gain . . . . .	200
6.4	Resolution divergence for the LQ-cost for MIMO systems . . . . .	204
6.4.1	Control problem and design . . . . .	204
6.4.2	Closed loop . . . . .	206
6.4.3	Stability . . . . .	207
6.4.4	Divergent performance . . . . .	209
<b>7</b>	<b>Resolution Scaling . . . . .</b>	<b>227</b>

7.1	Resolution scaling properties of the projection modification design . . .	229
7.1.1	Projection modification design . . . . .	229
7.1.2	Resolution scalability . . . . .	230
7.1.3	Resolution divergence . . . . .	231
7.2	Multi-resolution design for SISO systems . . . . .	231
7.2.1	Comparison of the multi-resolution and projection designs . . .	233
7.2.2	Extensions . . . . .	234
7.3	A resolution scaleable design for MIMO systems with decreasing dead-zone . . . . .	235
7.3.1	A semi-global design . . . . .	236
7.3.2	A global design . . . . .	241
<b>8</b>	<b>Strict Feedback Systems . . . . .</b>	<b>249</b>
8.1	Introduction . . . . .	249
8.2	An introduction to backstepping . . . . .	249
8.2.1	Backstepping a nominal system . . . . .	250
8.2.2	Backstepping a parametrically uncertain system I . . . . .	254
8.2.3	Backstepping a parametrically uncertain system II . . . . .	258
8.2.4	Backstepping with shared parameters . . . . .	262
8.2.5	Backstepping with bounded disturbances I . . . . .	263
8.2.6	Backstepping with bounded disturbances II . . . . .	267
8.2.7	A performance and stability result . . . . .	269
8.3	Worst-case formalism . . . . .	275
8.3.1	Systems and uncertainty . . . . .	275
8.4	Function approximator designs . . . . .	277
8.4.1	Performance and scaling . . . . .	278
8.4.2	Preliminary structural results . . . . .	279
8.4.3	Mixed $L^2/L^\infty$ uncertainty model and a globally adaptive controller utilising high-gain terms . . . . .	282
8.4.4	Global $L^2$ uncertainty and an adaptive controller utilising dynamic function approximating structures . . . . .	286
<b>9</b>	<b>Output Feedback Control . . . . .</b>	<b>293</b>
9.1	Introduction . . . . .	293
9.2	An introduction to observers . . . . .	294
9.3	Controller . . . . .	295
9.4	A performance and stability result . . . . .	299
9.4.1	Stability . . . . .	299
9.4.2	Performance . . . . .	314
9.5	Worst-case results . . . . .	319
9.5.1	Uncertainty model . . . . .	319
9.5.2	Semi-global result . . . . .	320
9.5.3	Global result . . . . .	323
<b>10</b>	<b>Comparison to Alternative Designs . . . . .</b>	<b>327</b>
10.1	Introduction . . . . .	327



10.2	A partial classification of control designs . . . . .	327
10.3	A comparison between adaptive and robust backstepping . . . . .	328
10.3.1	System, uncertainty model and cost functional . . . . .	328
10.4	A robust controller . . . . .	329
10.4.1	Performance of the robust design . . . . .	331
10.5	Adaptive controllers . . . . .	333
10.5.1	An adaptive damping controller . . . . .	334
10.5.2	Parametric adaptive control designs . . . . .	342
10.5.3	Approximate adaptive control designs . . . . .	342
10.6	When is a non-adaptive design better? . . . . .	344
10.6.1	Optimal worst-case designs . . . . .	344
10.6.2	Large initial conditions: robust vs adaptive damping . . . . .	344
10.7	Comparisons between adaptive designs . . . . .	349
10.7.1	A comparison between an adaptive design and an over-parameterised variant . . . . .	350
10.7.2	Spatial localisation of learning . . . . .	353
10.8	Summary . . . . .	356
<b>11</b>	<b>Conclusions and Outlook . . . . .</b>	<b>357</b>
11.1	Open questions . . . . .	358
11.1.1	Approximate adaptive control . . . . .	358
11.1.2	Worst-case design vs adaptive design . . . . .	359
11.1.3	Links to learning theory . . . . .	359
11.1.4	Time-varying adaptive control . . . . .	359
11.1.5	Robustness of adaptive controllers . . . . .	360
11.1.6	A performance order theory for constructive nonlinear control . . . . .	360
<b>Appendix A</b>	<b>Lyapunov's Direct Method . . . . .</b>	<b>363</b>
A.1	Introduction . . . . .	363
A.2	Definitions . . . . .	363
A.3	Lyapunov's direct method . . . . .	365
A.4	The La Salle-Yoshizawa Theorem for Lyapunov functions with negative semi-definite Lyapunov derivatives . . . . .	368
A.5	Barbalat's lemma . . . . .	370
A.6	Lyapunov lemma – stability of linear time invariant systems . . . . .	371
A.7	Filippov solutions to discontinuous differential equations . . . . .	377
<b>Appendix B</b>	<b>Functional Bounds from System Identification . . . . .</b>	<b>379</b>
<b>References</b>	<b>. . . . .</b>	<b>385</b>
<b>Index</b>	<b>. . . . .</b>	<b>389</b>

# Preface

In recent years there has been wide interest in nonlinear adaptive control using function approximator models, for either tracking or regulation, often described as 'neural network based control'. This is evidenced by the wide publication of such research in journals and many sessions in recent international conferences.

This field contrasts to the traditional domain of adaptive control, where the model is typically thought to contain uncertain 'physical' parameters. In function approximator designs systems are considered which contain uncertain nonlinear functions which are approximated using function approximator models. Many control researchers have cast doubt on the practicality of such approximate model based designs, and have further questioned whether there is any novelty in such an approach: surely such controls are just robust adaptive designs, and hence there is nothing new to be said? A few years ago we approached this subject from a similar critical viewpoint, but have become convinced that there are indeed many interesting features about the approximate adaptive approach that do not appear in the classical parametric theory. The purpose of this book is intended to describe this viewpoint. We will describe the approximate model philosophy and its setting, and *rigorously compare the performance of such controls against competing designs*. This comparison will thus highlight the situations in which the approximate model based designs are the most appropriate and conversely will indicate the scenarios in which other designs are more appropriate. Thus this book not only presents a description of a topical aspect of contemporary research and control practice, but also places the algorithms proposed in a wider comparative setting.

It is apparent that utilising an approximate model in an adaptive control design, and then using a robust modification to the adaptive laws to ensure stability in the face of approximation errors, yields a control design which is essentially just a semi-global robust adaptive design. However, this line of reasoning may obscure the implicit (and critical) choice that has necessarily been made: namely the choice of model. Different choices of models (even with identical approximation abilities) yield controllers with very different output and control transient performances. Examining this issue is one of the main themes of this book: the control designs are modifications to (now standard) Lyapunov designs for, e.g., matched, strict feedback and output feedback systems and the theme is to examine the performance of these designs for different choices of approximation theoretic models.

To develop a firm basis on which to judge the designs it is necessary to develop uncertainty models. As the designs are oriented towards systems with non-parametric, nonlinear, static uncertainties, these uncertainty models form subsets of function

spaces, and are characterised here by spatial (weighted)  $L^2$  norm bounds, spatial (weighted)  $L^\infty$  norm bounds and approximation theoretic smoothness constraints.

It is necessary to evaluate transient performance in a realistic but tractable manner. Throughout the book we primarily focus on LQ performance. This is modified in a variety of ways to suit the particular problem being considered (e.g. exponentially weighted or truncated for the tracking designs). The critical point is that all these costs penalise both the state/output transient *and the control effort*. Until the control effort is incorporated into the cost, many nonlinear designs can achieve an arbitrarily good output performance with respect to many singular cost functionals by tuning gains appropriately. Yet these gains lead to high control effort. Therefore, and for the first time in the nonlinear adaptive literature, we take this trade-off fully into account when evaluating designs. *The handling of the penalty on the control effort is the single most important contribution of this research*, and indeed the questions raised about controller comparison cannot be meaningfully answered in its absence.

Within this uncertainty/performance framework we demonstrate by a rigorously constructed example that a bad choice of function approximator model can cause an inevitable performance degradation, even when all free parameters of the system (adaptation gains, etc.) are tuned optimally. For example, it is shown that the performance of spline based adaptive controllers necessarily diverges when a uniform knot lattice is refined: this is in spite of the fact that finite performance can be achieved for any particular refinement if the gains are tuned appropriately. We also demonstrate that this scaling problem can be avoided if the model is chosen more appropriately, and constructions for such models and associated gains are given. These constructions essentially exploit multi-resolution properties of the models; this structure is absent in the spline example. These two complementary results then form the basis for the development of any approximate adaptive results: subsequently it is necessary to demonstrate both stability/asymptotic performance and also good scaling properties.

Finally we address the complexity issue. The rival algorithms (i.e. those which can operate on the same problem domains, e.g. robust backstepping, or adaptive stabilisation designs based on estimating the uncertainty level rather than physical parameters) are considerably simpler to implement. Thus the last theme of this book is to compare the approximate model-based designs against their competitors. In particular we show situations in which the approximate designs can be expected to out-perform other designs, and thus justify their complexity. Conversely we also show situations in which the rival algorithms are the better choice.

Besides the aim of critically evaluating a particular class of control designs, the second purpose of the book is to showcase a particular approach and the associated toolkit, which permits a quantitative comparison to be made between competing control designs. This toolkit is developed as required throughout the book, but on its own provides a constructive framework for estimating both upper and lower bounds of a variety of performance cost functionals for Lyapunov based control designs. We therefore believe that this book has a widespread appeal, of interest far beyond just those who are interested in the particular class of designs we consider in detail.

We intend that this book should appeal to a wide audience. We hope that it will be read by both control engineers, mathematicians with interests in control theory and computer scientists with interests in machine learning. The basic pre-requisite

is a certain level of mathematical maturity, in particular an understanding of basic analysis and functional analysis. The book [24] provides the required material at about the right level. We do not explicitly assume any knowledge of control theory, but a basic appreciation of the techniques and goals of the subject would undoubtedly be useful. A good introduction to control theory at an appropriate level of mathematical sophistication can be found in [48].

As with all researchers, we owe a great debt of gratitude to a number of people. In particular, our colleagues and students, both past and present, in Southampton, Budapest and elsewhere, have been a constant source of inspiration. The figures were drawn with the help of John Norton and the simulations were completed by a project student who wishes to remain anonymous. Kathryn Sharple at Wiley deserves a strong vote of thanks for supporting us, and for believing that the ever-receding delivery deadline would one-day be met! Most importantly our families have continued to earn our gratitude for their forbearance of this lengthy project.

Southampton and Budapest.  
March 2003.