Table 6.1 Summary of physical observations concerning the beginning of transition from laminar to turbulent flow

Flow Configuration	Condition[a] Necessary for the Existence of Laminar Flow	Source	Observations
Boundary layer flow (without longitudinal pressure gradient)	$Re < 3.5 \times 10^5$ $Re < 2 \times 10^4 – 10^6$	[3][b]	Re is the Reynolds number based on wall length and free-stream velocity
Duct flow	$Re < 2000$		Re is based on hydraulic diameter and duct-averaged velocity
Free-jet flow (axisymmetric)	$Re < 10–30$	[4]	Re is based on nozzle diameter and mean velocity through the nozzle
Wake flow (two-dimensional)	$Re < 32$	[5]	Re is based on the cylinder diameter and free-stream velocity

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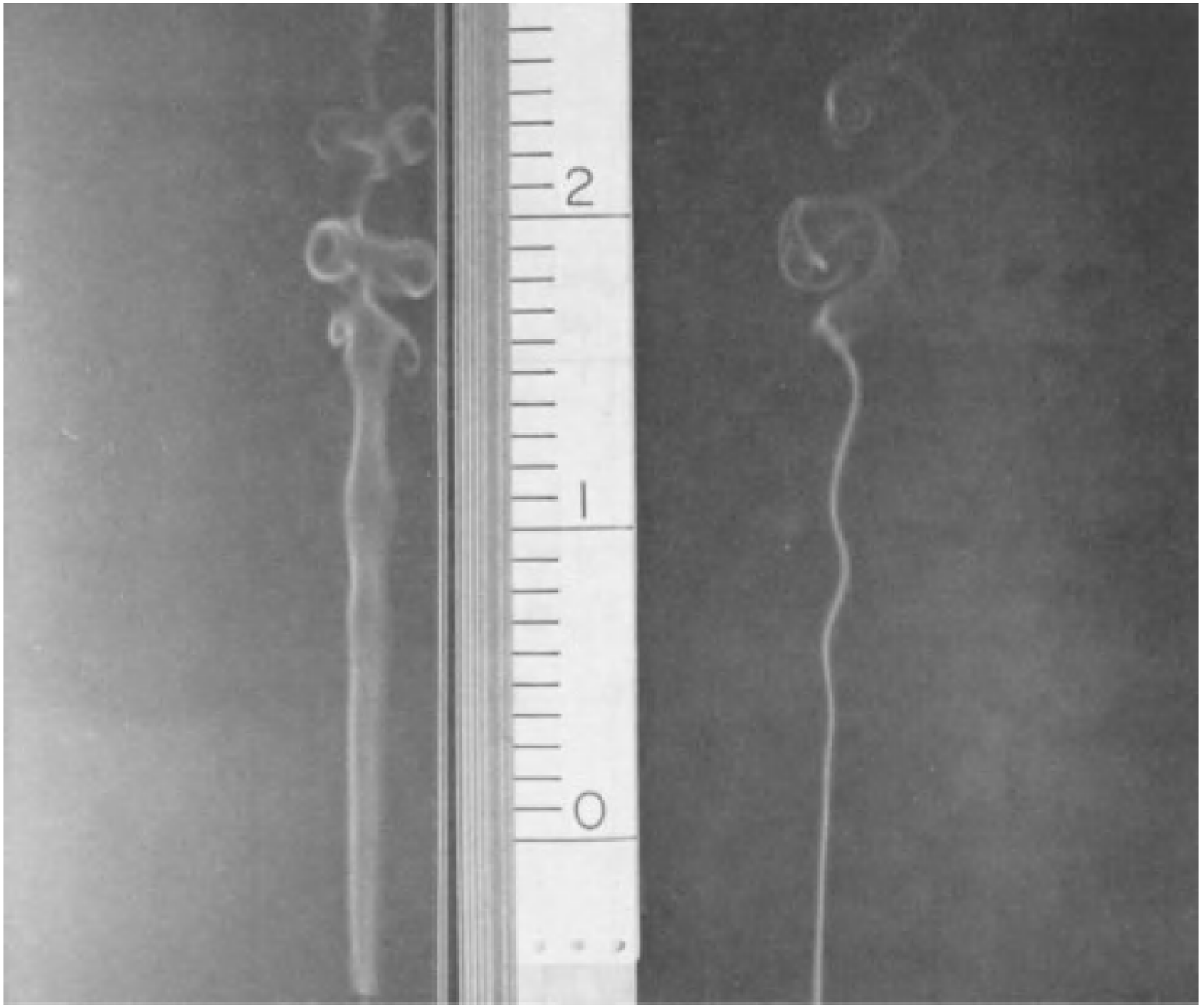
Natural convection boundary layer flow			
Isothermal wall	$Gr < 1.5 \times 10^9$ ($Pr = 0.71$) $Gr < 1.3 \times 10^9$ ($Pr = 6.7$)	[6][c] [6, 7]	The Grashof number Gr is based on wall height and wall–ambient temperature difference
Constant-heat-flux wall	$Gr_* < 1.6 \times 10^{10}$ ($Pr = 0.71$) $Gr_* < 6.6 \times 10^{10}$ ($Pr = 6.7$)	[6] [8]	Gr_* is the Grashof number based on heat flux and wall height [see eq. (4.70), where $Gr_* = Ra_*/Pr$]
Plume flow (axisymmetric)	$Ra_q < 10^{10}$ ($Pr = 0.71$)	[9]	Ra_q is the Rayleigh number based on heat source strength and plume height [see eq. (6.6)]
Film condensation on a vertical plate	$\frac{4\Gamma}{\mu} < 1800$	[10]	Γ is the condensate mass flow rate per unit of film width

[a] All numerical values are order-of-magnitude approximate and vary from one experimental report to another.
[b] The transition is triggered by velocity disturbances in excess of 18 percent of the free-stream velocity.
[c] Averaged from the data compiled in Ref. 6.



a sinusoidal shape of characteristic wavelength λ_B

$$\lambda_B \sim y_{\text{tr}}$$

TURBULENT BOUNDARY LAYER FLOW

Figure 7.1 Turbulent boundary layer in boiling water flowing from left to right over a flat surface; $U_\infty = 0.52$ m/s, $q_0'' = 4.8 \times 10^5$ W/m^2. (Reprinted with permission from J. H. Lienhard, *A Heat Transfer Textbook*, 1981, p. 417. Copyright © 1981 Prentice-Hall, Inc.)

TIME-AVERAGED EQUATIONS

mean values

$$u = \overline{u} + u', \qquad P = \overline{P} + P'$$

$$v = \overline{v} + v', \qquad T = \overline{T} + T'$$

$$w = \overline{w} + w'$$

$$\overline{u} = \frac{1}{\text{period}} \int_0^{\text{period}} u \, d(\text{time})$$

fluctuating

$$\int_0^{\text{period}} u' \, d(\text{time}) = 0$$

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$$\overline{u+v} = \overline{u} + \overline{v} \qquad \overline{uv} = \overline{u}\,\overline{v} + \overline{u'v'} \qquad \overline{\frac{\partial u}{\partial x}} = \frac{\partial \overline{u}}{\partial x} \qquad \frac{\partial \overline{u}}{\partial t} = 0 \qquad \overline{\frac{\partial u}{\partial t}} = 0$$

$$\overline{\overline{u}u'} = 0 \qquad \overline{u^2} = \overline{u}^2 + \overline{u'^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial w'}{\partial z} = 0$$

$$\boxed{\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0}$$

the x momentum equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \nabla^2 u$$

Averaging each term over time

$$\frac{\partial}{\partial x}(\overline{u^2}) + \frac{\partial}{\partial y}(\overline{uv}) + \frac{\partial}{\partial z}(\overline{uw}) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \nu \nabla^2 \overline{u}$$

$$\frac{\partial}{\partial x}(\overline{u}^2) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v}) + \frac{\partial}{\partial z}(\overline{u}\,\overline{w}) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \nu \nabla^2 \overline{u} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial y}(\overline{u'v'}) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \nu \nabla^2 \overline{u} - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) - \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - \frac{\partial}{\partial z}\left(\overline{u'w'}\right)$$

time-averaged of the momentum equations in the y and z directions are

$$\overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} + \nu \nabla^2 \overline{v} - \frac{\partial}{\partial x}\left(\overline{u'v'}\right) - \frac{\partial}{\partial y}\left(\overline{v'^2}\right) - \frac{\partial}{\partial z}\left(\overline{v'w'}\right)$$

$$\overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} + \overline{w}\frac{\partial \overline{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial z} + \nu \nabla^2 \overline{w} - \frac{\partial}{\partial x}\left(\overline{u'w'}\right) - \frac{\partial}{\partial y}\left(\overline{v'w'}\right) - \frac{\partial}{\partial z}\left(\overline{w'^2}\right)$$

the energy equation

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} = \alpha \nabla^2 \bar{T} - \frac{\partial}{\partial x}\ (\overline{u'T'}) - \frac{\partial}{\partial y}\ (\overline{v'T'}) - \frac{\partial}{\partial z}\ (\overline{w'T'})$$

17 unknowns

the unknowns are $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{P}$, $\bar{T}$, and the 12 terms of type of fluctuating quantities

hence, the *closure problem*

BOUNDARY LAYER EQUATIONS

in the boundary layer, we can neglect $(\partial/\partial x)(\overline{u'^2})$ relative to $(\partial/\partial y)(\overline{u'v'})$

$(\partial/\partial x)(\overline{u'T'})$ relative to $(\partial/\partial y)(\overline{v'T'})$

Applying the other simplifications

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{d\bar{P}}{dx} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}\ (\overline{u'v'})$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \alpha\frac{\partial^2 \bar{T}}{\partial y^2} - \frac{\partial}{\partial y}\ (\overline{v'T'})$$

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$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{d\bar{P}}{dx} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}(\overline{u'v'})$$

$$\underbrace{\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \alpha\frac{\partial^2 \bar{T}}{\partial y^2} - \frac{\partial}{\partial y}(\overline{v'T'})}_{\text{to rewrite eqs.}}$$

$\Rightarrow$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{d\bar{P}}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{1}{\rho c_P}\frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_P\overline{v'T'}\right)$$

the products $\overline{u'v'}$ and $\overline{v'T'}$ $\Rightarrow$ they are nonzero, are they negative or positive?

the product $u'v'$ emerges as a negative

$$-\rho\overline{u'v'} = \rho\epsilon_M\frac{\partial \bar{u}}{\partial y}$$ eddy shear stress

$$-\rho c_P\overline{v'T'} = \rho c_P\epsilon_H\frac{\partial \bar{T}}{\partial y}$$ eddy heat flux

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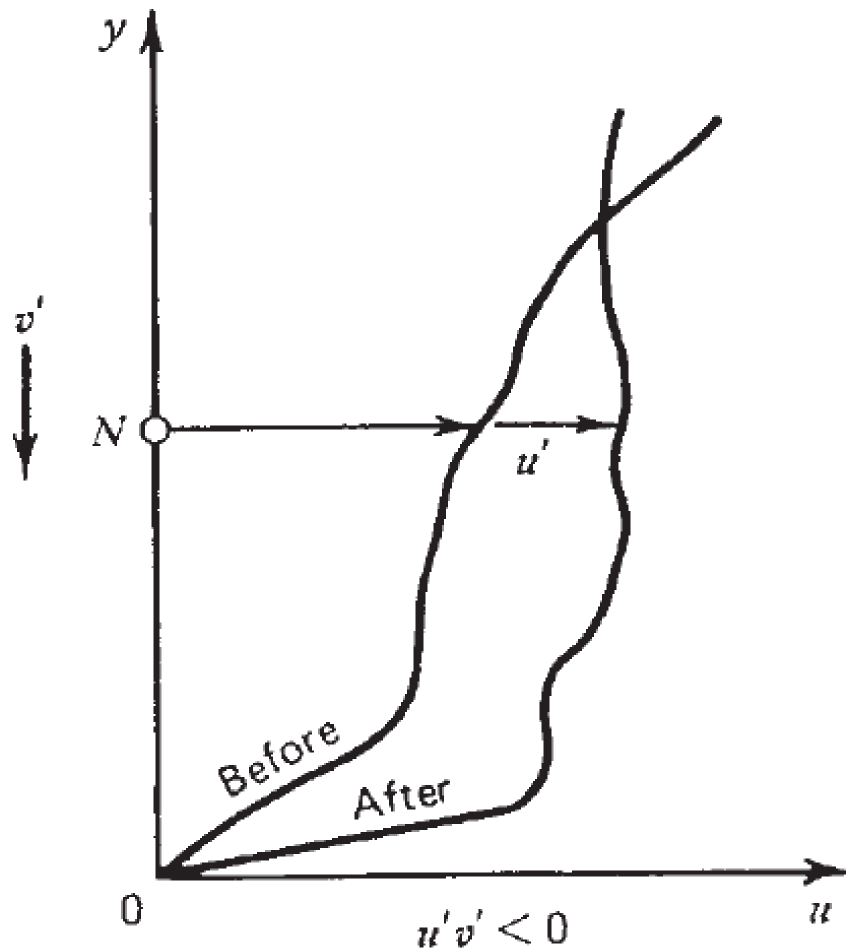

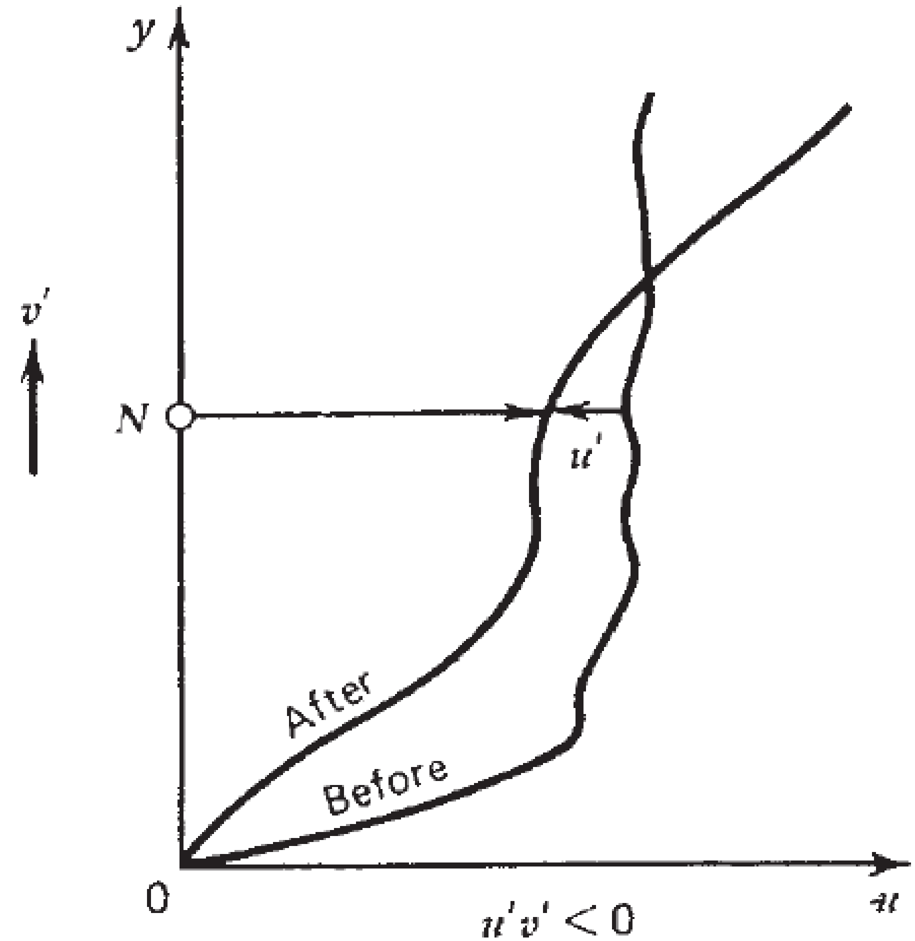

$v' < 0$; increase the longitudinal velocity to induce a positive fluctuation $u' > 0$.
the product $u'v'$ emerges as a negative

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$$\tau_{\text{app}} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} = \rho(\nu + \epsilon_M)\frac{\partial \overline{u}}{\partial y}$$ apparent shear stress

$$-q''_{\text{app}} = k\frac{\partial \overline{T}}{\partial y} - \rho c_P \overline{v'T'} = \rho c_P(\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y}$$ apparent heat flux

into the boundary layer equations

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{d\overline{P}}{dx} + \frac{\partial}{\partial y}\left[(\nu + \epsilon_M)\frac{\partial \overline{u}}{\partial y}\right]$$

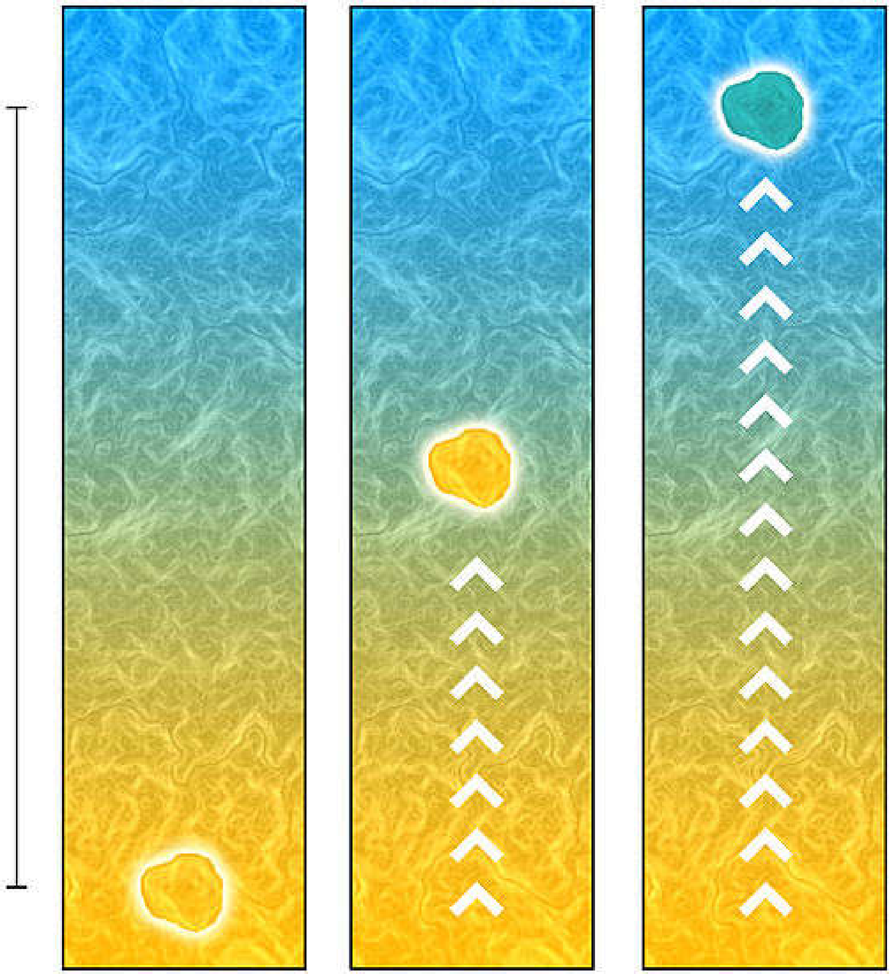
$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[(\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y}\right]$$

where ϵ_M and ϵ_H are two empirical functions known as *momentum eddy diffusivity* and *thermal eddy diffusivity*,

Note that ϵ_M and ϵ_H are *flow parameters*, not fluid properties.

five unknowns ($\overline{u}, \overline{v}, \overline{T}, \epsilon_M$, and ϵ_H).

MIXING LENGTH MODEL



a length called mixing length which is the average distance perpendicular to flow a small fluid mass will travel before its momentum is changed by new environment.

Fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v'. If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1}$$

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1}$$

$$\left|\overline{u'}\right| = \frac{1}{2}\left(\left|\Delta u_1\right| + \left|\Delta u_2\right|\right) = l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|_{y1}$$

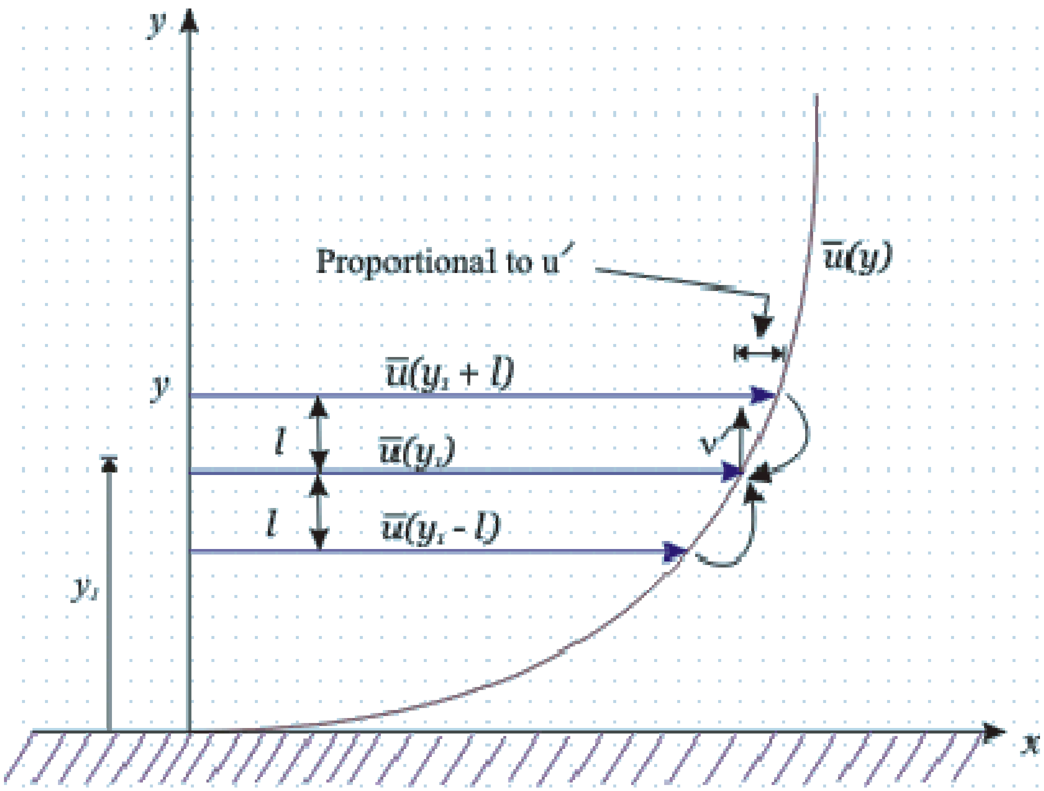
$$\left|\overline{v'}\right| \sim \left|\overline{u'}\right|$$

or, $$\left|\overline{v'}\right| = (const)\left|\overline{u'}\right| = (const)\, l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|$$

$$\overline{u'v'} = -C_1\left|\overline{u'}\right|\left|\overline{v'}\right|$$

$$\left.\begin{aligned} \overline{u'v'} &= -C_2 l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \\ \tau_t &= -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y} \end{aligned}\right\} \quad \mu_t = \rho l^2\left|\frac{\partial \bar{u}}{\partial y}\right|$$

$$\nu_t = l^2\left|\frac{\partial \bar{u}}{\partial y}\right|$$



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$$\epsilon_M = l^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \qquad \boxed{l = \kappa y}$$

κ is an empirical constant

for von Kármán's constant κ turns out to be 0.4

$$\epsilon_M = \kappa^2 y^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$$

uniform flow parallel to a flat wall

apparent shear stress $(\nu + \epsilon_M)(\partial \overline{u}/\partial y)$ that does not vary with y

$$(\nu + \epsilon_M)\frac{\partial \overline{u}}{\partial y} = \frac{\tau_0}{\rho}$$

at $y = 0$ → the Reynolds stress $-\rho\overline{u'v'}$ vanishes

at $y = 0$ → $(\nu + \overset{=0}{\cancel{\epsilon_M}})\dfrac{\partial \overline{u}}{\partial y}$ → τ_0 is the actual wall shear stress

the dimensions of $(\tau_0/\rho)^{1/2}$ = velocity

friction velocity $\boxed{u_* = \left(\dfrac{\tau_0}{\rho}\right)^{1/2}}$

nondimensionalization of the flow problem

$$u^+ = \frac{\overline{u}}{u_*}, \qquad v^+ = \frac{\overline{v}}{u_*}$$

$$x^+ = \frac{x u_*}{\nu}, \qquad y^+ = \frac{y u_*}{\nu}$$

$$\left.\begin{aligned} u^+ &= \frac{\overline{u}}{u_*}, \\ y^+ &= \frac{y u_*}{\nu} \\ u_* &= \left(\frac{\tau_0}{\rho}\right)^{1/2} \end{aligned}\right\} \quad \left(1 + \frac{\epsilon_M}{\nu}\right) \frac{du^+}{dy^+} = 1$$

depending on the relative size of ϵ_M and ν.

ϵ_M/ν must vary with the distance measured away from the wall

1. The *viscous sublayer* (VSL), where $\nu \gg \epsilon_M$
2. The *fully turbulent sublayer* (or the *turbulent core*), where $\epsilon_M \gg \nu$

$$\left.\begin{aligned} y^+_{\mathrm{VSL}} \Rightarrow \text{Neglecting the term } \epsilon_M/\nu \\ u^+(0) = 0, \end{aligned}\right\} \quad u^+ = y^+$$

In the fully turbulent sublayer

$$\left.\begin{aligned} \left(1 + \frac{\epsilon_M}{\nu}\right) \frac{du^+}{dy^+} = 1 \Rightarrow \frac{\epsilon_M}{\nu}\frac{du^+}{dy^+} = 1 \\ \epsilon_M = \kappa^2 y^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \end{aligned}\right\} \quad \kappa^2 (y^+)^2 \left(\frac{du^+}{dy^+}\right)^2 = 1$$

Integrating this equation from the sublayer interface y^+_{VSL} to any y^+

$$\int \kappa^2 (y^+)^2 \left(\frac{du^+}{dy^+}\right)^2 = 1 \quad \Rightarrow \quad u^+ = \frac{1}{\kappa} \ln y^+ + y^+_{VSL} - \frac{1}{\kappa} \ln y^+_{VSL}$$

$$\boxed{u^+ = A \ln y^+ + B}$$

where A and B are two empirical constants.

experimental measurements $\Rightarrow$ $A \cong 2.5$ and $B \cong 5.5$

$\kappa \cong 0.4$ and $y^+_{VSL} \cong 11.6$

$y^+ = y^+_{VSL}$, where neither ν nor ϵ_M can be neglected

The experimental observation that the transition from the viscous sublayer to the fully turbulent sublayer takes place around $y^+_{VSL} = O(10)$

The laminar shear layer grows until the local Reynolds number based on local thickness becomes of order 10^2

$$\frac{y_{VSL} U_\infty}{\nu} \sim 10^2$$

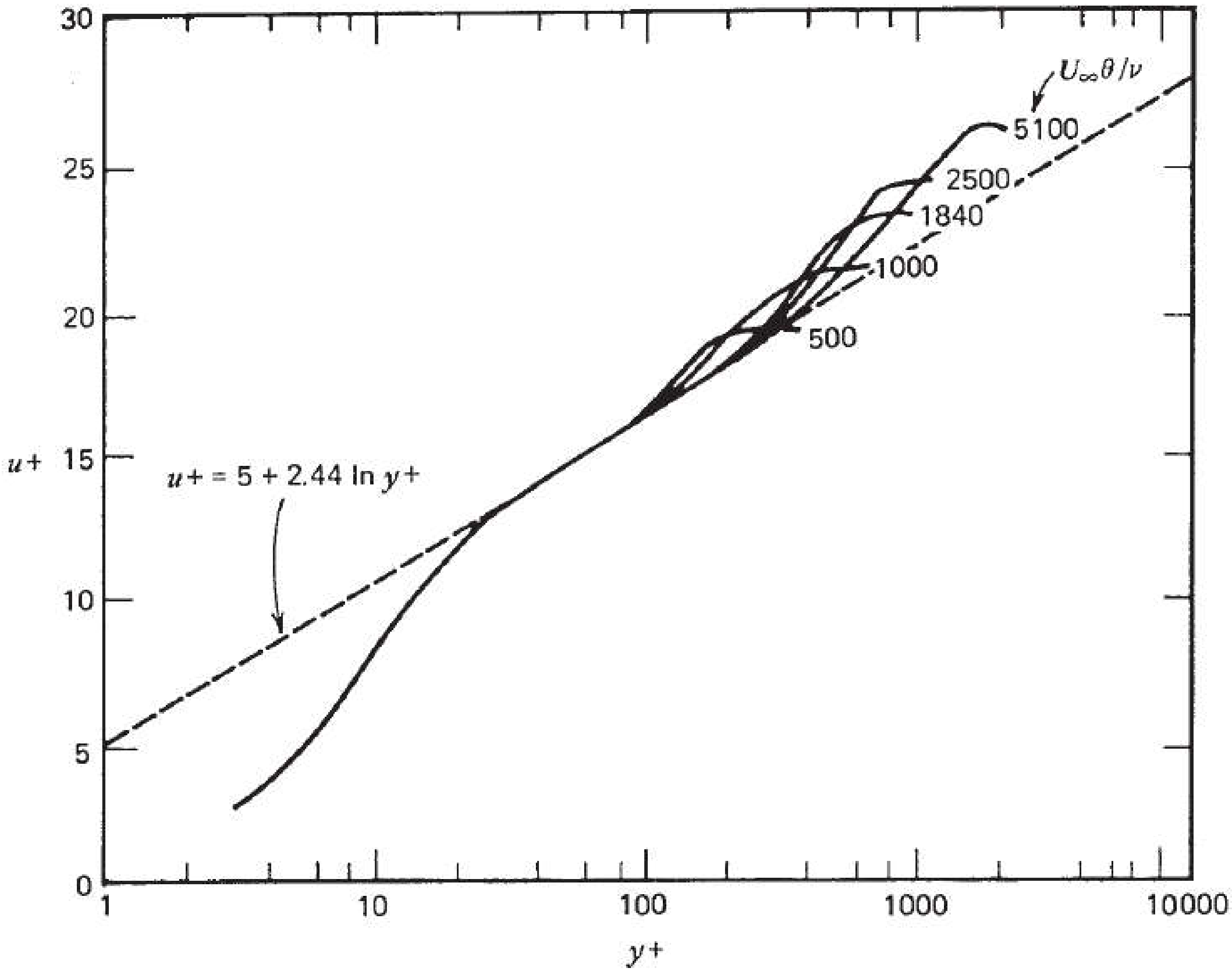

Figure 7.4 Example of $u^+(y^+)$ velocity measurements in turbulent boundary layer flow without longitudinal pressure gradient. Note the use of $A \cong 2.44$ and $B \cong 5$ to fit the data. (Reprinted with permission from L. P. Purtell et al., *Physics of Fluids*, Vol. 24, pp. 802–811, May 1981. Copyright © 1981 American Institute of Physics.)

Table 7.1 Summary of longitudinal velocity expressions for the inner region of a turbulent boundary layer

$u^+(y^+)$	Range	Reference
$u^+ = y^+$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 11.6$ $y^+ > 11.6$	Prandtl and Taylor [9]
$u^+ = y^+$ $u^+ = 5 \ln y^+ - 3.05$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 5$ $5 < y^+ < 30$ $y^+ > 30$	von Kármán [12]
$u^+ = 14.53 \tanh(y^+/14.53)$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 27.5$ $y^+ > 27.5$	Rannie [13]
$\frac{du^+}{dy^+} = \frac{2}{1 + \{1 + 4\kappa^2 y^{+2}[1 - \exp(-y^+/A^+)]^2\}^{1/2}}$ $\kappa = 0.4 \qquad A^+ = 26$	All y^+	van Driest [14]
$u^+ = 2.5 \ln(1 + 0.4y^+)$ $+ 7.8[1 - \exp(-y^+/11)$ $- (y^+/11) \exp(-0.33y^+)]$	All y^+	Reichardt [15]

$\frac{du^+}{dy^+} = \frac{1}{1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]}$ $n = 0.124$ $u^+ = 2.78 \ln y^+ + 3.8$	$0 < y^+ < 26$	Deissler [16]
$y^+ = u^+ + A[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2 - \frac{1}{6}(Bu^+)^3 - \frac{1}{24}(Bu^+)^4]$ (last term in u^{+4} may be omitted)	All y^+ $A = 0.1108$ $B = 0.4$	Spalding [17]

Source: After Ref. 10.

the wall shear stress scales as $\mu U_\infty / y_{\mathrm{VSL}}$

y^+_{VSL} at transition $\rightarrow$ $y^+_{\mathrm{VSL}} = \frac{y_{\mathrm{VSL}}}{\nu}\left(\frac{\tau_0}{\rho}\right)^{1/2} = \left(\frac{y_{\mathrm{VSL}} U_\infty}{\nu}\right)^{1/2}$ $\rightarrow$ $y^+_{\mathrm{VSL}} \sim 10$

the y^+_{VSL} ~10 theory presented *first eddy* that forms immediately after the laminar shear layer

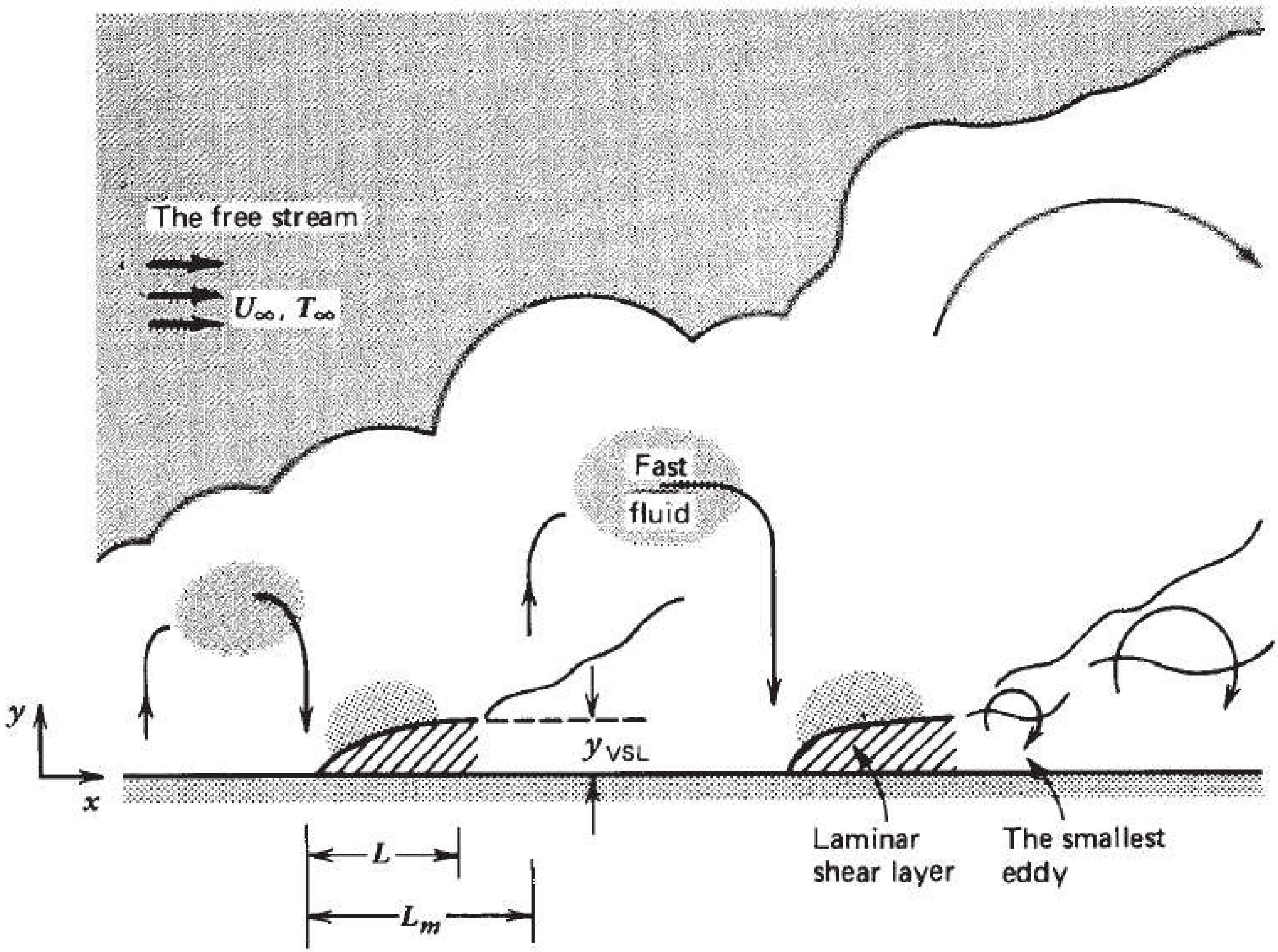

Figure 7.5 Formation of the viscous sublayer as the time-averaged superposition of laminar shear layers with *local Reynolds numbers* no greater than $\sim 10^2$.

$$\epsilon_M = \kappa^2 y^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$$

$$\left(1 + \frac{\epsilon_M}{\nu}\right) \frac{du^+}{dy^+} = 1$$

$$\left[1 + \kappa^2 (y^+)^2 \frac{du^+}{dy^+}\right] \frac{du^+}{dy^+} = 1 \Rightarrow \kappa u^+ = \frac{\cos\alpha - 1}{\sin\alpha} + \ln\left[\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]$$

$$\alpha = \arctan(2\kappa y^+)$$

$$u^+ \to y^+ \quad \text{as} \quad y^+ \to 0$$

$$u^+ \to \frac{1}{\kappa} \ln y^+ + \frac{2 \ln 2 + \ln \kappa - 1}{\kappa} \qquad \text{as} \quad y^+ \to \infty$$

if $\kappa \cong 0.4$, $\Rightarrow$ $u^+ = 2.5 \ \ln \ y^+ - 1.325$

WALL FRICTION IN BOUNDARY LAYER FLOW

$$C_{f,x} = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} \qquad \frac{U_\infty}{(\tau_0/\rho)^{1/2}} \cong f_u\left[\frac{\delta}{\nu}\left(\frac{\tau_0}{\rho}\right)^{1/2}\right]$$

$$u^+ \cong f_u(y^+)$$

$$d\overline{P}/dx = 0 \,;$$

$$\frac{d}{dx}\int_0^\infty \overline{u}(U_\infty - \overline{u}) \ dy = \frac{\tau_0}{\rho}$$

Prandtl's one-seventh power law as the fit for the $u^+(y^+)$ data ➡ $f_u = 8.7(y^+)^{1/7}$

$$\frac{\tau_0}{\rho U_\infty^2} = 0.0225\left(\frac{U_\infty \delta}{\nu}\right)^{-1/4} \qquad \frac{\delta}{x} = 0.37\left(\frac{U_\infty x}{\nu}\right)^{-1/5} \qquad \delta = 8\delta^* = \frac{72}{7}\ \theta$$

$$\frac{\tau_0}{\rho U_\infty^2} = \frac{1}{2}C_{f,x} = 0.0296\left(\frac{U_\infty x}{\nu}\right)^{-1/5}$$

$$\frac{\tau_{0-x}}{\rho U_\infty^2} = \frac{1}{2}C_{f,0-x} = 0.037\left(\frac{U_\infty x}{\nu}\right)^{-1/5}$$

Schultz-Grunow's empirical correlation [18]

$$C_{f,x} = 0.37\left[\log_{10}\left(\frac{U_\infty x}{\nu}\right)\right]^{-2.584}$$

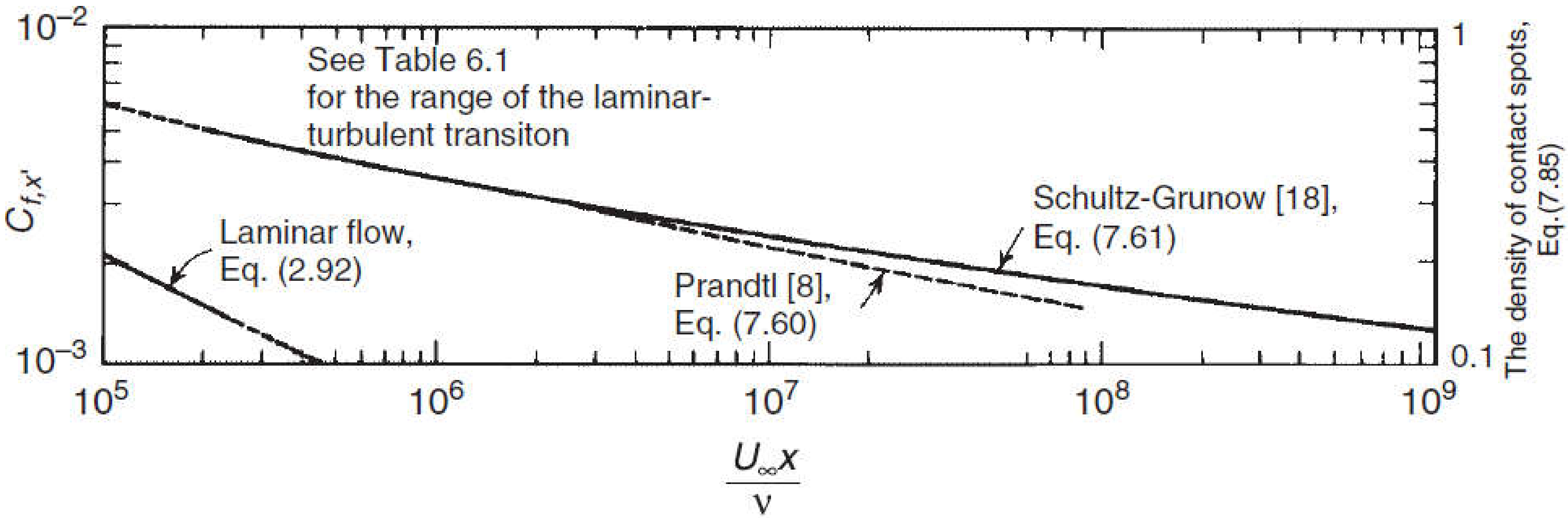

Figure 7.6 Local skin friction coefficient for turbulent boundary layer flow over a plane wall.

HEAT TRANSFER IN BOUNDARY LAYER FLOW

the apparent heat flux q''_{app} does not depend on y

$$(\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y} = \left[(\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y}\right]_{y=0} \Rightarrow (\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y} = \frac{-q''_0}{\rho c_P}$$

$$\frac{\rho c_P u_*}{-q''_0}\frac{\partial}{\partial y^+}(\overline{T} - T_0) = \frac{1}{\alpha/\nu + \epsilon_H/\nu}$$

$$T^+(x^+,\ y^+) = (T_0 - \overline{T})\frac{\rho c_P u_*}{q_0''}$$ the integral ➡ $$T^+ = \int_0^{y^+} \frac{dy^+}{1/\mathrm{Pr} + (1/\mathrm{Pr}_t)(\epsilon_M/\nu)}$$

the *turbulent Prandtl number* ($\mathrm{Pr}_t = \epsilon_M/\epsilon_H$)

assumptions regarding Pr, Pr_t, and ϵ_M/ν

in the fully turbulent region $$\frac{\epsilon_M}{\nu} = \frac{dy^+}{du^+} = \kappa y^+$$

$$T^+ = \int_0^{y^+_{\mathrm{CSL}}} \frac{dy^+}{\dfrac{1}{\mathrm{Pr}} + \begin{pmatrix}\text{negligible}\\ \text{term}\end{pmatrix}} + \int_{y^+_{\mathrm{CSL}}}^{y^+} \frac{dy^+}{\begin{pmatrix}\text{negligible}\\ \text{term}\end{pmatrix} + \dfrac{1}{\mathrm{Pr}_t}\dfrac{\epsilon_M}{\nu}}$$

y^+_{CSL} is the dimensionless thickness of a *conduction sublayer* (CSL) the molecular mechanism outweighs the eddy transport of heat.

$$T^+ = \begin{cases} \mathrm{Pr}\ y^+, & y^+ < y^+_{\mathrm{CSL}} \\ \mathrm{Pr}\ y^+_{\mathrm{CSL}} + \dfrac{\mathrm{Pr}_t}{\kappa} \ln \dfrac{y^+}{y^+_{\mathrm{CSL}}}, & y^+ > y^+_{\mathrm{CSL}} \end{cases}$$

$\mathrm{Pr}_t \cong 0.9$,

$\kappa \cong 0.41$,

$y^+_{\mathrm{CSL}} \cong 13.2$

$y^+ > y^+_{\mathrm{CSL}}$ portion of the T^+ profile $\Rightarrow T^+ = 2.195 \ln y^+ + 13.2\mathrm{Pr} - 5.66$

Prandtl number of the fluid is in the range 0.5–5.

Setting $\overline{T} = T_\infty$ at $y = \delta$

$$T^+(x^+, \ y^+) = (T_0 - \overline{T})\frac{\rho c_P u_*}{q_0''}$$

$$T^+ = \begin{cases} \mathrm{Pr} \ y^+, & y^+ < y^+_{\mathrm{CSL}} \\ \mathrm{Pr} \ y^+_{\mathrm{CSL}} + \dfrac{\mathrm{Pr}_t}{\kappa} \ln \dfrac{y^+}{y^+_{\mathrm{CSL}}}, & y^+ > y^+_{\mathrm{CSL}} \end{cases}$$

$$\rho c_P u_* \frac{T_0 - T_\infty}{q_0''} = \mathrm{Pr} \ y^+_{\mathrm{CSL}} + \frac{\mathrm{Pr}_t}{\kappa} \ln \frac{\delta u_*/\nu}{y^+_{\mathrm{CSL}}}$$

$$h = q_0''/(T_0 - T_\infty)$$

$$\frac{U_\infty}{u_*} = \frac{1}{\kappa} \ln \frac{\delta u_*}{\nu} + B$$

$$\frac{U_\infty}{u_*} = \left(\frac{2}{C_{f,x}}\right)^{1/2}$$

$$\frac{h}{\rho c_P U_\infty} = \frac{\frac{1}{2}C_{f,x}}{\mathrm{Pr}_t + \left(\frac{1}{2}C_{f,x}\right)^{1/2}[\mathrm{Pr}\, y^+_{\mathrm{CSL}} - B\,\mathrm{Pr}_t - (\mathrm{Pr}_t/\kappa) \ln y^+_{\mathrm{CSL}}]}$$

$$\mathrm{St}_x = \frac{h}{\rho c_P U_\infty} = \frac{\mathrm{Nu}_x}{\mathrm{Pe}_x} = \frac{\mathrm{Nu}_x}{\mathrm{Re}_x \mathrm{Pr}}$$ local *Stanton number*

$$B \cong 5.1$$

$$\boxed{\mathrm{St}_x = \frac{\frac{1}{2}C_{f,x}}{0.9 + \left(\frac{1}{2}C_{f,x}\right)^{1/2}(13.2\mathrm{Pr} - 10.25)}}$$