Complex Analysis – Test I C Term 2006

Do any 5 of the following.

1. a) Find all solutions of $z^5 = 3i - 3$ and indicate where they lie in the complex plane.

Solution: $|3i-3| = 3\sqrt{2}$ and $\mathrm{Arg}(3i-3) = 3\pi/4$. So the fifth roots are $(3\sqrt{2})^{1/5}e^{3\pi i/20+2k\pi i/5}$, with $k = 0,1,2,3,4$

b) Find all solutions to $z^2 - 2iz + 1 = -2i$ and show their positions on the z plane.

Solution: $z = \dfrac{2i \pm \sqrt{-4-4(1+2i)}}{2} = i \pm \sqrt{-2-2i} = i \pm \sqrt{8e^{5\pi i/4}} = i \pm 2\sqrt{2}e^{5\pi i/8}$.

2. Let $z = 3i + 3$ and $w = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. Compute $\bar{w}/z$, $w - |w|$, Re(zw), and z^9 in either rectangular or polar form.

Solution: $|z| = 3\sqrt{2}$, $\mathrm{Arg}(z) = \pi/4$, $|w| = 1$, $\mathrm{Arg}(w) = -\pi/3$.

$\bar{w}/z = 3\sqrt{2}e^{\pi i/12}$,

$w - |w| = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$,

$\mathrm{Re(zw)} = \frac{3}{2} + 3\frac{\sqrt{3}}{2} = \frac{3}{2}(1+\sqrt{3})$, and

$z^9 = (3\sqrt{2})^9 e^{\pi i/4}$.

3. Find all solutions to $\sin(z) = 3$.

Solution: $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$. So either $\cos(x) = 0$ or $\sinh(y) = 0$.

If $\sinh(y) = 0$, then $\cosh(y) = 1$, and $\sin(z) = \sin(x) \neq 3$ for real x, so $\cos(x) = 0$, and $x = \frac{(2k+1)\pi}{2}$. $\sin(\frac{(2k+1)\pi}{2}) = (-1)^k$, so $\cosh(y) = 3(-1)^k$. Since $\cosh(y) > 1$, we have $x = \frac{(4k+1)\pi}{2}$ and $y = \pm\mathrm{arccosh}(3) = \ln(3 \pm \sqrt{8})$.

Note: $(3+\sqrt{8})(3-\sqrt{8}) = 1$, so $\ln(3\pm\sqrt{8}) = \pm\ln(3+\sqrt{8})$.

Alternate Solution: $\sin(z) = 3$ is equivalent to $e^{iz} - e^{-iz} = 6i$, or $(e^{iz})^2 - 6ie^{iz} - 1 = 0$, so $e^{iz} = \dfrac{6i \pm \sqrt{-6^2+4}}{2} = 3i \pm \sqrt{-3^2+1} = 3i \pm i\sqrt{8} = i(3\pm\sqrt{8}) = ie^{\pm\ln(3+\sqrt{8})} = e^{i\pi/2 \pm \ln(3+\sqrt{8})} = e^{i(\pi/2 \pm i\ln(3+\sqrt{8}))}$
So $z = \pi/2 + 2k\pi \pm i\ln(3+\sqrt{8})$.

4. Suppose $x > 0$ and set set $w = u + iv$ with $u = \dfrac{\ln(x^2+y^2)}{2}$ and $v = \arctan\left(\dfrac{y}{x}\right)$.

Show that w is analytic.

Solution: We check the Cauchy-Riemann equations.

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} \\
\frac{\partial w}{\partial y} &= \frac{-y/x^2}{1+(y/x)^2} + i\frac{1/x}{1+(y/x)^2}
\end{aligned}$$

$$= \frac{-y}{x^2+y^2} + i\frac{x}{x^2+y^2}$$

5. As in problem 1, suppose $x > 0$ and set set $w = u + iv$ with $u = \dfrac{\ln(x^2+y^2)}{2}$ and $v = \arctan\left(\dfrac{y}{x}\right)$.

Consider the (half) polar grid $z = ne^{it}$, with $n = 1, 2, 3, \ldots$ and $-\pi/2 < t < \pi/2$ and $z = se^{m\pi/8}$, with $m = \{-3, -2, -1, 0, 1, 2, 3\}$ and $0 < s < \infty$.

Sketch the image of this grid in the w plane.

Solution: For the rays $z = se^{m\pi/8}$, $u = \ln(s)$ and $v = m\pi/8$, so these map to equally spaced horizontal lines with y-intercepts $m\pi/8$.

For the semi-circles $z = ne^{it}$, $u = \ln(n)$ and $v = t$, $-\pi/2 < t < \pi/2$, giving vertical line segments at $\ln(n)$.

6. a) Show that $\sin(\bar{z}) = \overline{\sin(z)}$ and that $e^{\bar{z}} = \overline{e^{z}}$.

b) Let $f(z)$ be analytic in for all z in the complex plane. Either prove that $f(\bar{z}) = \overline{f(z)}$ or give an example of an $f(z)$, analytic in the whole plane, which violates the property.

Solution: $e^{\bar{z}} = e^{x}(\cos(-y) + i\sin(-y)) = e^{x}(\cos(y) - i\sin(y)) = \overline{(e^{z})}$.

Since this is true, $\overline{\sin(\bar{z})} = \dfrac{\overline{e^{i\bar{z}}} - \overline{e^{-i\bar{z}}}}{-2i} = \dfrac{e^{-iz} - e^{iz}}{-2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = \sin(z)$

The general result is not true, however. Even the constant function $f(z) = i$ does not satisfy it.

7. Let $u = xy(x^2 - y^2 + 1)$. Find v such that $u + iv$ is analytic for all $z = x + iy$.

Solution: $u = x^3y - y^3x + xy$. $u_x = 3x^2y - y^3 + y = v_y$, so $v = 3x^2y^2/2 - y^4/4 + y^2/2 + p(x)$.

Also $-u_y = -x^3 + 3y^2x - x = v_x$, so $v = -x^4/4 + 3x^2y^2/2 - x^2/2 + q(y)$.

So we can take $v = y^4/4 - x^4/4 + 3x^2y^2/2 + y^2/2 - x^2/2 + C$.

(In this case $w = i(C - 2z^2 - z^4)/4$.)