MTH 562 Complex Analysis

Spring 2007

Abstract

Rigorous development of theory of functions. Topology of plane, complex integration, singularities, conformal mapping.

1 Complex Numbers

Definition. We define

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$$

equipped with the following rules of addition and multiplication:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac - bd, bc + ad)\end{aligned}$$

It is easy and boring to check that

1. $\mathbb{C}$ is an Abelian group under addition with identity $(0, 0)$.
2. $\mathbb{C} \setminus \{(0, 0)\}$ is an Abelian group under multiplication with identity $(1, 0)$.
3. Multiplication distributes over addition; that is,

$$\begin{aligned}(a, b)((c, d) + (e, f)) &= (a, b)(c, d) + (a, b)(e, f)\\ ((a, b) + (c, d))(e, f) &= (a, b)(e, f) + (c, d)(e, f)\end{aligned}$$

These three conditions imply that $\mathbb{C}$ is a **field**.

The set $\{(a, 0) : a \in \mathbb{R}\}$ is a subfield of $\mathbb{C}$, and the mapping

$$\mathbb{R} \ni a \mapsto (a, 0) \in \mathbb{C}$$

is an isomorphism of fields. This shows that $\mathbb{R}$ sits inside $\mathbb{C}$. (From now on, forget about the identification and write a instead of $(a, 0)$.)

The special number $i := (0, 1)$ satisfies $i^2 = -1$.

Any complex number $z = (a, b)$ can be expressed uniquely as

$$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1),$$

which using our identification above, we can write as

$$z = a + ib.$$

From now on, we will write complex numbers in this so-called Cartesian form. For $z = a + ib$, a is called the real part of z, $\operatorname{Re} z$, and b is called the imaginary part of z, $\operatorname{Im} z$.

Two important operations on $\mathbb{C}$ (which are *not* field operations) are

1. Conjugation: If $z = a + ib$, then the conjugate $\overline{z}$ is given by
$$\overline{z} = a - ib.$$
2. Modulus (or Absolute Value): If $z = a + ib$, then the modulus or absolute value $|z|$ is given by
$$|z| = \sqrt{a^2 + b^2}.$$

The following are some easy facts about these two operations:

- $z\overline{z} = |z|^2$.
- If $z \neq 0$, then $1/z = \overline{z}/|z|^2$.
- $$\operatorname{Re} z = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$
- $(\overline{z + w}) = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$.
- $|zw| = |z|\,|w|$.
- $|z/w| = |z|\,/\,|w|$ provided $w \neq 0$.
- $|\overline{z}| = |z|$.
- The triangle inequality: $|z + w| \leqslant |z| + |w|$.
- The reverse triangle inequality: $|z - w| \geqslant ||z| - |w||$.

Since a point $(a, b) \in \mathbb{R}^2$ can also be represented in terms of polar coordinates (r, θ), where
$$r = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \tan\theta = \frac{b}{a}$$
or
$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta,$$
we can represent a complex number $z = a + ib$ as
$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$
where
$$r = \sqrt{a^2 + b^2} = |z| \qquad \text{and} \qquad \tan\theta = \frac{b}{a}.$$
The number θ is called the argument of z, $\operatorname{Arg} z$. Note that $\operatorname{Arg} z$ is a only defined up to an integer multiple of 2π. (We will meet the more usual form $z = re^{i\theta}$ later, after we have done some work on complex power series.)

A Little Topology

Definition. We define the open disk centered at $z \in \mathbb{C}$ of radius r as

$$D(z,r) = \{w \in \mathbb{C} : |w - z| < r\}.$$

Definition. $U \subset \mathbb{C}$ is said to be **open** if for every $z \in U$, there exists $\varepsilon > 0$ such that $D(z,\varepsilon) \subset U$.

Definition. $F \subset \mathbb{C}$ is **closed** if $\mathbb{C} \setminus F$ is open.

Proposition. *$F \subset \mathbb{C}$ is closed if and only if every convergent sequence of points in F has its limit in F.*

Definition. $K \subset \mathbb{C}$ is **compact** if given any cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of K by open sets, there is a finite subcover.

Theorem. *The following are equivalent:*

1. *$K \subset \mathbb{C}$ is compact.*
2. *Every sequence of points in K has a convergent subsequence.*
3. *K is closed and bounded.*

Theorem. *If f is a continuous real-valued function on a compact set K, then f attains its maximum and minimum values on K.*

2 Differentiability

2.1 Definition. Let $U \subset \mathbb{C}$ be open. Then $f : U \to \mathbb{C}$ is **differentiable** at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If the limit exists, we call it the **derivative** of f at z_0. We write $f'(z_0)$ or $\mathrm{d}f/\mathrm{d}z\,|_{z=z_0}$. We say that f is **differentiable** in U if f is differentiable at each point in U. We say that f is **holomorphic** in U if it is differentiable in U and f' is continuous in U.

2.2 Proposition. *If f is differentiable at z_0, then f is continuous at z_0.*

2.3 Proposition. *If f and g are differentiable at z_0, $c \in \mathbb{C}$, so are $f+g$, $f-g$, cf, fg, and f/g provided $g(z_0) \neq 0$, and the usual calculus formulae hold for these derivatives.*

2.4 Proposition. *Let g be the inverse of f at $f(z_0)$. Suppose f is continuous at $g(z_0)$, then g is continuous at z_0. Then if f is differentiable at $g(z_0)$, and if $f'(g(z_0)) \neq 0$, g is differentiable at z_0 and*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

2.5 Proposition (Chain Rule)**.** *Let f, g be differentiable on open sets U and V, respectively. Suppose f is differentiable at $z_0 \in U$ and g is differentiable at $f(z_0) \in V$. Then $g \circ f$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0).$$

Proof. Let $r > 0$ be such that $D(z, r) \subset U$ where U is open. Without loss of generality, it suffices to show that given any sequence $\{h_n\}$ with $0 < |h_n| < r$ for all $n \in \mathbb{N}$ and $h_n \to 0$ as $n \to \infty$, we have to show

$$\lim_{n\to\infty} \frac{g(f(z_0 + h)) - g(f(z_0))}{h_n} = g' f(z_0)) \cdot f'(z_0).$$

Case 1: Assume $f(z_0 + h_n) \neq f(z_0)$ for all but finitely many n. Then

$$\frac{g(f(z_0 + h)) - g(f(z_0))}{h_n} = \overbrace{\frac{g(f(z_0 + h_n)) - g(f(z_0))}{f(z_0 + h_n) - f(z_0)}}^{(2.1)} \cdot \overbrace{\frac{f(z_0 + h_n) - f(z_0)}{h_n}}^{(2.2)}.$$

We can do this since $f(z_0 + h_n) \neq f(z_0)$ provided n is sufficiently large. Since f is differentiable at z_0, it is continuous at z_0, so $f(z_0 + h_n) \to f(z_0)$. Hence (2.1) tends to $g'(f(z_0))$ while (2.2) tends to $f'(z_0)$. This completes Case 1.

Case 2: Assume $f(z_0 + h_n) = f(z_0)$ for infinitely many n. Split $\{h_n\}$ into two sequences $\{k_n\}$ and $\{\ell_n\}$, where

$$\begin{aligned} f(z_0 + k_n) &\neq f(z_0) \qquad \text{for all } n \in \mathbb{N}, \\ f(z_0 + \ell_n) &= f(z_0) \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Note that $\{k_n\}$ may be finite or empty. f is differentiable at z_0, so

$$\lim_{n\to\infty} \frac{f(z_0 + \ell_n) - f(z_0)}{\ell_n} = f'(z_0),$$

and since $f(z_0 + \ell_n) = f(z_0)$ for all n, this means that $f'(z_0) = 0$. Also, similarly,

$$\lim_{n\to\infty} \frac{g(f(z_0 + \ell_n)) - g(f(z_0))}{\ell_n} = 0.$$

Now, apply Case 1 to $\{k_n\}$ to get

$$\lim_{n\to\infty} \frac{g(f(z_0 + k_n)) - g(f(z_0))}{k_n} = g'(f(z_0)) \cdot f'(z_0) = 0$$

since $f'(z_0) = 0$. We get the same answer for $\{k_n\}$ and $\{\ell_n\}$, which finishes the proof. ∎

3 The Cauchy-Riemann Equations

3.1 Proposition. *If $f(z) = u(z) + iv(z)$ is differentiable at z_0, then the first-order partial derivatives of u and v exist at z_0 and satisfy the Cauchy-Riemann Equations:*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

Proof. If h is real and $h \to 0$, then

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = f_x(z_0) = u_x(z_0) + iv_x(z_0).$$

If $h = ik$ is imaginary and $h \to 0$, then

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+ik) - f(z_0)}{ik} \\ &= \frac{1}{i} \lim_{h\to 0} \frac{f(z_0+ik) - f(z_0)}{k} \\ &= \frac{1}{i}(u_y(z_0) + iv_y(z_0)) = -iu_y(z_0) + v_y(z_0). \end{aligned}$$

So

$$f_x(z_0) = \frac{1}{i} f_y(z_0).$$

Equating real and imaginary parts

$$u_x(z_0) = v_y(z_0),\ v_x(z_0) = -u_y(z_0)$$

or $u_y(z_0) = -v_x(z_0)$. ■

3.2 Proposition. *Satisfying the Cauchy-Riemann equations does not imply differentiability.*

Example. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = f(x+iy) = \begin{cases} \frac{xy}{x^2+y^2}(x+iy), & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Note that $f = 0$ on both axes, so all partial derivatives are 0 and satisfy the Cauchy-Riemann Equations. But if $y = \alpha x$ for some $\alpha \in \mathbb{R}$,

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{x\to 0} \frac{f(x(1+i\alpha)) - 0}{x(1+i\alpha)} = \lim_{x\to 0} \frac{x\alpha x}{x^2 + i^2\alpha^2} = \frac{\alpha}{1+\alpha^2},$$

which is non-constant. Hence this f is not differentiable even though it satisfies the Cauchy-Riemann Equations.

3.3 Proposition. *If the first order partial derivatives of $f = u + iv$ exist on a neighborhood of a point $z_0 = x_0 + iy_0$ are continuous at z_0 and satisfy the Cauchy-Riemann Equations at z_0, then f is differentiable at z_0.*

Proof. We want to show that

$$\frac{f(z_0+h)-f(z_0)}{h} \to f_x(z_0) = u_x(z_0) + iv_x(z_0)$$

as $h \to 0$. We will use the Mean Value Theorem. Let $h = \xi + i\eta$. Then

$$\frac{u(z_0+h)-u(z_0)}{h} = \frac{u(x_0+\xi, y_0+\eta) - u(x_0,y_0)}{\xi + i\eta},$$

which then equals

$$\frac{u(x_0+\xi, y_0+\eta) - u(x_0+\xi) + u(x_0+\xi) - u(x_0,y_0)}{\xi + i\eta}.$$

By the Mean Value Theorem, the latter equals

$$\frac{\eta}{\xi+i\eta} u_y(x_0+\xi, y_0+\theta_1\eta) + \frac{\xi}{\xi+i\eta} u_x(x_0+\theta_2\xi, y_0),$$

where $0 < \theta_1, \theta_2 < 1$. Similarly

$$\frac{v(z_0+h)-v(z_0)}{h} = \frac{\eta}{\xi+i\eta} v_y(x_0+\xi, y_0+\theta_3\eta) + \frac{\xi}{\xi+i\eta} v_x(x_0+\theta_4\xi, y_0)$$

for some $0 < \theta_3, \theta_4 < 1$. For convenience, set

$$z_1 = x_0 + \xi + i(y_0 + \theta_1\eta),\ z_2 = x_0 + \theta_2\xi + iy_0,\ z_3 = \cdots,\ z_4 = \cdots.$$

Then

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\eta}{\xi+i\eta}[u_y(z_1) + iv_y(z_3)] + \frac{\xi}{\xi+i\eta}[u_x(z_2) + iv_x(z_4)] \quad (1)$$

By the Cauchy-Riemann equations at z_0,

$$f_y(z_0) = if_x(z_0).$$

Then

$$f_x(z_0) = \frac{\xi+i\eta}{\xi+i\eta} f_x(z_0) = \frac{\eta}{\xi+i\eta} f_y(z_0) + \frac{\xi}{\xi+i\eta} f_x(z_0).$$

Subtract this from (**??**) to get

$$\frac{f(z_0+h)-f(z_0)}{h} - f_x(z_0) = \cdots$$

$$\cdots = \frac{\eta}{\xi+i\eta}[(u_y(z_1) - u_y(z_0)) + i(v_y(z_3) - v_y(z_0))] + \cdots$$

$$\cdots + \frac{\xi}{\xi+i\eta}[(u_x(z_2) - u_x(z_0)) + i(v_x(z_4) - v_x(z_0))].$$

Since

$$0 \leqslant \left|\frac{\eta}{\xi+i\eta}\right|, \left|\frac{\xi}{\xi+i\eta}\right| \leqslant 1$$

and $z_1, z_2, z_3, z_4 \to z_0$ as $h \to 0$ and the partial derivatives u_x, u_y, v_x, v_y are continuous at z_0, this right hand side tends to zero as $h \to 0$, which gives us what we want. ■

Suppose f is differentiable on $U \subset \mathbb{C}$ open. Suppose also that f is C^2 on U; that is, partial derivatives up to and including second-order exist and are continuous on U. Then

$$u_x = v_y,\ u_y = -v_x \text{ on } U.$$

Also,

$$u_{xx}(u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}.$$

So, $u_{xx} + u_{yy} = 0$ on U. Similarly, $v_{xx} + v_{yy} = 0$. Such functions are called **harmonic**. (Note: we need C^2 for the step in which $v_{xy} = v_{yx}$.)

4 Power Series

4.1 Definition. A power series about $z_0 \in \mathbb{C}$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There is a different power series for each different value of z. We say that the power series at some particular value of z converges if the corresponding sequence of partial sums converges. The power series **converges absolutely** at z if

$$\sum_{n=0}^{\infty} |a_n (z - z_0)^n| = \sum_{n=0}^{\infty} |a_n|\,|z - z_0|^n$$

converges. Absolute convergence at z implies ordinary convergence at z. The converse is not true; consider

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

The series converges at -1 but not absolutely convergent at 1.

For a sequence of real numbers $\{b_n\}$, we define the limit superior and limit inferior, respectively, as

$$\varlimsup_{n\to\infty} b_n = \lim_{n\to\infty}\left(\sup_{m\geqslant n} b_m\right)$$

and

$$\varliminf_{n\to\infty} b_n = \lim_{n\to\infty}\left(\inf_{m\geqslant n} b_m\right).$$

The following are important facts about limit superior and limit inferior.

1. If $\varlimsup b_n = L$, then for all $N \in \mathbb{N}$ and all $\varepsilon > 0$, there exists $n \geqslant N$ such that $b_n > L - \varepsilon$.
2. If $\varlimsup b_n = L$, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $b_n < L + \varepsilon$ for all $n \geqslant N$.

3. $\overline{\lim}\, kb_n = k\,\overline{\lim}\, b_n$ for $k \geqslant 0$.

4. $\overline{\lim}\, b_n \geqslant \underline{\lim}\, b_n$.

5. $\overline{\lim}\, b_n = \underline{\lim}\, b_n$ if and only if $\lim b_n$ exists.

From now on, for the sake of simplicity, we shall work (mostly) with power series about zero. The proofs for series about other points are the same.

For a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

let

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}}\, |a_n|^{1/n}}.$$

R is called the radius of convergence for the power series.

4.2 Theorem (Radius of Convergence Theorem - Abel)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence R.

1. *If $R = 0$, the series converges only at $z = 0$.*

2. *If $R = \infty$, the series converges absolutely for all $z \in \mathbb{C}$.*

3. *If $0 < R < \infty$, the series converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

Proof. Case 1: Let $R = 0$. Then $\overline{\lim}\, |a_n|^{1/n} = \infty$. For all $z \neq 0$,

$$|a_n|^{1/n} \geqslant \frac{1}{|z|} \text{ for infinitely many } n.$$

Hence $|a_n z^n| \geqslant 1$ for infinitely many n. Then the series diverges by the divergence test.

Case 2: Assume $R = \infty$. Then

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n} = 0.$$

Hence

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n}|z| = 0 \text{ for any fixed } z \in \mathbb{C}.$$

Hence for all $z \in \mathbb{C}$, there exists $N \in \mathbb{N}$ (possibly depending on z) such that

$$\left|a_n^{1/n} z\right| < \frac{1}{2} \text{ for all } n \geqslant N.$$

That implies that

$$|a_n z^n| < \frac{1}{2^n} \text{ for all } n \geqslant N.$$

If we use the comparison test to compare this with the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

we get absolute convergence.

Case 3: Assume $0 < R < \infty$. Suppose first that $|z| < R$ and let $\delta > 0$ be such that

$$|z| = R(1 - 2\delta).$$

Then

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} |z| = |z| \cdot \frac{1}{R} = (1 - 2\delta).$$

Hence by Fact 2, there exist $N \in \mathbb{N}$ such that

$$|a_n|^{1/n}|z| < (1 - \delta) \text{ for all } n \geqslant N.$$

Now compare with the convergent geometric series

$$\sum_{n=0}^{\infty} (1 - \delta)^n$$

to get the desired result. Now suppose $|z| > R$. Then

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} |z| > \frac{R}{R}.$$

Hence for infinitely many n, $|a_n|^{1/n}|z| > 1$ and $|a_n z^n| > 1$. Hence the series diverges by the divergence test. ■

4.3 Example.

$$i) \sum_{n=0}^{\infty} z^n \qquad ii) \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad iii) \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

All three series have radius of convergence 1, so all three converge for $|z| < 1$ and diverge for $|z| > 1$. On the unit circle $C(0, 1)$,

i) diverges everywhere;

ii) diverges at 1, but converges everywhere else on $C(0, 1)$;

iii) converges everywhere.

$$iv) \sum_{n=0}^{\infty} n! z^n \text{ has radius of convergence } 0.$$

$$v) \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ has radius of convergence } \infty.$$

4.4 Lemma (M-Test)**.** *Let $U \subset \mathbb{C}$ be open and suppose for each $n \geqslant 1$, f_n is continuous in U. If $|f_n| \leqslant M_n$ on U and*

$$\sum_{n=1}^{\infty} M_n$$

converges, then

$$\sum_{n=1}^{\infty} f_n$$

is convergent to a function f which is continuous in U.

Proof. Let

$$f_n = \sum_{m=1}^{n} f_m \qquad \text{and} \qquad S_n = \sum_{m=1}^{n} M_m.$$

If $m < n$,

$$|S_n - S_m| \leqslant \left| \sum_{i=m+1}^{n} f_i \right| \leqslant \sum_{i=m+1}^{n} |f_i| \leqslant \sum_{i=m+1}^{n} M_i \text{ on } U.$$

Since $\sum M_n$ is convergent, S_n is convergent. So S_n is Cauchy. Hence f_n is uniformly Cauchy in U, and so $f_n(z)$ converges for every $z \in U$ to a limit function $f(z)$. $f(z)$ is a uniform limit of continuous functions, so it is continuous itself. ∎

4.5 Corollary. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence $R > 0$. Then f is continuous in $D(0, R)$.

Proof. In any smaller disk $D(0, R-\delta)$, we see that the convergence of the power series was dominated by the convergence of the geometric series. By the M-test, the power series gives a function which is continuous in $D(0, R-\delta)$. Letting $\delta > 0$ gives the result. ∎

Differentiation and Uniqueness of Power Series

Suppose

$$\sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Formally, we can differentiate this series term by term to get

$$\sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Can we do this?

We start by observing that for this series has the same radius of convergence. Recall that
$$\lim_{n\to\infty} n^{1/(n-1)} = 1$$
by L'Hôpital's Rule. Then
$$\overline{\lim_{n\to\infty}}\, |na_n|^{1/(n-1)} = \overline{\lim_{n\to\infty}}\, n^{1/(n-1)}\, |a_n|^{1/(n-1)} = \overline{\lim_{n\to\infty}}\, |a_n|^{1/(n-1)}$$
from above. $\overline{\lim}\, |a_n|^{1/(n-1)}$ is the radius of convergence for the series
$$\sum_{n=1}^{\infty} a_n z^{n-1}.$$
Suppose this series has radius of convergence R'. Since
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z \sum_{n=1}^{\infty} a_n z^{n-1},$$
$R' \leqslant R$. Since
$$\sum_{n=1}^{\infty} a_n z^{n-1} = -\frac{a_0}{z} + \frac{1}{z}\sum_{n=0}^{\infty} a_n z_n,\ z \neq 0,\ z \in D(0,R),$$
$R \leqslant R'$. Hence $R = R'$ and we're done. That is,
$$\sum_{n=1}^{\infty} n a_n z^{n-1}$$
also has radius of convergence.

4.6 Theorem. *Suppose*
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
has radius of convergence $R > 0$*. Then* f *is differentiable on* $D(0,R)$ *and*
$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \text{ on } D(0,R).$$

Proof. Without loss of generality, suppose $0 < R < \infty$. (If $R = \infty$, take a suitably large disk.) Let $|z| = R - 2\delta$ for some $\delta > 0$ and let $|h| < \delta$. We want to show that
$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

or equivalently that

$$\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} \to 0 \text{ as } h \to 0.$$

Observe

$$\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} = \frac{\sum_{n=0}^{\infty} a_n (z_n+h)^n - \sum_{n=0}^{\infty} a_n z^n}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Subtract term by term to obtain that the latter equals

$$\frac{\sum_{n=0}^{\infty} a_n((z_n+h)^n - z_n)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

By the Binomial theorem, the above equals

$$\begin{aligned}
&= \frac{\sum_{n=0}^{\infty}(a_n \sum_{k=0}^{\infty} \binom{n}{k} h^k z^{n-k} - z^n)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} \\
&= \frac{\sum_{n=0}^{\infty} a_n (\sum_{k=1}^{\infty} \binom{n}{k} h^k z^{n-k})}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{\infty} \binom{n}{k} h^{k-1} z^{n-k} \right) - \sum_{n=1}^{\infty} n a_n z^{n-1} \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{\infty} \binom{n}{k} h^{k-1} z^{n-k} \right) \\
&= \sum_{n=0}^{\infty} a_n b_n,
\end{aligned}$$

where

$$b_n = \sum_{k=2}^{\infty} \binom{n}{k} h^{k-1} z^{n-k}.$$

If $z = 0$, $b_n = h^{n-1}$ and

$$\frac{f(z+h)-f(z)}{h} = \frac{f(h)-f(z)}{h} = \sum_{n=1}^{\infty} a_n h^{n-1} \to 0 \text{ as } h \to 0$$

by continuity of power series at zero since $\sum a_n z^{n-1}$ has radius of convergence

$R > 0$. If $z \neq 0$, first note that

$$\begin{aligned}
\binom{n}{k} &= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots(2)(1)} \\
&\leqslant \frac{n(n-1)\cdots(n-k+3)n\cdot n}{(k-2)(k-3)\cdots(2)(1)} \\
&= n^2\binom{n}{k-2}.
\end{aligned}$$

Hence for $z \neq 0$,

$$\begin{aligned}
b_n &\leqslant \frac{n^2|h|}{|z^2|}\sum_{k=2}^{n}\binom{n}{k-2}|h|^{k-2}|z|^{n-(k-2)} \\
&= \frac{n^2|h|}{|z^2|}\sum_{j=0}^{n-2}\binom{n}{j}|h|^j|z|^{n-j} \\
&\leqslant \frac{n^2|h|}{|z^2|}\sum_{j=0}^{n}\binom{n}{j}|h|^j|z|^{n-j} \\
&= \frac{n^2|h|}{|z|^2}\left(|h|+|z|\right)^2 \leqslant \frac{n^2|h|}{|z|^2}(R-\delta)^n.
\end{aligned}$$

Hence

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1}\right| \leqslant \frac{|h|}{|z|^2}\sum_{n=1}^{\infty} n^2|a_n|(R-\delta)^n.$$

The series on the right also has radius of convergence R, so the series on the right hand side is bounded by some constant A. Hence

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1}\right| \leqslant A\frac{|h|}{|z|^2}.$$

Let $h \to 0$ and we get our result. (Note that A does not depend on h and z.) ■

4.7 Corollary. *Power series are infinitely differentiable inside their radii of convergence.*

4.8 Corollary. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

have radius of convergence $R > 0$. *Then*

$$a_n = \frac{f^{(n)}(0)}{n!}, \qquad n \geqslant 0.$$

4.9 Corollary. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

has radius of convergence $R > 0$, then

$$F(z) = \sum_{n=0}^{\infty} \frac{a}{n+1} a_n z^{n+1}$$

also has radius of convergence R and $F' = f$ on the disk of convergence.

4.10 Definition. If a function f has a power series representation on some disk $D(z_0, r)$ of radius r about z_0; i.e. there exists a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ on } D(z_0, r),$$

then we say that f is (complex) **analytic** at z_0. If $U \subset \mathbb{C}$ is open and $f : U \to \mathbb{C}$ has a power series representation about every point of U, we say f is **analytic** on U.

To clarify

$$\text{analytic} \implies \text{holomorphic} \implies \text{differentiable}.$$

In fact, they are equivalent, as we will show later.

4.11 Corollary. *We define the exponential function e^z by*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Since e^z has infinite radius of convergence, e^z is infinitely differentiable on all of $\mathbb{C}$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}[e^z] = e^z.$$

Note that if $z = x \in \mathbb{R}$, $e^z = e^x$. Also

$$e^0 = 1 \text{ and } e^{z+w} = e^z e^w \text{ for all } z, w \in \mathbb{C}.$$

We can also define

$$\sin z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Both series have infinite radius of convergence. It is easy to see that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$e^{iz} = \cos z + i \sin z.$$

If $z = \theta$ and $\theta \in \mathbb{R}$, then $\sin z = \sin\theta$ and $\cos z = \cos\theta$, and

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

For $z = x + iy$, $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$. So

$$|e^z| = e^x = e^{\operatorname{Re} z}.$$

Try to solve $e^w = z = re^{i\theta}$ for w. Let $w = x + iy$. Then

$$e^w = e^{x+iy} = e^x e^{iy} = re^{i\theta}.$$

So comparing moduli, $e^z = r$. So, there is no solution at all if $z = 0$. Otherwise

$$x = \log r = \log |z|.$$

Also,

$$\cos y + i \sin y = e^{iy} = e^{i\theta} = \cos\theta = i \sin\theta.$$

So y is only defined up to an integer multiple of 2π. Hence, we can say that

$$y = \operatorname{Arg} z.$$

So for $z \neq 0$,

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$$

is only defined up to an integer multiple of 2π.

4.12 Definition. A subset Ω of $\mathbb{C}$ which is open and connected is called a **domain** or a **region**.

4.13 Definition. Let Ω be a domain and f be holomorphic in Ω with $f(z) \neq 0$ for all $z \in \Omega$; that is, f is non-vanishing. A holomorphic function g on Ω is called a branch of the logarithm of f in Ω if

$$e^{g(z)} = f(z) \text{ for all } z \in \Omega.$$

4.14 Example. Let $\Omega = \mathbb{C} \setminus \mathbb{R}_-$; i.e.

$$\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0,\ \operatorname{Im} z = 0\}.$$

On Ω, define a branch of the logarithm of z, $\log z$, by

$$\log z := \log r + i\theta,\ z = re^{i\theta}$$

where $-\pi \leqslant \theta \leqslant \pi$. This is usually referred to as the **principal branch of the logarithm**.

Note that if $z = x \in \mathbb{R}$, $x > 0$, then $\log z = \log x$. Also, if $z_1, z_2 \in \Omega$ and $z_1 z_2 \in \Omega$, then

$$\log(z_1 z_2) = \log z_1 + \log z_2.$$

4.15 Proposition (Uniqueness of Power Series)**.** *Suppose*

$$\sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$ and is zero at points of a nonzero sequence $\{z_n\}$ which tends to infinity. Then the power series is zero on the whole disk of convergence and $a_n = 0$ for all $n \geqslant 0$.

Proof. Set

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By 4.6, continuity of power series,

$$a_0 = f_0(0) = \lim_{z \to 0} f_0(z) = \lim_{n \to \infty} f_0(z_n) = 0.$$

Now set

$$f_1(z) = \sum_{n=1}^{\infty} a_n z^{n-1} = \frac{f_1(z)}{z}, \quad \text{if } z \neq 0.$$

This series has the same radius of convergence and

$$a_1 = f_1(0) = \lim_{z \to 0} f_1(z) = \lim_{n \to \infty} \frac{f_1(z_n)}{z_n} = 0.$$

Next set

$$f_2 = \sum_{n=2}^{\infty} a_n z^{n-2} = \frac{f_2(z)}{z^2}, \ z \neq 0.$$

We find that $a_2 = 0$, and so on. ∎

4.16 Corollary. *If a power series is zero at all points of an infinite set with an accumulation point at zero, then the power series is identically zero.*

4.17 Corollary. *If*

$$\sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n z^n$$

each have radius of convergence greater than zero and agree on an infinite set with an accumulation point at zero, then $a_n = b_n$ for all $n \geqslant 0$.

Proof. Subtract term by term within a common disk of convergence. ∎

Of course, one can make similar conclusions for power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

about any point $z_0 \in \mathbb{C}$.

4.18 Abel's Limit Theorem (Non-Tangential Convergence)**.** *If*

$$\sum_{n=0}^{\infty} a_n$$

is convergent and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$f(z) \to f(1) = \sum_{n=0}^{\infty} a_n \text{ as } z \to 1$$

in such a way that

$$\frac{|1-z|}{1-|z|}$$

remains bounded. [Note that $|1-z| \geqslant 1-|z|$ by the reverse triangle inequality.]

Proof. Without loss of generality, assume

$$\sum_{n=0}^{\infty} a_n = 0.$$

Let S_n be the partial sum

$$S_n = a_0 + \cdots + a_n$$

and

$$S_n(z) = a_0 + \cdots + a_n z^n.$$

Then

$$\begin{aligned} S_n(z) &= a_0 + (S_1 - S - 0)z + \cdots + (S_n - S_{n-1})z^n \\ &= S_0(1-z) + S_1(z - z^2) + \cdots + S_{n-1}(z^{n-1} - z^n) + S_n z^n. \end{aligned}$$

But $S_n z^n \to 0$ as $n \to \infty$ by the convergence of $\sum a_n$, and formally we have

$$f(z) = (1-z) \sum_{n=0}^{\infty} S_n z^n. \tag{2}$$

Well

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$$

is convergent, and so since $\{S_n\}$ is in particular bounded, the series

$$\sum_{n=0}^{\infty} S_n z^n$$

has radius of convergence greater than 1 by Abel's Theorem. Hence the series (**??**) does converge for $|z| < 1$ and multiplying by $(1-z)$ gives $f(z)$. Now we assume that $|1-z| \leqslant K(1-|z|)$ for some suitable K. Since

$$\sum_{n=0}^{\infty} a_n = 0,$$

$S_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and choose M such that $|S_n| < \varepsilon$ for all $n \geqslant M$. Then

$$\sum_{n=0}^{\infty} S_n z^n$$

is dominated by

$$\varepsilon \sum_{n=0}^{\infty} z^n,$$

and so

$$|f(z)| = \left|(1-z)\sum_{n=0}^{\infty} S_n z^n\right|,$$

by $(*)$

$$\begin{aligned}
&\leqslant |1-z|\left|\sum_{n=0}^{n-1} S_n z^n\right| + |1-z|\left|\sum_{n=m}^{\infty} S_n z^n\right| \\
&\leqslant |1-z|\left|\sum_{n=0}^{n-1} S_n z^n\right| + K(1-|z|)\sum_{n=m}^{\infty} \varepsilon |z|^n \\
&\leqslant |1-z|\left|\sum_{n=0}^{n-1} S_n z^n\right| + K(1-|z|)\frac{\varepsilon |z|^n}{1-|z|} \\
&= |1-z|\left|\sum_{n=0}^{n-1} S_n z^n\right| + K\varepsilon |z|^n \\
&\leqslant |1-z|\left|\sum_{n=0}^{n-1} S_n z^n\right| + K\varepsilon \\
&\leqslant |1-z|\,.
\end{aligned}$$

■

5 Line Integrals

5.1 Definition. Let $f(t) = u(t) + iv(t)$, $a \leqslant t \leqslant b$, be a continuous complex valued function of the real variable t defined on $[a, b]$. Define

$$\int_a^b f(t)\,\mathrm{d}t,$$

the integral of f over $[a, b]$ to be

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

5.2 Definition. Let $z(t) = a(t) + ib(t)$ be a complex function on $a \leqslant t \leqslant b$.

a) The curve γ determined by $z(t)$ is piecewise differentiable and we set

$$z'(t) = a'(t) + ib'(t).$$

If x, y are continuous on $[a, b]$ and C^1 on each subinterval $[a, t_1], [t_1, t_2], \ldots, [t_{n-1}, b]$ of a partition of $[a, b]$ (i.e. C^1 on the interior of each subinterval and the derivatives have one-sided limits at the endpoints).

b) The curve γ is also piecewise smooth if $z'(t) \neq 0$ except at (possibly) finitely many points in $[a, b]$.

5.3 Definition. Let γ be a piecewise smooth curve given by $z(t)$, $a \leqslant t \leqslant b$, and let f be complex valued continuous at each point $z(t)$, $a \leqslant t \leqslant b$ (this is called the track of γ, written $[\gamma]$). We define the integral of f along γ by

$$\int_\gamma f(t)\,\mathrm{d}t := \int_a^b f(z) \cdot z'(t)\,\mathrm{d}t.$$

That is,

$$\int_\gamma f(t)\,\mathrm{d}t := \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(z) \cdot z'(t)\,\mathrm{d}t$$

where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ is a partition as in Definition 5.2.

5.4 Definition. The two curves $\gamma_1 : z(t)$, $a \leqslant t \leqslant b$, and $\gamma_2 : w(t)$, $c \leqslant t \leqslant d$, are **smoothly equivalent** and write $\gamma_1 \sim \gamma_2$ if there exists a one-to-one mapping $\lambda(t) : [c, d] \to [a, b]$ such that $\lambda(c) = a$ and $\lambda(d) = b$, $\lambda'(t) > 0$, $c < t < d$ and $w(t) = z(\lambda(t))$. It is easy to show that $\sim$ is an equivalence relation.

5.5 Proposition. *Suppose $\gamma_1 \sim \gamma_2$ both piecewise differentiable and f is continuous at all points in $[\gamma_1] = [\gamma_2]$. Then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. Let $f(z) = u(z) + iv(z)$, $\gamma_1 : z(t)$, $\gamma_2 : z(\lambda(t))$ on $a \leqslant t \leqslant b$. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_a^b u(z(t)) \cdot x'(t)\,\mathrm{d}t - \int_a^b v(z(t)) \cdot y'(t)\,\mathrm{d}t + \cdots$$
$$\cdots + i\int_a^b u(z(t)) \cdot y'(t)\,\mathrm{d}t + i\int_a^b v(z(t)) \cdot x'(t)\,\mathrm{d}t.$$

Also,

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \int_c^d [u(z(\lambda(t))) + iv(z(\lambda(t)))] \cdot [x'(\lambda(t)) + iy'(\lambda(t))] \cdot \lambda'(t)\,\mathrm{d}t.$$

We can also multiply this out to again get four terms. Now compare each of them term by term with the corresponding one from the first integral over γ_1. For example,

$$\int_c^d u(z(\lambda(t))) \cdot x'(\lambda(t)) \cdot \lambda'(t)\,\mathrm{d}t = \int_a^b u(z(s)) \cdot x'(s)\,\mathrm{d}s$$

by substitution where $s = \lambda(t)$, $\mathrm{d}s = \lambda'(t)\,\mathrm{d}t$. Similarly we show the other three terms and equal. ∎

5.6 Definition. Given a curve γ defined by $z(t)$, $a \leqslant t \leqslant b$, we call $-\gamma$ the curve defined by $z(b + a - t)$, $a \leqslant t \leqslant b$.

5.7 Proposition. *Let $\gamma : z(t)$, $a \leqslant t \leqslant b$ be piecewise differentiable and f continuous on $[\gamma]$. Then*

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

Proof. Writing out the integral

$$\int_{-\gamma} f(z)\,\mathrm{d}z = \int_a^b f(z(b + a - t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(z(b + a - t)) \cdot \mathrm{d}t.$$

Let $u = b + a - t$, $\mathrm{d}u = -\mathrm{d}t$. Then the latter equals

$$\begin{aligned}
\int_a^b f(z(u)) \cdot \frac{\mathrm{d}}{\mathrm{d}u}(z(u)) \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \cdot \mathrm{d}u &= \int_a^b f(z(u)) \cdot z'(u)\,\mathrm{d}u \\
&= -\int_a^b f(z(u)) \cdot z'(u)\,\mathrm{d}u \\
&= \int_\gamma f(z)\,\mathrm{d}z.
\end{aligned}$$

∎

5.8 Example.

1. Let $f(z) = x^2 + iy^2$ for $z = x + iy$. Define $\gamma : z(t) = t + it$, $0 \leqslant t \leqslant 1$. Then

$$\int_\gamma f(z)\ \mathrm{d}z = \int_0^1 (t^2 + it^2)(1+i)\ \mathrm{d}t = (1+i)^2 \int_0^1 t^2\ \mathrm{d}t = 2i \int_0^1 t^2\ \mathrm{d}t = \frac{2i}{3}.$$

2. Consider

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}.$$

Define $\gamma : z(t) = R\cos(t) + iR\sin(t)$ for $0 \leqslant t \leqslant 2\pi$ and $R \neq 0$. Then

$$\begin{aligned}\int_\gamma f(z)\ \mathrm{d}z &= \int_0^{2\pi} \left(\frac{\cos t}{R} - \frac{i\sin t}{R}\right)(-R\sin t + iR\cos t)\ \mathrm{d}t \\ &= \int_0^{2\pi} i\ \mathrm{d}t = 2\pi i.\end{aligned}$$

3. Let $f(z) = 1$ and γ be piecewise differentiable. Then

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b z'(t)\ \mathrm{d}t = z(b) - z(a).$$

5.9 Proposition. *Let γ be piecewise differentiable, f, g be continuous on $[\gamma]$ and let $\alpha \in \mathbb{C}$ be any constant, then*

1. $$\int_\gamma (f(z) \pm g(z))\ \mathrm{d}z = \int_\gamma f(z)\ \mathrm{d}z \pm \int_\gamma g(z)\ \mathrm{d}z.$$

2. $$\int_\gamma \alpha f(z)\ \mathrm{d}z = \alpha \int_\gamma f(z)\ \mathrm{d}z.$$

5.10 Lemma. *Let $G(t)$ be a continuous complex valued function on $[a, b]$. Then*

$$\left|\int_a^b G(t)\ \mathrm{d}t\right| \leqslant \int_a^b |G(t)|\ \mathrm{d}t.$$

Proof. Suppose

$$\int_a^b G(t)\ \mathrm{d}t = Re^{i\theta}, \qquad R \geqslant 0.$$

From 5.9,

$$\int_a^b e^{-i\theta} G(t)\ \mathrm{d}t = R.$$

Write $e^{-i\theta}G(t) = A(t) + iB(t)$. Then

$$\begin{aligned} R = \left|\int_a^b A(t)\ \mathrm{d}t\right| &= \left|\int_a^b \mathrm{Re}(e^{-\theta}G(t))\ \mathrm{d}t\right| \\ &\leqslant \int_a^b \left|\mathrm{Re}(e^{-i\theta}G(t))\right|\ \mathrm{d}t \\ &\leqslant \int_a^b \left|e^{-i\theta}G(t)\right|\ \mathrm{d}t \\ &= \int_a^b |G(t)|\ \mathrm{d}t, \end{aligned}$$

as $|\mathrm{Re}\, z| \leqslant |z|$ and $|e^{-i\theta}| = 1$. ∎

5.11 Definition. For a piecewise differentiable curve γ, we define the length of γ, $\ell(\gamma)$ by

$$\ell(\gamma) = \int_a^b |z'(t)|\ \mathrm{d}t.$$

5.12 Proposition (M-L Formula)**.** *Suppose γ is piecewise differentiable of length L and f is continuous on $[\gamma]$ with $|f| \leqslant M$ on $[\gamma]$. Then*

$$\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leqslant ML.$$

Proof. Let $\gamma : z(t) = x(t) + iy(t)$ on $a \leqslant t \leqslant b$. Then

$$\begin{aligned} \left|\int_\gamma f(z)\ \mathrm{d}z\right| &= \left|\int_a^b f(z(t)) \cdot z'(t)\ \mathrm{d}t\right| \\ &\leqslant \int_a^b |f(z(t))| \cdot |z'(t)|\ \mathrm{d}t \\ &\leqslant \int_a^b M \cdot |z'(t)|\ \mathrm{d}t \\ &= M \int_a^b |z'(t)|\ \mathrm{d}t = ML. \end{aligned}$$

∎

5.13 Example. Let $\gamma = C(0,1)$ oriented counter clockwise. Let $|f| \leqslant 1$ on $C(0,1)$. Then

$$\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leqslant 2\pi.$$

5.14 Proposition. *Let γ be piecewise differentiable and let f_n be a sequence of continuous functions on $[\gamma]$ which converges uniformly to a function on $[\gamma]$. Then*

$$\int_\gamma f(z)\ \mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n(z)\ \mathrm{d}z.$$

Proof. Observe

$$\left|\int_\gamma f(z)\,\mathrm{d}z - \int_\gamma f_n(z)\,\mathrm{d}z\right| = \left|\int_\gamma (f(z) - f_n(z))\,\mathrm{d}z\right| \leqslant \int_\gamma |f(z) - f_n(z)|\,\mathrm{d}z.$$

Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be such that

$$|f(x) - f_n(x)| \leqslant \varepsilon \text{ on } [\gamma] \text{ for all } n \geqslant n_0.$$

Then by 5.12,

$$\left|\int_\gamma f(z)\,\mathrm{d}z - \int_\gamma f_n(z)\,\mathrm{d}z\right| \leqslant \varepsilon\ell(\gamma).$$

Since $\varepsilon > 0$ was arbitrary, the result follows. ■

5.15 Proposition. *Let $U \subset \mathbb{C}$ be open. Let f be differentiable on U and γ be a piecewise smooth curve lying inside U. Then if f has a primitive F on U,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(z(a)) - F(z(b)) \text{ where } \gamma : z(t),\ a \leqslant t \leqslant b.$$

Proof. Suppose without loss of generality that γ is smooth. Let $H(t) := F(z(t))$. Then

$$H'(t) = \lim_{h\to 0} \frac{F(z(t+h)) - F(z(t))}{h}.$$

Differentiate complex-valued fuctions of a real variable by differentiating real and imaginary parts, respectively, in the sense of single-variable calculus.

$$= \lim_{h\to 0} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}$$

Since $z'(t) \neq 0$ since z is smooth by the mean value theorem of calculus, we can find $\delta > 0$ such that if $|h| < \delta$ and $z(t) = u(t) + iv(t)$, then

$$\begin{aligned} z(t+h) &= u(t+h) - u(t) + iv(t+h) - iv(t) \\ &= hu'(t+\theta_1 h) + ihv'(t+\theta_2 h) \end{aligned}$$

for some $0 < \theta_1, \theta_2 < 1$. Since at least one of u', v' is nonzero on a δ-neighborhood of t (since γ is C^1), it follows from above that for $|h| < \delta$,

$$z(t+h) \neq z(t).$$

Thus, by letting $h \to 0$, we see that

$$\begin{aligned} H'(t) &= F'(z(t)) \cdot z'(t) \\ &= f(z(t)) \cdot z'(t). \end{aligned}$$

Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t.$$

By the Fundamental Theorem of Calculus, since $H(t)$ is a primative for $f(z(t)) \cdot z'(t)$, the above is simply

$$H(b) - H(a) = F(z(b)) - F(z(a))$$

as required. ■

6 Cauchy's Theorem for a Rectangle

Let us denote by R a filled and closed rectangle with sides parallel to the real and imaginary axes. By ∂R, we denote a closed curve which traces out the perimeter of R in a counter clockwise direction starting at the bottom right corner. n.b. This is a well-defined concept.

6.1 Theorem. *Let f be differentiable on an open set U which contains the rectangle R. Then*

$$\int\limits_{\partial R} f(z)\ \mathrm{d}z = 0.$$

Proof using the Bisection argument. For any rectangle $S \subset U$, define

$$\eta(S) = \int\limits_{\partial S} f(z)\ \mathrm{d}z.$$

If we divide R into four congruent rectangles $R_{(1)}, \ldots, R_{(4)}$, then

$$\eta(R) = \sum_{i=1}^{4} \eta(R_{(i)}.$$

By the triangle inequality, for at least one i,

$$\left|\eta(R_{(i)})\right| \geqslant \frac{1}{4}\left|\eta(R)\right|.$$

Otherwise,

$$\begin{aligned}
|\eta(R)| &= \left|\eta(R_{(1)}) + \cdots + \eta(R_{(4)})\right| \\
&\leqslant \left|\eta(R_{(1)})\right| + \cdots + \left|\eta(R_{(4)})\right| \\
&< \frac{1}{4}\left|\eta(R)\right| + \cdots + \frac{1}{4}\left|\eta(R)\right| \\
&= \left|\eta(R)\right|,
\end{aligned}$$

which is a contradiction.

Call this chosen rectangle R_1. Now repeat the argument to divide R_1 into four congruent rectangles to get R_2. Etc. Then we get a nested sequence of rectangles $R_1 \subset R_2 \subset \cdots$ such that

$$\left|\eta(R_n)\right| \geqslant \frac{1}{4}\left|\eta(R_{n-1})\right|.$$

Hence $|\eta(R_n)| \geqslant 4^{-n}\,|\eta(R)|$. Then since $\operatorname{diam} R_n = 2^{-n} \operatorname{diam} R$, and each R_n is closed,

$$\bigcap_n R_n = \{z^*\} \text{ for some } z^* \in \mathbb{C}.$$

> To see this, pick a point $z_n \in R_n$ for each n. Since the rectangles are nested and $\operatorname{diam} R_n = 2^{-n} \operatorname{diam} R$. The sequence $\{z_n\}$ is Cauchy. Hence $z_n \to z^*$. It is easy to see that z^* will not depend on our choice of $\{z_n\}$.

Now let $\delta > 0$ be such that $f(z)$ is differentiable on $D(z^*, \delta)$. Given $\varepsilon > 0$, by differentiability of f at z^*, we can make δ smaller if needed so that

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon$$

if $|z - z^*| < \delta$. Then

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon|z - z^*|$$

if $|z - z^*| < \delta$. Now by 5.15, for each n,

$$\int_{\partial R_n} \mathrm{d}z = 0 \qquad \text{and} \qquad \int_{\partial R_n} z\,\mathrm{d}z = 0.$$

Since if g, G are such that $G' = g$ on U, then

$$\int_{\partial S} g(z)\,\mathrm{d}z = 0 \text{ if } S \subset U \text{ is a closed path.}$$

Hence

$$\begin{aligned} \eta(R_n) &= \int_{\partial R_n} f(z)\,\mathrm{d}z \\ &= \int_{\partial R_n} (f(z) - f(z^*) - (z - z^*)f'(z^*))\,\mathrm{d}z. \end{aligned}$$

Hence by 5.10,

$$\begin{aligned} |\eta(R_n)| &\leqslant \int_{R_n} |z - z^*||\mathrm{d}z| \\ &\leqslant \varepsilon \operatorname{diam} R_n \cdot \int_{R_n} |\mathrm{d}z| \\ &= \varepsilon \cdot 2^{-n} \operatorname{diam} R \cdot \ell(\partial R_n) \\ &= \varepsilon \cdot 2^{-n} \operatorname{diam} R \cdot 2^{-n} \ell(\partial R). \end{aligned}$$

Hence
$$4^{-n}\,|\eta(R)| \leqslant |\eta(R_n)| \leqslant \varepsilon \cdot 4^{-n}\,\mathrm{diam}(R)\ell(\partial R),$$
and so
$$|\eta(R)| < \varepsilon \cdot \mathrm{diam}(R) \cdot \ell(\partial R).$$
Since $\varepsilon > 0$ was arbitrary, $\eta(R) = 0$ as required. ■

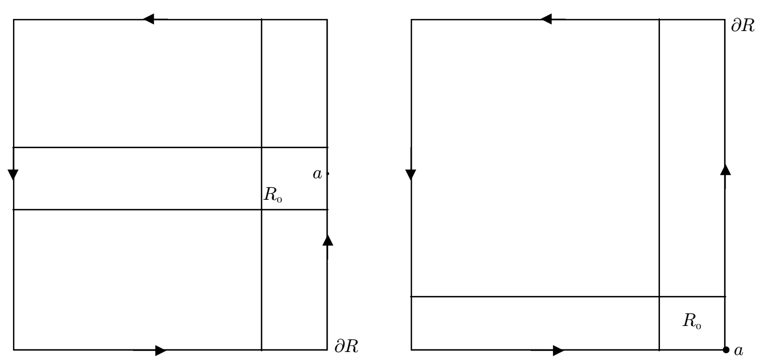
Note that we only used that f was differentiable. We didn't need that f was holomorphic or analytic. We need to generalize the result.

6.2 Theorem. *Let f be differentiable on an open set U containing a rectangle R except at possibly finitely many points where it is continuous. Then*
$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

Proof. Without loss of generality, we need only consider when there is just one point, a, where f may not be differentiable. There are three cases.

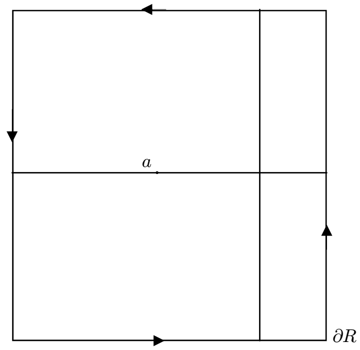
Case 1: Suppose $a \notin R$. Just make U smaller and apply the previous result.

Case 2: Suppose $a \in \partial R$. Divide up R as follows:



By the previous result and cancellation, if
$$\int_{\partial R_0} f(z)\,\mathrm{d}z = 0, \text{ then } \int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

Let $\varepsilon > 0$ and assume without loss of generality that $\ell(\partial R_0) < \varepsilon$. Now if f is continuous at a so there exists $\delta > 0$ such that we can find some M such that
$$|f(z)| \leqslant M \text{ if } |z - a| < \delta.$$
Then
$$\left| \int_{\partial R_0} f(z)\,\mathrm{d}z \right| \leqslant M\varepsilon \text{ by 5.12.}$$



Since $\varepsilon > 0$ was arbitrary, the result follows in this case.

Case 3: Suppose $a \in \text{Int}(R)$. Then the case reduces to Case 2, and the result follows. ∎

6.3 Corollary. *Let f be differentiable on $U \subset \mathbb{C}$ open, let $a \in U$, and define $g(z)$ on U by*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a. \end{cases}$$

Then

$$\int_{\partial R} g(z)\,\mathrm{d}z = 0 \text{ for all rectangles } R \subset U.$$

Primitives - (First Version)

6.4 Theorem. *Let f be continuous on an open disk $D = D(a, R)$ and suppose*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0$$

for every rectangle $R \subset D$. Then f has a primitive F on D.

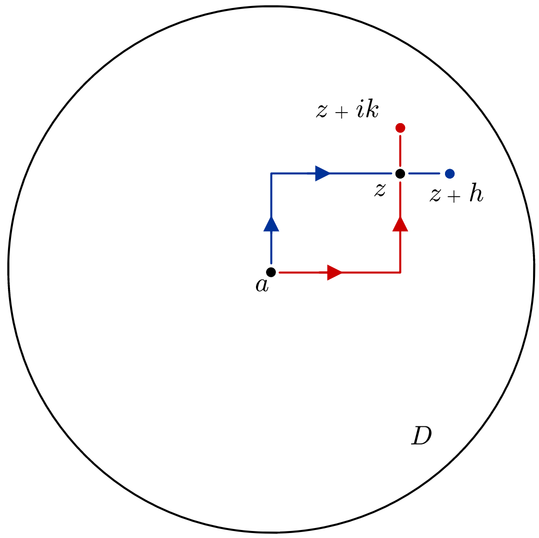
Proof. Define $F(z)$ on D by

$$F(z) = \int_\gamma f(z)\,\mathrm{d}z$$

where γ is any path from a to z in D containing one vertical and one horizontal line segment. Since

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0 \text{ for all rectangles } R \subset U,$$

$F(z)$ is well-defined. Let $f(z) = u(z) + iv(z)$ and $F(z) = U(z) + iV(z)$. Let h



be real. Then if $z_0 \in D$,

$$\begin{aligned} \frac{F(z_0+h)-F(z)}{h} - f(z_0) &= \frac{1}{h} \int\limits_{[z_0, z_0+h]} f(z)\,\mathrm{d}z - f(z_0) \\ &= \frac{1}{h} \int\limits_{[z_0, z_0+h]} (f(z) - f(z_0))\,\mathrm{d}z. \end{aligned}$$

By continuity of f at z_0, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

Hence if $|h| < \delta$,

$$\left| \frac{F(z_0+h)-F(z_0)}{h} - f(z_0) \right| \leqslant \frac{1}{h} \int\limits_{[z_0, z_0+h]} \varepsilon\,\mathrm{d}z = \varepsilon.$$

Since $h \in \mathbb{R}$, $z_0 \in D$ was arbitrary, the partial derivatives U_x and V_x exist on D and

$$U_x = u \qquad \text{and} \qquad V_x = v \text{ on } D.$$

Since f is continuous on D, so are U_x and V_x. Similarly if we look at

$$\frac{F(z_0+ik)-F(z_0)}{ik} - f(z_0)$$

for $k \in \mathbb{R}$, we see that U_y and V_y exist on D and

$$V_y = u \qquad \text{and} \qquad U_y = v \text{ on } D.$$

So, in particular, U_y and V_y are continuous on D. Hence the first-order partial derivatives of U and V exist on D, are continuous on D, and satisfy the Cauchy-Riemann equations on D. Hence by Proposition 3.3, F is differentiable on D and

$$F'(z) = F_x(z) \cdot f(z) \text{ on } D.$$

■

6.5 Corollary. *Let f be differentiable on an open disk $D = D(a, R)$. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \textit{ for all piecewise smooth closed paths } \gamma \subset D.$$

Proof. Let $\gamma : z(t)$, $a \leqslant t \leqslant b$ be a closed piecewise smooth path in D. By 6.1,

$$\int\limits_{\partial R} f(z)\,\mathrm{d}z = 0 \text{ for all rectangles } R \subset D,$$

and f is continuous on D since f is differentiable on D. Hence by 5.15, f has a primitive F on D. Then by 5.15,

$$\int_\gamma f(z)\,\mathrm{d}z = F(z(b)) - F(z(a)) = 0.$$

■

6.6 Corollary. *Let Ω be a domain and f be continuous and complex-valued on Ω. Then if*

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \textit{ for all piecewise smooth closed curves } \gamma \textit{ in } \Omega,$$

then f has a primitive F in Ω.

Proof. Pick $a \in \Omega$. For any $z \in \Omega$, define

$$F(z) = \int_\gamma f(z)\,\mathrm{d}z$$

where γ is any piecewise smooth path in Ω from a to z. By hypothesis, F is well-defined (doesn't depend on the choice of γ). If we now let $\delta > 0$ be such that $D(z, \delta) \subset \Omega$, then the same proof as for 6.4 shows that $F'(z) = f(z)$. Since $z \in \Omega$ was arbitrary, the result follows. ■

7 Cauchy's Integral Formula in a Disk, Taylor's Theorem, Applications

Notation. $C(a, R)$ denotes the circle centered at a with radius R with a counter clockwise orientation.

7.1 Lemma. *For any $a \in \mathbb{C}$, $R > 0$ and any $b \in D(a,R)$,*

$$\int\limits_{C(a,R)} \frac{\mathrm{d}z}{z-b} = 2\pi i.$$

Proof. Suppose $z \in C(a,R)$. Then

$$\frac{1}{z-b} = \frac{1}{z-b-(b-a)} = \frac{1}{z-a} \cdot \frac{1}{1-\left(\frac{b-a}{z-a}\right)}.$$

Since $|b-a| < R = |z-a|$,

$$\frac{|b-a|}{|z-a|} < 1$$

and we can expand this expression as a geometric series which converges uniformly for every $z \in C(a,R)$.

$$\begin{aligned} &= \frac{1}{z-a}\left(1+\left(\frac{b-a}{z-a}\right)+\left(\frac{b-a}{z-a}\right)^2+\cdots\right. \\ &= \frac{1}{z-a}+\frac{(b-a)}{(z-a)^2}+\frac{(b-a)^2}{(z-a)^3}+\cdots. \end{aligned}$$

By Proposition 5.14 on path integrals for uniform convergent sequences of functions,

$$\begin{aligned} \int\limits_{C(a,R)} \frac{\mathrm{d}z}{b-a} &= \int\limits_{C(a,R)} \frac{\mathrm{d}z}{z-a} + \int\limits_{C(a,R)} \frac{(b-a)}{(z-a)^2}\,\mathrm{d}z + \cdots \\ &= 2\pi i + 0 + 0 + \cdots = 2\pi i. \end{aligned}$$

■

7.2 Cauchy's Integral Formula in a Disk. *Let f be differentiable on an open set U containing the closed disk $D = \overline{D}(a,R)$ $(0 < R < \infty)$. Then if $b \in D(a,R)$ (open disk),*

$$\frac{1}{2\pi i} \int\limits_{C(a,R)} \frac{f(z)}{z-b}\,\mathrm{d}z = f(b).$$

Proof. By the compactness of $\overline{D}(a,R)$, we can find a bigger radius $R' > R$ such that

$$D(a,R) \subset \overline{D}(a,R) \subset D(a,R') = D' \subset U.$$

For $z \in U$, let

$$g(z) = \begin{cases} \frac{f(z)-f(b)}{z-b}, & z \neq b \\ f'(b), & z = b. \end{cases}$$

By 6.3,

$$\int_{\partial R} g(z)\,\mathrm{d}z = 0 \text{ for all rectangles } R \subset D'.$$

So by Theorem 6.4, g has a primitive G on D'. Hence by 5.15,

$$\int_\gamma g(z)\,\mathrm{d}z = 0 \text{ for all piecewise smooth closed curves } \gamma \subset D'.$$

In particular,

$$\int_{C(a,R)} g(z)\,\mathrm{d}z = 0.$$

Hence

$$\int_{C(a,R)} \frac{f(z)-f(b)}{z-b}\,\mathrm{d}z = 0,$$

which implies

$$\int_{C(a,R)} \frac{f(z)}{z-b}\,\mathrm{d}z = \int_{C(a,R)} \frac{f(z)}{z-b}\,\mathrm{d}z = 2\pi i f(b).$$

■

7.3 Taylor's Theorem. *Let f be differentiable on an open set U containing $D(a,R)$, $0 < R \leqslant \infty$. Then f has power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

This series converges on $D(a,R)$ and converges uniformly on $\overline{D}(a,\rho)$ for any $\rho \leqslant R$, and the series has radius of convergence greater or equal to R. Also for any $0 < \rho < R$,

$$a_n = \frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z, \qquad n \geqslant 0.$$

In particular, f is infinitely differentiable on $D(a,R)$.

Proof. Let $0 < \rho < R$ and let $z \in D(a,\rho)$. Use the same trick as for Lemma 7.1. By the Cauchy integral formula 7.2,

$$f(z) = \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Now

$$\frac{1}{w-z} = \frac{1}{w-a(z-a)} = \frac{1}{w-a}\cdot\frac{1}{1-\left(\frac{z-a}{w-a}\right)}$$
$$= \frac{1}{w-a}\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}},$$

and the series converges uniformly on $C(a,\rho)$. So,

$$f(z) = \frac{1}{2\pi i}\int\limits_{C(a,\rho)}\left\{\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right\} f(w)\,\mathrm{d}w.$$

By 5.14, we can write this as

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int\limits_{C(a,\rho)}\frac{(z-a)^n}{(w-a)^{n+1}} f(w)\,\mathrm{d}w$$
$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left\{\int\limits_{C(a,\rho)}\frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w\right\}(z-a)^n.$$

We can write the latter as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int\limits_{C(a,\rho)}\frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w, \qquad n \geqslant 0.$$

By reversing the argument, we see that this series converges for every $z \in D(a,\rho)$ (and in fact the convergence is uniform). By the radius of convergence theorem 4.3, the radius of convergence of $\sum a_n(z-a)^n$ is greater than or equal to ρ. Since $a_n = f^{(n)}(a)/n!$ or by the uniqueness of power series, we see that a_n does not depend on ρ. Since $\rho < R$ was arbitrary, the radius of convergence of

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

is greater or equal to R. Since power series converge uniformly on smaller disks than the disk of convergence,

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

converges uniformly on $D(a,\rho)$ for any $\rho < R$. ■

Corollaries to Taylor

7.4 Corollary. *Let f be differentiable on a neighborhood of $\overline{D}(a, R)$. Then f is analytic on $D(a, R)$ and in particular is infinitely differentiable on $D(a, R)$.*

Proof. Combine Taylor with infinitely differentiable inside the disk of convergence. ∎

This shows how much stronger complex differentiability is compared to Riemann differentiability.

7.5 Morera's Theorem. *Let U be an open set and let $f : U \to \mathbb{C}$ be a continuous complex-valued function such that*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0$$

for all rectangles $R \subset U$. Then f is analytic on U.

Proof. Suffices to show that if a is any point in U and $D(a, r) \subset U$, then f is analytic on $D(a, r)$. By 6.4, f has a primitive F on $D(a, r)$. By 7.4, F is analytic on $D(a, r)$ and infinitely differentiable. Since $F' = f$, f is infinitely differentiable on $D(a, r)$. Hence f is infinitely differentiable and hence analytic on $D(a, r)$ by 7.4. ∎

7.6 Corollary. *Let f be differentiable on an open set U, let $a \in U$, and let $g(z)$ be given by*

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a. \end{cases}$$

Then g is differentiable; in fact, g is analytic.

Proof. By 6.3,

$$\int_{\partial R} g(z)\,\mathrm{d}z = 0 \text{ for all rectangles } R \subset U.$$

Hence g is analytic by Morera's Theorem. ∎

7.7 Goursat's Theorem. *Let $U \subset \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be differentiable. Then f is analytic on U.*

Proof. Immediate from 7.4. ∎

7.8 Cauchy's Estimates. *Let f be differentiable on a neighborhood of $\overline{D}(a, R)$ and suppose $|f(z)| \leqslant M$ on $C(a, R)$. Then for all $n \geqslant 0$ and $z \in D(a, R)$,*

$$\left|f^{(n)}(z)\right| \leqslant \frac{Mn!}{R^n}.$$

Proof. Recall that by Taylor, f has a power series

$$\sum_{n=0}^{\infty} a_n(z-a)^n$$

on $D(a,R')$ for some $R' > R$. Then by Taylor again:

$$|a_n| = \left|\frac{f^{(n)}(a)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{C(a,R)} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z\right| \leqslant \frac{1}{2\pi}\int_{C(a,R)} \frac{|f(z)|}{|(z-a)^{n+1}|}\,\mathrm{d}z.$$

Now by 5.12,

$$\leqslant \frac{1}{2\pi}\frac{M}{R^{n+1}}\cdot 2\pi R = \frac{M}{R^n}.$$

The result follows. ■

7.9 Liouville's Theorem. *If f is a bounded entire function, then f is constant.*

Proof. Suppose $|f(z)| \leqslant M$ on $\mathbb{C}$. Let $z \in \mathbb{C}$. Then if $R > 0$, f is analytic on $D(z,R)$. Hence, by Cauchy,

$$|f'(z)| \leqslant \frac{M}{R}.$$

Letting $R \to \infty$ and using the fact that z was arbitrary shows that $f' \equiv 0$ on $\mathbb{C}$. Hence f is constant. ■

7.10 Fundamental Theorem of Algebra. *Let P be a polynomial with complex coefficients and degree $d \geqslant 1$. Then P has a root in $\mathbb{C}$.*

Proof. Suppose not. Write

$$P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$$

with $a_d \neq 0$. The latter can be rewritten

$$P(z) = a_d z^d\left(1 + \frac{a_{d-1}}{a_d}\cdot\frac{1}{z} + \cdots + \frac{a_0}{a_d z^d}\right).$$

Thus shows that for R sufficiently large,

$$|P(z)| \geqslant \frac{|nd|R^d}{z} \text{ if } |z| \geqslant R.$$

Now consider

$$Q(z) = \frac{1}{P(z)}.$$

Q is entire since P has no zeros. Also

$$|Q(z)| \leqslant \frac{z}{|a_d R^d|} \text{ for } |z| \geqslant R.$$

Q is also bounded on $\overline{D}(0,R)$ since it is continuous and $\overline{D}(0,R)$ is compact. Hence Q is bounded and entire. By Liouville's Theorem, Q is constant. Hence P is constant, which is impossible as $\deg P \geqslant 1$. ■

8 A Mixed Bag

8.1 The Identity Principle. *Let f be analytic on a domain Ω and let $a \in \Omega$. The following are equivalent:*

(1) $f \equiv 0$ on Ω,

(2) $f^{(n)}(a) = 0$ for all $n \geqslant 0$.

(3) There exists a sequence of points z_n with $z_n \neq a$ for all n, $z_n \to a$ and $f(z_n) = 0$ for all n.

Proof. (1) $\Rightarrow$ (2) trivially. (2) $\Rightarrow$ (3): Suppose that $\overline{D}(a,r) \subset \Omega$. Then f has a Taylor series about a on the disk $D(a,r)$. Hence $f \equiv 0$ on the disk. (3) $\Rightarrow$ (1): As above, f has a Taylor series on $D(a,r)$. So by uniqueness of power series, $f \equiv 0$ on the disk. Now let

$$A = \{z \in \Omega : z \text{ is a limit of zeros of } f\}.$$

The same argument as above shows that A is open in Ω. By the continuity of f, A is closed in Ω. Since A is both open and closed, it is either empty or the entire set. Since $a \in A$, $A \neq \varnothing$. Hence $A = \Omega$. By continuity again, $f \equiv 0$ on Ω. ∎

8.2 Corollary. *If two functions f and g are analytic on a domain Ω agree on an infinite set of points which has an accumulation point in Ω, then*

$$f \equiv g \text{ on } \Omega.$$

Proof. Apply 8.1 to $f - g$. ∎

8.3 Mean Value Theorem. *Suppose $U \subset \mathbb{C}$ is open, f is analytic on U, $a \in U$ and $r > 0$ such that $\overline{D}(a,r) \subset U$. Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, \mathrm{d}\theta.$$

Proof. Cauchy Integral Formula on a disk

$$f(a) = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(z)}{z-a}\, \mathrm{d}z.$$

On $C(a,r)$, let $z = a + re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$ and $\mathrm{d}z = ire^{i\theta}\, \mathrm{d}\theta$. So,

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}\, \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, \mathrm{d}\theta.$$

∎

8.4 Maximum Modulus. *Let f be analytic on a domain Ω and suppose $a \in \Omega$ is such that*

$$|f(a)| \geqslant |f(z)| \text{ for all } z \in \Omega.$$

Then f is constant.

Proof. Let $A = \{z \in \Omega : |f(z)| = |f(a)|\}$. By 8.3, if $\overline{D}(a,r) \subset \Omega$ and $0 \leqslant \rho \leqslant r$,

$$|f(a)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\,\mathrm{d}\theta \leqslant \frac{1}{2\pi}|f(a)|\cdot 2\pi = |f(a)|.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} (|f(a)| - |f(a+re^{i\theta})|)\,\mathrm{d}\theta = 0.$$

This is the integral of a continuous nonnegative function and so

$$|f(z)| = |f(a)| \text{ on } C(a,\rho).$$

Hence since $0 \leqslant \rho \leqslant r$ was arbitrary,

$$|f(z)| = |f(a)| \text{ on } D(a,r).$$

This argument shows A is open in Ω ans A is closed by the continuity of f. Since Ω is connected and $a \in A$, $|f|$ is constant on Ω. Then, f is constant on Ω. ∎

8.5 Corollary. *Let Ω be a domain, f be analytic and non-constant on Ω and continuous on $\overline{\Omega}$. Then if a is such that*

$$|f(a)| \geqslant |f(z)| \text{ for all } z \in \Omega,$$

then $a \in \partial\Omega$.

8.6 Minimum Modulus. *If f is a non-constant analytic function on a domain Ω, then no point $z \in \Omega$ can be minimum for $|f|$ unless $f(z) = 0$.*

Proof. Suppose note. Apply the Maximum Modulus Theorem to $g = 1/f$. ∎

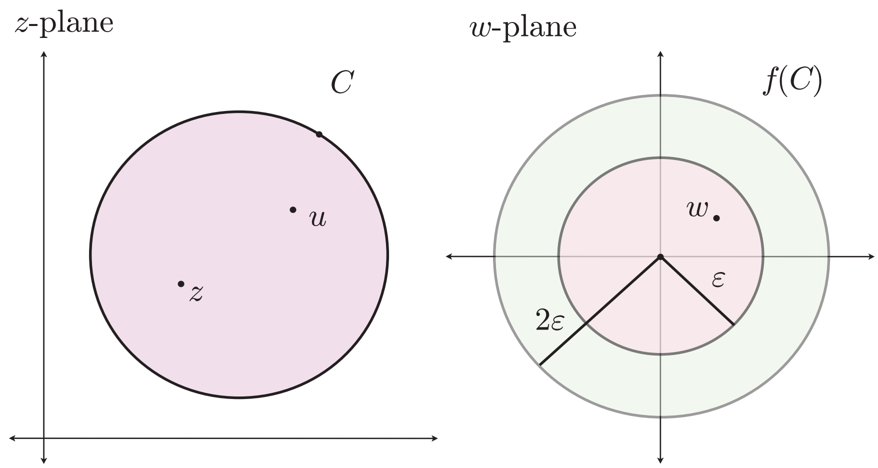
8.7 Open Mapping Theorem. *The image of a domain Ω under a non-constant analytic function f is again a domain.*

Proof. $f(\Omega)$ is connected since Ω is connected and f is continuous. To show $f(\Omega)$ is open, let $a \in \Omega$ and we show the image of a disk around a contains a disk around $f(a)$. Without loss of generality, assume $f(a) = 0$. By the Identity Principle, there exists a circle $C(a,r) \subset \Omega$ such that $f(z) \neq 0$ on $C(a,r)$. Let

$$2\varepsilon = \min_{z \in C(a,r)} |f(z)|.$$

We want to show that

$$f(D(a,r)) \supset D(0,\varepsilon).$$



Let $w \in D(0, \varepsilon)$ and consider $f(z) - w$. For $z \in C(a, r)$,

$$|f(z) - w| \geqslant |f(z)| - |w| \geqslant 2\varepsilon - \varepsilon = \varepsilon.$$

While at a,

$$|f(z) - w| = |0 - w| = |w| < \varepsilon.$$

Hence, $|f(z) - w|$ assume its minimum value inside $C(a, r)$. By Minimum Modulus Theorem,

$$f(z) - w = 0 \text{ somwhere inside } C(a, r).$$

Thus, w is in the range of $f|_{D(a,r)}$. ■

8.8 Corollary. *Suppose $\Omega \in \mathbb{C}$ is open is open and $f : \Omega \to G$ is bijective and analytic. Then $f^{-1} : G \to \Omega$ is analytic and*

$$(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}.$$

Proof. By Open Mapping Theorem, G is a domain and f^{-1} is continuous on G. Result follows from Proposition 2.4. ■

8.9 Schwarz' Lemma. *Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic with $f(0) = 0$. Then*

(i) $|f(z)| \leqslant |z|$;

(ii) $|f'(0)| \leqslant 1$,

with equality in either of the above if and only if $f(z) = ze^{i\theta}$ for some $\theta \in \mathbb{R}$.

Proof. Define

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(z), & z = 0. \end{cases}$$

By 7.6, g is analytic on $\mathbb{D}$. If $0 < r < 1$, then

$$|g| \leqslant \frac{1}{r} \text{ on } C(0, r)$$

by Maximum Modulus (8.4). If we now let r increase to 1, then we see that $|g| \leqslant 1$ on $\mathbb{D}$ which proves (i) and (ii). Also if $|g(z_0)| = 1$ for some $z_0 \in \mathbb{D}$, then $|g| = 1$ on $\mathbb{D}$ by Maximum Modulus and so $g(z) \equiv e^{i\theta}$ on $\mathbb{D}$ for some $\theta \in \mathbb{R}$. Thus, $f(z) = ze^{i\theta}$. ∎

9 The Homotopic Version of Cauchy's Theorem, Simple Connectedness, Integrals Over a Continuous Path

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \to U$ be a continuous path. From now on, for the sake of convenience, we shall assume that $\gamma : [0, 1] \to U$. We can always ensure this anyway by reparameterizing. From now on, denote $[0, 1]$ by I.

9.1 Definition. Let $U \subset \mathbb{C}$ be open. Two closed continuous paths $\gamma_0, \gamma_1 : I \to U$ are said to be **homotopic in** U if there exists a continuous function $\Gamma : I \times I \to U$ which satisfies

$$\Gamma(s, 0) = \gamma_0(s) \text{ and } \Gamma(s, 1) = \gamma_1(s) \text{ for } 0 \leqslant s \leqslant 1,$$

and $\Gamma(0, t) = \Gamma(1, t)$ for all $t \in I$. t measures which path you are on and s measures how far along a given path you are. Two continuous paths $\gamma_0, \gamma_1 : I \to U$ (not necessarily closed) are **fixed-endpoint homotopic in** U if they have the same endpoints and there exists $\Gamma : I \times I \to U$ where Γ is continuous and satisfies

$$\Gamma(s, 0) = \gamma_0(s) \text{ and } \Gamma(s, 1) = \gamma_1(s) \text{ for } 0 \leqslant s \leqslant 1$$

and

$$\Gamma(0, t) = \gamma_0(0) = \gamma_1(0) \text{ and } \Gamma(1, t) = \gamma_0(1) = \gamma_1(t) \text{ for } 0 \leqslant t \leqslant 1.$$

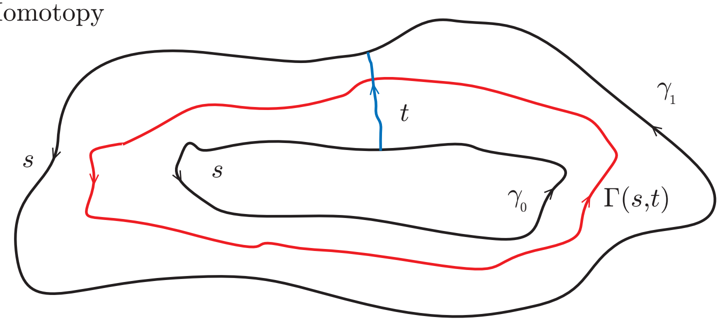
From now on, abbreviate fixed-endpoint homotopic by FEH.

9.2 Proposition. *For any $U \subset \mathbb{C}$ open, homotopy in U and FEH in U are equivalence relations.*

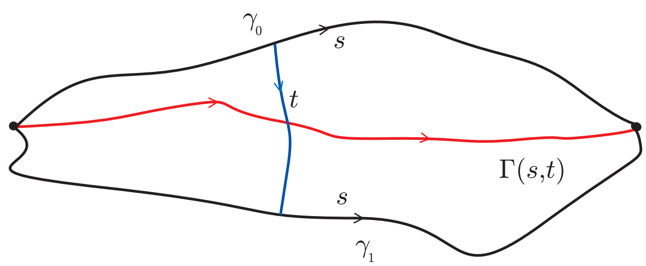
Proof. We shall only prove the case for homotopy. Reflexivity: If $\gamma_0 \approx \gamma_1$ in U via the homotopy Γ, then $\gamma_1 \approx \gamma_0$ in U via the homotopy $\Gamma'(s, t) = \Gamma(s, 1 - t)$. Symmetry: If γ is a loop in U , then $\gamma \approx \gamma$ via $\Gamma : I^2 \to U$ where $\Gamma(s, t) = \gamma(s)$. Transitivity: Suppose $\gamma_0 \approx \gamma_1$ in U via Γ and $\gamma_1 \approx \gamma_2$ in U via Λ. Then $\Phi : I^2 \to U$ defined by

$$\Phi(s, t) = \begin{cases} \Gamma(s, 2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ \Lambda(s, 2t - 1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Homotopy



Fixed-Endpoint Homotopy



is a homotopy from γ_0 to γ_2.

■

9.3 Definition (Composition of Paths)**.** For two paths (not necessarily closed) γ, η with $\gamma(1) = \eta(0)$, we define the composition $\eta \circ \gamma$ to be the path

$$(\eta \circ \gamma)(t) = \begin{cases} \gamma(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ \eta(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Note that if γ, η are piecewise differentiable and f is continuous on $[\gamma], [\eta]$, then

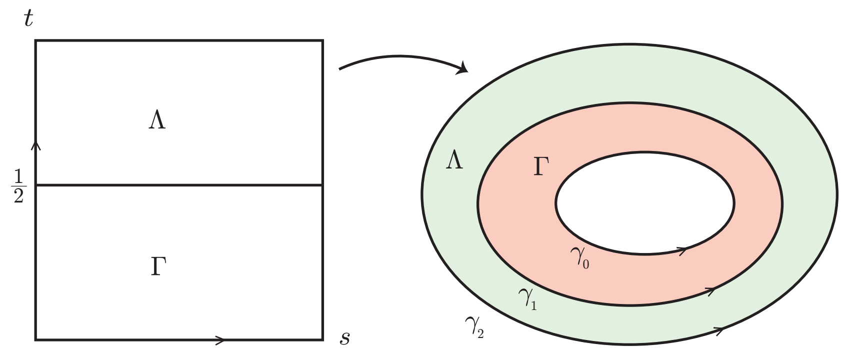
$$\int\limits_{\eta\circ\gamma} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z + \int_\gamma f(z)\,\mathrm{d}z.$$

9.4 Proposition. *Let $U \subset \mathbb{C}$ be open. If $\gamma_0, \gamma_1, \eta_0, \eta_1$ are continuous paths in U (not necessarily closed) with*

$$\gamma_0(1) = \eta_0(0) \qquad \textit{and} \qquad \gamma_1(1) = \eta_1(0).$$

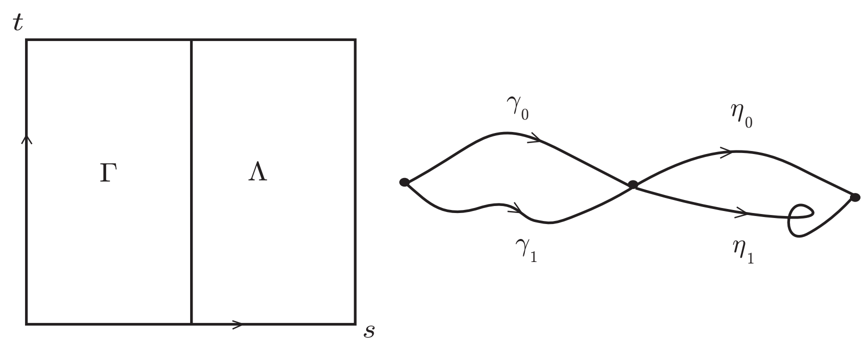
If γ_0 is FEH to γ_1 in U and η_0 is FEH to η_1 in U, then

$$\eta_0 \circ \gamma_0 \textit{ is FEH to } \eta_1 \circ \gamma_1 \textit{ in } U.$$



Proof. Let Γ, Λ be FEHs between γ_0, γ_1 and η_0, η_1, respectively. Define $\Phi : I^2 \to U$ by

$$\Phi(s,t) = \begin{cases} \Gamma(2s,t), & 0 \leqslant s \leqslant \frac{1}{2} \\ \Lambda(2s-1,t), & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$



Then Φ is continuous since

$$\Phi\left(\frac{1}{2}, t\right) = \Gamma(1,t) = \gamma_0(1) = \eta_0(0) = \Lambda(2s-1,t).$$

Also

$$\begin{aligned} \Phi(s,0) &= \begin{cases} \Gamma(2s,0), & 0 \leqslant s \leqslant \frac{1}{2} \\ \Lambda(2s-1,0), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \\ &= \begin{cases} \gamma_0(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ \eta_0(2s-1), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \\ &= \eta_0 \circ \gamma_0. \end{aligned}$$

Similarly, $\Phi(s,1) = (\eta_1 \circ \gamma_1)(s)$. Finally,

$$\Phi(0,t) = \Gamma(0,t) = \gamma_0(0) = \gamma_1(0) = (\eta_0 \circ \gamma_0)(0) = (\eta_0 \circ \gamma_0)(0)$$

and

$$\Phi(1,t) = \Lambda(1,t) = \eta_0(1) = \eta_1(1) = (\eta_0 \circ \gamma_0)(1) = (\eta_1 \circ \gamma_1)(1).$$

Hence Φ is a FEH in U from $\eta_0 \circ \gamma_0$ to $\eta_1 \circ \gamma_1$. ■

9.5 Definition. If $U \subset \mathbb{C}$ is open and γ is a closed path (loop) in U, we say γ is null-homotopic in U or homotopic to zero in U if γ is FEH in U to the constant path $\gamma(0) = \gamma(1)$. Write $\gamma \approx 0$.

9.6 Proposition. *If $U \subset \mathbb{C}$ is open and γ is a path in U (not necessarily closed), then*

$$\begin{aligned} -\gamma \circ \gamma &\approx 0 \text{ in } U \\ \gamma \circ -\gamma &\approx 0 \text{ in } U. \end{aligned}$$

9.7 Proposition. *Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Then the set of equivalence classes under FEH in U of loops starting and ending at z_0, together with the operation of composition of paths, forms a group. Call this the fundamental group of U with basepoint z_0 and write $\pi_1(U, z_0)$.*

9.8 Proposition. *Let $U \subset \mathbb{C}$ be open and let z_1, z_2 be points in U for which we can find a path γ in U with $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Then $\pi_1(U, z_1) \simeq \pi_1(U, z_2)$.*

Hence, we usually omit the basepoint from the notation.

9.9 Definition. A domain $\Omega \subset \mathbb{C}$ is **simply connected** if every loop in Ω is null-homotopic in Ω.

9.10 Cauchy's Theorem (Homotopic Version for Piecewise Differentiable Paths)**.** *If $U \subset \mathbb{C}$ is open and γ_0, γ_1 are two piecewise differentiable paths in U which are homotopic in U, then*

$$\int_{\gamma_0} f(z)\, \mathrm{d}z = \int_{\gamma_1} f(z)\, \mathrm{d}z$$

for all analytic functions f in U.

Proof. Let $\Gamma : I^2 \to U$ be the homotopy from γ_0 to γ_1. Since Γ is continuous and I^2 is compact, Γ is uniformly continuous on I^2 and $\Gamma(I^2)$ is a compact subset of U. Hence

$$r = \operatorname{dist}(\Gamma(I^2), \mathbb{C} \setminus U) = \inf_{z \in \Gamma(I^2)} \inf_{w \in \mathbb{C} \setminus U} |z - w| > 0.$$

By uniform continuity, we can find $n \in \mathbb{N}$ such that $s, s', t, t' \in I$ and

$$\sqrt{(s-s')^2 + (t-t')^2} < \frac{2}{n},$$

then
$$|\Gamma(s,t) - \Gamma(s',t')| < r.$$
Let
$$Z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right) \text{ for } 0 \leqslant j,k \leqslant n$$
and set
$$J_{jk} = \left[\frac{j}{n}, \frac{j+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ for } 0 \leqslant j,k \leqslant n-1.$$
Now
$$\operatorname{diam}(J_{jk}) = \frac{\sqrt{2}}{n} < \frac{2}{n},$$
and so
$$\Gamma(J_{jk}) \subset D(Z_{jk}, r) \subset U$$
by the definition of Γ.

If we let P_{jk} be the closed polygon
$$[Z_{jk}, Z_{j+1,k}, Z_{j+1,k+1}, Z_{j,k+1}, Z_{jk}] \text{ for } 0 \leqslant j,k \leqslant n-1,$$
then the vertices of P_{jk} lie in $D(z_{jk}, r)$, and since disks are convex, $P_{jk} \subset D(Z_{jk}, r)$. Then by Corollary 6.5, if f is analytic in U, then
$$\int_{P_{jk}} f(z)\,\mathrm{d}z = 0. \tag{3}$$
We now show
$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$
for any f analytic in U by the "ladder" we have constructed one rung at a time. For $0 \leqslant k \leqslant n$, let Q_k be the polygon
$$Q_k = [Z_{0k}, Z_{1k}, \ldots, Z_{nk} = Z_{0k}].$$
We show that
$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{Q_0} f(z)\,\mathrm{d}z = \int_{Q_1} f(z)\,\mathrm{d}z = \cdots = \int_{Q_n} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z.$$
To see that
$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$
for $0 \leqslant j \leqslant n-1$, set
$$\sigma_j(t) = \gamma_0(t) \text{ for } \frac{j}{n} < t < \frac{j+1}{n}.$$
Then $\sigma_j - [Z_{j0}, Z_{j+1,0}]$ is a closed path in $D(P_{jk}, r)$ where
$$\int\limits_{\sigma_j - [Z_{j0}, Z_{j+1,0}]} f(z)\,\mathrm{d}z = 0.$$

Note: This is where we use the fact that γ_0 is piecewise differentiable so that $\gamma_0 - [Z_{j0}, Z_{j+1,0}]$ is piecewise differentiable. Remember, we only know how to integrate over piecewise differentiable paths.

Hence,

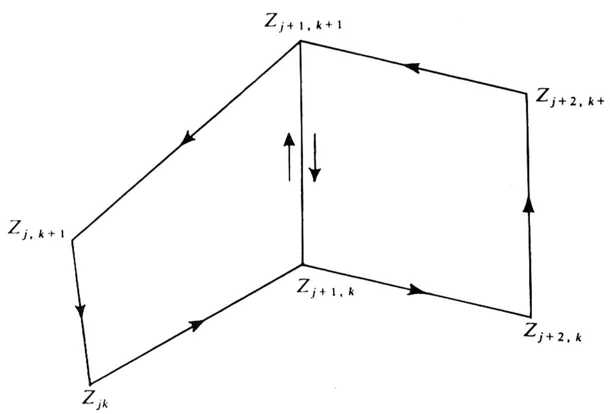
$$\int_{\sigma_j} f(z)\,\mathrm{d}z = \int\limits_{[Z_{j0}, Z_{j+1,0}]} f(z)\,\mathrm{d}z.$$

Summing over j gives

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{Q_0} f(z)\,\mathrm{d}z.$$

The same argument gives that

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{Q_n} f(z)\,\mathrm{d}z.$$



To show

$$\int_{Q_k} f(z)\,\mathrm{d}z = \int_{Q_{k+1}} f(z)\,\mathrm{d}z \text{ for } 0 \leqslant k \leqslant n-1,$$

first note that by (**??**),

$$\sum_{j=0}^{n-1} \int_{P_{jk}} f(z)\,\mathrm{d}z = 0. \tag{4}$$

Each of the "vertical" sides appears twice, once as $[Z_{jk}, Z_{j,k+1}]$ and once as $[Z_{j,k+1}, Z_{0k}]$, and so the integrals over these sides cancel.

The integral over $[Z_{0,k+1}, Z_{0k}]$ cances with that over $[Z_{nk}, Z_{n,k+1}]$ by the definition of homotopy where

$$\Gamma(0,t) = \Gamma(1,t) \text{ for } 0 \leqslant t \leqslant 1.$$

Each of the "horizontal" sides $[Z_{jk}, Z_{j,k+1}]$ is traversed exactly once. Each of the horizontal sides
$[Z_{j,k+1}, Z_{j+1,k+1}]$ is also traversed exactly once but in the opposite direction as $[Z_{j+1,k+1}, Z_{jk}]$. Combining these three observations with (**??**) gives

$$\int\limits_{Q_k - Q_{k+1}} f(z)\,\mathrm{d}z = 0,$$

so

$$\int_{Q_k} f(z)\,\mathrm{d}z = \int_{Q_{k+1}} f(z)\,\mathrm{d}z$$

as required and the proof is complete. ■

9.11 Corollary (Independence of Paths Theorem)**.** *If $U \subset \mathbb{C}$ is open and γ_0, γ_1 are two piecewise differentiable paths with the same endpoints (not necessarily closed) which are FEH in U, then*

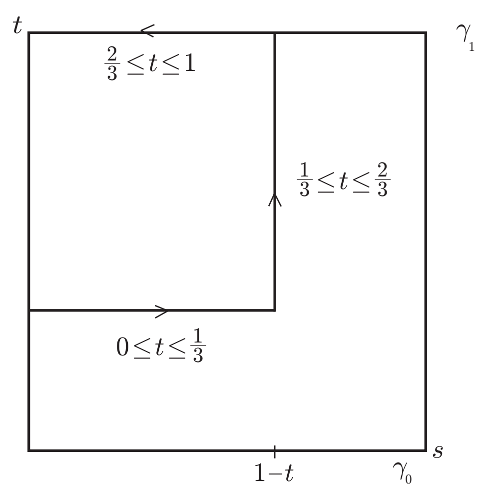
$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

for all f analytic in U.

Proof. Let $\Gamma : I^2 \to U$ be the FEH. Then $\Lambda : I^2 \to U$ given by

$$\Lambda(s,t) = \begin{cases} \Gamma(3s(1-t), t), & 0 \leqslant t \leqslant \frac{1}{3} \\ \Gamma(1-t, t + (3s-1)(1-t)), & \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \\ \gamma((3-3s)(1-t)), & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

is a FEH from $-\gamma_1 \circ \gamma_0$ to $\gamma_0(0) = \gamma_1(0)$. Now apply 9.10. ■



Integrals over General Continuous Paths

How to integrate an analytic function over a path which may not be piecewise differentiable.

9.12 Lemma. *Let $U \subset \mathbb{C}$ be open and let $\gamma : I \to U$ be a continuous path. Then there exists another piecewise differentiable path γ' consisting of horizontal and vertical line segments which has the same start and endpoints of γ and which is FEH to γ in U.*

Proof. Since γ is continuous and I is compact, γ is uniformly continuous on I and $\gamma(I)$ is a compact subset of U. Hence

$$r = \operatorname{dist}(\gamma(I), \mathbb{C} \setminus U) > 0.$$

(Note that $\gamma(I) = [\gamma]$.) By uniform continuity, we can find n such that if $|s - s'| < 1/n$, then

$$|\gamma(s) - \gamma(s')| < r.$$

Set $z_j = \gamma(j/n)$ for $0 \leqslant j \leqslant n$ and $\sigma_j(t) = \gamma(t)$ for $j/n \leqslant t \leqslant (j+1)/n$ and $0 \leqslant j \leqslant n-1$. Then

$$[\sigma_j] \subset D(z_j, r).$$

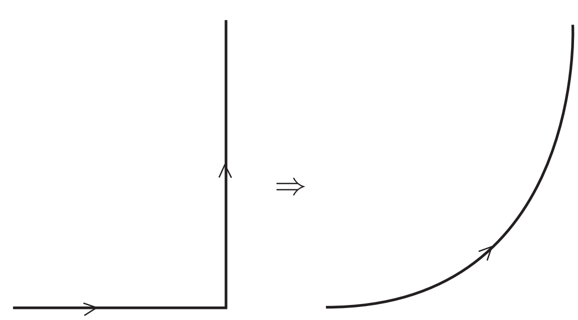
Since disks are simply connected, we can replace σ_j by a path σ_j' in $D(z_j, r)$ which consists of one horizontal and vertial line segment which runs from z_j to z_{j+1}. Then clearly σ_j is FEH to σ_j' in $D(z_j, r)$.

Then the path

$$\gamma' = \sigma_{n-1}' \circ \sigma_{n-1}' \circ \cdots \circ \sigma_1 \circ \sigma_0$$

is a path in U which is FEH in U to γ and is clearly piecewise smooth. ∎

Note that in fact we can take our path to be smooth and not just piecewise smooth.



We can now define the integral of an analytic function defined on an open set over a general continuous path.

9.13 Definition. Let $U \subset \mathbb{C}$ be open and let $\gamma : I \to U$ be a continuous path. If $f : U \to \mathbb{C}$ is analytic, we define the integral of f over γ by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma'} f(z)\,\mathrm{d}z$$

where γ' is a piecewise differentiable path which is FEH to γ in U.

Lemma 9.12 shows we can always find a suitable path γ'. However, we still need to show this is well-defined. As the integral over γ' does not depend on the choice of γ'.

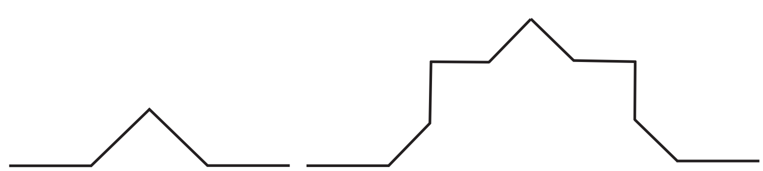
So suppose γ', γ'' are two such paths which both are FEH to γ in U. Then by Proposition 9.2, γ' and γ'' are FEH in U, and so by 9.11,

$$\int_{\gamma'} f(z)\,\mathrm{d}z = \int_{\gamma''} f(z)\,\mathrm{d}z.$$

Note that we can now integrate analytic functions over paths which are:

- nowhere differentiable and
- infinite-length (non-rectifiable).

Example. Consider the Koch curve:



This is nowhere differentiable. It also has Hausdorff dimension

$$\frac{\log 4}{\log 3} = 1.26;$$

whereas, it has infinite length.

9.14 Cauchy's Theorem (The General Version of the Homotopic Form)**.** *If $U \subset \mathbb{C}$ is open and γ_0 and γ_1 are two continuous closed paths which are homotopic in U, then*

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

for all analytic functions f on U.

Proof. By 9.12, we can find piecewise differentiable paths γ_0' and γ_1' in U such that γ_0' is FEH to γ_0 in U and γ_1' is FEH to γ_1 in U. Then since for *closed* paths if γ and η are FEH in U, then they are homotopic, we see by 9.2 that γ_0' and γ_1' are homotopic in U. Then if f is analytic in U,

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_0'} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_1'} f(z)\,\mathrm{d}z$$

by 9.10. ■

9.15 Corollary (The General Version of Independence of Paths)**.** *If $U \subset \mathbb{C}$ is open and γ_0 and γ_1 are continuous paths in U with the same endpoints which are FEH in U, then*

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

for all analytic functions f in U.

Proof. Similar to 9.14. ■

9.16 Theorem. *If Ω is simply connected and f is analytic on Ω, then*

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for all closed continuous paths γ in Ω.

9.17 Corollary. *If Ω is simply connected and $f : \Omega \to \mathbb{C}$ is analytic, then f has a primitive on Ω.*

Proof. By preceding result,

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for all continuous paths γ in Ω. In particular, this holds for all piecewise smooth closed curves and so by Corollary 6.5, f has a primitive F on Ω. ■

9.18 Proposition. *Let Ω be simply connected and f be analytic and non-vanishing on Ω. Then there exists an analytic branch of* $\log(f)$ *on Ω; i.e. there exists $g : \Omega \to \mathbb{C}$ analytic with*

$$f(z) = e^{g(z)}.$$

If $z_0 \in \Omega$ and $e^{w_0} = f(z_0)$, we can choose g such that $g(z_0) = w_0$.

Proof. Since f never vanishes, $f'/f = [\log(f)]'$ is analytic on Ω and so by the Corollary 9.17, f'/f has a primitive g on Ω. If $h(z) = e^{g_1(z)}$, then h is analytic on Ω and never vanishes (as $e^z \neq 0$ for all $z \in \mathbb{C}$). So, f/h is analytic and its derivative is

$$\frac{hf' - h'f}{h^2}.$$

But

$$h' = g_1' h = \frac{f' h}{f}.$$

So,

$$hf' - h'f = hf' - \frac{f'h}{f} \cdot f = 0.$$

Hence f/h is a constant c on Ω. Hence

$$f = ch = ce^{g_1} = e^{g_1 + c_1} \text{ for some } c_1.$$

Now let $g(z) = g_1(z) + c$. For the last part, we have

$$e^{w_0} = z_0 - e^{g(z_0)}.$$

Hence by adding a suitable integer multiple of $2\pi i$ to $g(z)$, we get a primitive satisfying $g(z_0) = w_0$. ■

10 Winding Numbers

10.1 Definition. If γ is a continuous closed path which does not pass through a point a, we define the **winding number** or **index** of γ about a, $n(\gamma, a)$

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - a}.$$

Note that in view of the results of the previous section, without loss of generality, we can take γ to be piecewise smooth when doing any calculations.

10.2 Lemma. *If a continuous path γ that does not pass through a, then $n(\gamma, a)$ is an integer.*

Attempt at proof:

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{z - a} &= \int_\gamma \mathrm{d}(\log(z - a)) \\
&= \int_\gamma \mathrm{d}(\log(z - a)) + i \int_\gamma \mathrm{d}(\mathrm{Arg}(z - a)) \\
&= 0 + \int_\gamma \mathrm{d}(\mathrm{Arg}(z - a)).
\end{aligned}$$

So the answer should be given by how many times the vector $\gamma(t) - a$ turns 'around' a, when we let t go from 0 to 1.

This can be made precise, but it easier to do the following.

Proof. Let $\gamma : I \to \mathbb{C}$ be our path. Remember that we are allowed to assume that γ is piecewise smooth. Define $g : I \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} \, \mathrm{d}s.$$

Hence $g(0) = 0$ and

$$g(1) = \int_\gamma \frac{\mathrm{d}z}{z-a} = 2\pi i \cdot n(\gamma, a).$$

Also,

$$g'(t) = \frac{\gamma'(s)}{\gamma(s) - a} \text{ for } 0 \leqslant t \leqslant 1$$

by the Fundamental Theorem of Calculus (second part). Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(e^{-g(t)} \cdot (\gamma(t) - a) &= -e^{-g(t)} g'(t)(\gamma(t) - a) + e^{-g(t)}\gamma'(t) \\
&= e^{-g(t)} \cdot \frac{-\gamma'(t)}{\gamma(t) - a}(\gamma(t) - a) + e^{-g(t)}\gamma'(t) \\
&= 0.
\end{aligned}$$

So $e^{-g(t)}(\gamma(t) - a)$ is equal to the constant

$$e^{-g(t)}(\gamma(0) - a) = \gamma(0) - a.$$

Hence

$$\begin{aligned}
& e^{-g(t)}(\gamma(1) - a) = \gamma(0) - a \\
\Rightarrow\ & e^{-g(t)}(\gamma(0) - a) = \gamma(0) - a \\
\Rightarrow\ & e^{-g(1)} = 1 \\
\Rightarrow\ & g(1) = 2\pi i k \text{ for } k \in \mathbb{Z} \\
\Rightarrow\ & n(\gamma, a) = k.
\end{aligned}$$

■

10.3 Remark. It is clear that $n(-\gamma, a) = -n(\gamma, a)$.

10.4 Lemma. *If the continuous closed path γ lies inisde some disk, then $n(\gamma, a) = 0$ for all points a outside the disk.*

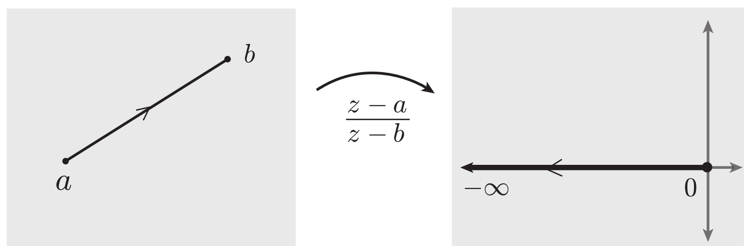
Proof. Suppose $[\gamma] \subset D(z, r)$ for some z and r. If $a \in D(z, r)$, then $1/z - a$ is analytic and non-vanishing on this disk which is simply connected. Hence by 9.18, there exists a branch of $\log(z - a)$ on the disk and so $n(\gamma, a) = 0$. ■

10.5 Definition. If $\gamma : I \to \mathbb{C}$ is a continuous path, $[\gamma]$ is compact, and so $\mathbb{C} \setminus [\gamma]$ is open and consists of a number of connected components. We call these connected components the **regions determined by** γ.

10.6 Theorem. *As a function of a, the winding number $n(\gamma, a)$ is constant on each of the regions determined by γ and 0 on the unbounded region.*

Proof. Any two points in the same region determined by γ can be joined by a polygonal path consisting of horizontal and vertical line segments (9.12). Hence it suffices to show the winding number is constant on a line segment connecting two points in some region determined by γ. (i.e. $[\gamma]$ does not meet the line segment.) Let a and b be two such consider the function

$$\frac{z-a}{z-b}.$$



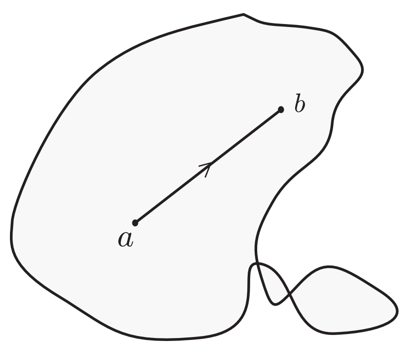
Off the line segment $[a,b]$, this function is never zero and real. Hence we can use the principal branch of the logarithm to define a branch of

$$\log\left(\frac{z-b}{z-a}\right) \text{ off } [a,b].$$

The derivative of

$$\log\left(\frac{z-a}{z-b}\right) = \frac{1}{z-a} - \frac{1}{z-b}$$

and hence if $\gamma \cap [a,b] = \varnothing$,

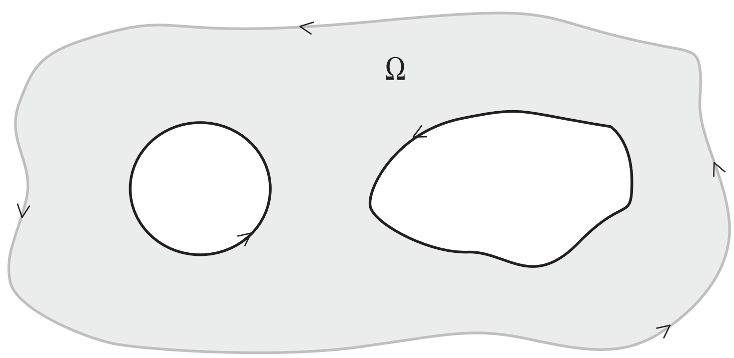


$$\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right)\,\mathrm{d}z = 0$$

and $n(\gamma, a) = n(\gamma, b)$. Finally, if $|a|$ is sufficiently large, γ is contained in $D(0, \rho)$ where $\rho = |a|$ and so $n(\gamma, a) = 0$ by Lemma 10.3. ■

11 Homology

Motivation



Often we have domains which look like the above. $\partial\Omega$ consists of several pieces and we would like to treat it so one object rather than several closed curves. This leads to the following definition.

11.1 Definition. A **chain** is a formal sum of the form

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

where $\gamma_1, \ldots, \gamma_n$ are continuous paths in $\mathbb{C}$.

We wish to integrate over chains and so if f is analytic on a neighborhood of each $[\gamma_i]$, then we define

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\,\mathrm{d}z.$$

Technically speaking, the continuous paths are elements of the free abelian group on 1-simplices (chain group C_1).

Guiding principle is that two chains are considered equivalent if they yield the same line integral for all suitable analytic functions f. Given this, the integral above doesn't change if we

1. permute two paths.
2. subdivide a path.
3. compose two paths.

4. reparameterize a path.
5. cancel opposite paths.

Can make these conditions give us an equivalence relation and then formally consider chians as equivalence classes with this relation.

Chains can be added in the obvious way and using (1) - (5), we can write any chian in the form
$$\Gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$$
where the γ_i are distinct and $a_i \in \mathbb{N}$. Also, if f is analytic on a neighborhood of each $[\gamma_i]$, then
$$\int_\Gamma f(z)\,\mathrm{d}z = a_1\int_{\gamma_1} f(z)\,\mathrm{d}z + \cdots + a_n\int_{\gamma_n} f(z)\,\mathrm{d}z.$$
A chain is a **cycle** if it can be represented as a sum of closed curves. Easy to see that the following are equivalent:

1. Γ is a cycle.
2. The initial and endpoint of Γ are in one-to-one correspondence.
3. $\displaystyle\int_\gamma \mathrm{d}z = 0.$

Technically, the mapping
$$\int_\Gamma \mathrm{d}z$$
is the boundary mapping ∂_1, which sends 1-chains to 0-chains (which are formal sums of 0-simplices or points). Note that if f is analytic and has a primitive on an open set U containing the curves of Γ, then
$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$
The winding number for cycles is defined in the obvious way. Clearly if Γ_1 and Γ_2 are cycles, $a \in [\Gamma_1], [\Gamma_2]$, then
$$n(\Gamma_1 + \Gamma_2, a) = n(\Gamma_1, a) + n(\Gamma_2, a).$$
Definition. If Ω is a domain and Γ is a cycle in Ω, then we say Γ is homologous to zero in Ω if
$$n(\Gamma, a) = 0 \text{ for all } a \in \mathbb{C} \setminus \Omega.$$
Write $\Gamma \approx 0$. Two cycles are homologous to each other in Ω if $\Gamma_1 - \Gamma_2 \approx 0$ in Ω.

11.2 Lemma. *If γ_1 and γ_2 are two continuous closed paths in $U \subset \mathbb{C}$ open and γ_1 and γ_2 are FEH in U, then*
$$\gamma_1 \approx \gamma_2 \textit{ in } U.$$

Proof. Let $a \notin U$. Then $1/z - a$ is analytic on U and so by Cauchy (9.15),

$$\int_{\gamma_1} \frac{\mathrm{d}z}{z-a} = \int_{\gamma_2} \frac{\mathrm{d}z}{z-a}.$$

Hence, $n(\gamma_1 - \gamma_2, a) = 0$. ■

11.3 Corollary. *Let $\Omega \subset \mathbb{C}$ be a domain. Then if Ω is simply connected, $\Gamma \approx 0$ in Ω for all cycles Γ in Ω.*

Proof. Every loop γ in Ω is FEH to the constant path. Apply 11.2. ■

11.4 Lemma. *Suppose γ is a piecewise smooth path in $\mathbb{C}$ and φ is continuous on $[\gamma]$. Then for each $n \geqslant 1$, the function F_n,*

$$F_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\,\mathrm{d}\zeta$$

is analytic on each of the regions determined by γ and

$$F_n'(z) = nF_{n+1}(z).$$

Proof. Show first that F_1 is continuous. Let $z_0 \notin [\gamma]$ and let $\delta > 0$ be such that $D(z_0, \delta) \cap [\gamma] = \varnothing$. If $z \in D(z_0, \delta/2)$, then

$$|\zeta - z| > \frac{\delta}{2} \text{ for all } \zeta \in [\gamma].$$

Then if $z \in D(z_0, \delta/2)$, from

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)(\zeta - z_0)}\,\mathrm{d}\zeta,$$

we obtain by the M-L formula (5.12),

$$\begin{aligned} |F_1(z) - F_1(z_0)| &< |z - z_0| \cdot \frac{2}{\delta^2} \int_\gamma |\varphi(\zeta)||\mathrm{d}\zeta| \\ &\leqslant |z - z_0| \cdot \frac{2}{\delta^2} \max_{\zeta \in [\gamma]} |\varphi(\zeta) \cdot \ell(\gamma)|\,. \end{aligned}$$

As $|z - z_0| \to 0$ and the integral on the right is bounded, continuity of F_1 at z_0 is proved.

Now apply the above continuity to the function

$$\frac{\varphi(\zeta)}{\zeta - z_0}$$

to conclude that

$$\begin{aligned} \frac{F_1(z) - F_1(z_0)}{z - z_0} &= \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)(\zeta - z_0)}\,\mathrm{d}\zeta \\ &\to \int_\gamma \frac{\varphi(\zeta)}{\zeta - z_0}\mathrm{d}\zeta \\ &= F_2(z_0) \text{ as } z \to z_0. \end{aligned}$$

Hence F_1 is analytic on each of the regions determined by γ and $F_1'(z) = F_2(z)$. The general case is shown by induction.

Suppose we have that $F_{n-1}'(z) = (n-1)F_n(z)$. Then one can check that

$$F_n(z) - F_n(z_0) = \left[\int_\gamma \frac{\varphi(\zeta)\mathrm{d}\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int_\gamma \frac{\varphi(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^n}\right] + (z - z_0)\int_\gamma \frac{\varphi(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^n(\zeta - z_0)}. \quad (5)$$

The first part in brackets goes to zero by the continuity of F_{n-1} for

$$\frac{\varphi(\zeta)}{\zeta - z_0}.$$

The second part goes to zero by a similar argument, by using the M-L formula, to that. For the continuity of F_1 in the base case. Now divide both sides of (**??**) by $z - z_0$. The first part of the right hand side of (**??**) by the induction hypothesis applied to F_{n-1} for

$$\frac{\varphi(\zeta)}{\zeta - z_0}$$

gives nF_n for

$$\frac{\varphi(\zeta)}{\zeta - z_0}.$$

The second part of the right hand side of (**??**) gives

$$\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,\mathrm{d}\zeta,$$

and by continuity we've just proved applied to F_n for

$$\frac{\varphi(\zeta)}{\zeta - z_0},$$

as $z \longrightarrow z_0$, we get

$$\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,\mathrm{d}\zeta = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta.$$

Adding these gives

$$(n+1)\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^{n+1}} = (n+1)F_{n+1}(z_0).$$

This completes the induction and the proof. ∎

11.5 Homological Cauchy's Integral Formula. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be analytic. If Γ is a cycle which is homologous to zero in U, then for $a \in \mathbb{C} \setminus [\gamma]$,*

$$n(\Gamma, a)f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - a}\,\mathrm{d}z.$$

Proof. Without loss of generality, Γ is piecewise smooth. Define $\varphi : U \times U \to \mathbb{C}$ by

$$\varphi(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w}, & z \neq w \\ 0, & z = w. \end{cases}$$

Then φ is continuous on $U \times U$ and by 7.6, for each fixed $w \in U$, $z \mapsto \varphi(z,w)$ is analytic.

Let

$$H = \{z \in \mathbb{C} : n(\Gamma, z) = 0\}.$$

Since $[\Gamma]$ is closed and $n(\Gamma, z) = 0$ is continuous and integer-valued on each of the (open) regions determined by Γ, it follows that H is open. Moreover, if $z \notin U$, then $n(\Gamma, z) = 0$ since $\Gamma \approx 0$ in U by definition. Hence, $z \in H$ and $H \cup U = \mathbb{C}$.

Now define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\Gamma \varphi(z,w)\,\mathrm{d}w, & z \in U \\ \int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w, & z \in H. \end{cases}$$

We need to show that g is well-defined. Let $z \in H \cap U$. Then

$$\begin{aligned} \int_\Gamma \varphi(z,w)\,\mathrm{d}w &= \int_\Gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}w \\ &= \int_\Gamma \frac{f(z)}{z-w}\,\mathrm{d}w - \int_\Gamma \frac{f(w)}{z-w}\,\mathrm{d}w \\ &= f(z)\int_\Gamma \frac{\mathrm{d}w}{z-w} - \int_\Gamma \frac{f(w)}{z-w}\,\mathrm{d}w \\ &= f(z)2\pi i n(\Gamma,z) - \int_\Gamma \frac{f(w)}{z-w}\,\mathrm{d}w \\ &= 0 + \int_\Gamma \frac{f(w)}{z-w}\,\mathrm{d}w, \end{aligned}$$

and g is well-defined.

By 11.4 applied to $f(w)/(w-z)$ and by the fact that a uniform limit of analytic functions is analytic, g is entire on $\mathbb{C}$. By Lemma 10.5, we can find $R > 0$ such that

$$\mathbb{C} \setminus \overline{D}(0,R) \subset H.$$

(Say H contains a neighborhood of infinity.) Since f is bounded on Γ and

$$\lim_{z\to\infty} \frac{1}{w-z} = 0$$

uniformly on $[\Gamma]$,

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\Gamma \frac{f(w)}{w-z} = 0. \tag{6}$$

Hence, we can find $R' > 0$ such that

$$|g(z)| \leqslant 1 \text{ if } |z| > R'.$$

Since g is continuous it is bounded on $\overline{D}(0, R')$. Hence g is bounded on $\mathbb{C}$ and entire, and so constant by Liouville's Theorem. It then follows from (**??**) that $g(z) = 0$ on $\mathbb{C}$.

Hence if $a \in U \setminus [\Gamma]$,

$$\begin{aligned} 0 = g(a) &= \int_\Gamma \frac{f(z) - f(a)}{z - a} \, \mathrm{d}z \\ &= \int_\Gamma \frac{f(z)}{z - a} \, \mathrm{d}z - \int_\Gamma \frac{f(a)}{z - a} \, \mathrm{d}z. \end{aligned}$$

Hence

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} \, \mathrm{d}z = f(a) n(\Gamma, a)$$

as required. ■

11.6 Homological Cauchy's Theorem. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be analytic. If $\Gamma \approx 0$ in U, then*

$$\int_\Gamma f(z) \, \mathrm{d}z = 0.$$

Proof. Apply 11.5 to $f(z)(z - a)$ for any point $a \in U \setminus [\Gamma]$. ■

11.7 Corollary. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be analytic. Then if $\Gamma \approx 0$ in U and a is any point in $U \setminus [\Gamma]$*

$$f^{(n)}(a) n(\Gamma, a) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - a)^{(n+1)}} \, \mathrm{d}z.$$

Proof. Use 11.4 on term under the integral sign to diffeerentiate the equation in the statement of 11.5 n times (with respect to a). (We assume without loss of generality that Γ is piecewise smooth.) ■

11.8 Corollary. *Let $U \subset \mathbb{C}$ be open and let Γ be a cycle in U. Thus*

$$\int_\Gamma f(z) \, \mathrm{d}z = 0 \text{ for all } f \text{ analytic on } U$$

if and only if

$$\Gamma \approx 0 \text{ in } U.$$

Proof. ($\Rightarrow$) Let $a \in \mathbb{C} \setminus U$ and $f(z) = 1/z - a$. ($\Leftarrow$) 11.6. ■

12 Counting Zeroes

Let f be analytic on $U \subset \mathbb{C}$ open. Suppose $f(a) = 0$ for some $a \in U$. We saw earlier that using Taylor that the function

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f(a), & z = a \end{cases}$$

is analytic on U. We can then write

$$f(z) = (z-a)g(z)$$

a formula which remains true at $z = a$ since $f(a) = 0$.

Continuing this way if $a_1, \dots, a_m$ are all the zeroes of f in U are all the zeroes of f in U (possibly repated for multiple roots), then we can write

$$f(z) = (z-a_1)\dots(z-a_m)g(z)$$

where $g(z)$ is analytic on U and never vanishes on U. Hence, by logarithmic differentiation, we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \dots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)} \tag{7}$$

for $z \neq a_i$. We then have the following theorem.

12.1 Theorem. *Let $U \subset \mathbb{C}$ be open and let f be analytic on U with finitely many roots $a_1, \dots, a_m$ (repeated according to multiplicity). If Γ is a cycle on U which doesn't pass through any of the a_i, and if $\Gamma \approx 0$ in U, then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{i=1}^m n(\Gamma, a_i).$$

Proof. Integrate both sides of (**??**) over Γ. ■

12.2 Corollary. *Let U, f, Γ be as in the last theorem, except that $a_1, \dots, a_m$ are points where $f(z) = \alpha$ for some $\alpha \in \mathbb{C}$. Then:*

$$\int_\Gamma \frac{f'(z)}{f(z)-\alpha}\,\mathrm{d}z = \sum_{i=1}^m n(\Gamma, a_i).$$

13 Isolated Singularities and Laurent Series

Suppose f is analytic on $U \subset \mathbb{C}$ open except at $a \in U$. What does f do near a?

13.1 Proposition. *A function f has an **isolated singularity** at $z = a$ if there exists $r > 0$ such that f is analytic on $D(a, r) \setminus \{a\}$ but not on $D(a, r)$. a is called a **removable singularity** if there exists g analytic on $D(a, r)$ where $g(z) = f(z)$ on $D(a, r) \setminus \{a\}$.*

13.2 Theorem. *If f is analytic on $U \subset \mathbb{C}$ open except for an isolated singularity at $a \in U$ then $z = a$ is a removable singularity if and only if*

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose $\lim\limits_{z \to a} (z-a) f(z) = 0$. Let $h(z)$ be defined as

$$h(z) = \begin{cases} (z-a) f(z), & z \neq a \\ 0, & z = a \end{cases}.$$

If h is analytic on U, then we're done since by Taylor we can find $g(t)$ such that

$$h(t) = (z-a) g(t) \to f(z) = g(z), \text{ for } z \neq a.$$

By 6.2, given any rectangle R contained in U (including the inside), h is differentiable on $U \setminus \{a\}$ and at a, so

$$\int_{\partial R} h(z)\, \mathrm{d}z = 0 \text{ for all rectangles } R \subset U.$$

Hence, by Morera, h is analytic in U and we're done. ∎

13.3 Definition. If $z = a$ is an isolated singularity of f, then a is a **pole** of f if

$$\lim_{z \to a} |f(z)| = \infty$$

and if $z = a$ is an isolated singularity, which is neither removable nor a pole, it is an **essential singularity of** f.

Suppose that f has a pole at $z = a$. Then $1/f(z)$ has a removeable singularity at $z = a$ since

$$\lim_{z \to a} \left| (z-a) \frac{1}{f(z)} \right| = \lim_{z \to a} |z-a| \lim_{z \to a} \frac{1}{|f(z)|} = 0 \cdot \frac{1}{\infty} = 0.$$

Then $1/f$ has a removeable singularity by 13.2. Hence

$$g(z) = \begin{cases} 1/f(z), & z \neq a \\ 0, & z = a \end{cases}$$

is analytic in $D(a, r)$ for some $r > 0$, and it follows that we can write

$$g(z) = (z-a)^m h(z)$$

for some $m \geqslant 1$, where h is analytic on $D(a, r)$ and $h(z) \neq 0$ on $D(a, r)$ (may need to make r smaller). Hence

$$(z-a)^m f(z) = \frac{1}{h(z)}$$

has a removeable singularity at $z = a$. We then get the following:

13.4 Proposition. *Let $U \subset \mathbb{C}$ be open, $a \in U$, and f analytic on $U \setminus \{a\}$ with a pole at $z = a$. Then there is a natural number $m \geqslant 1$ and an analytic function $g(z)$ on U such that*

$$f(z) = \frac{g(z)}{(z-a)^m} \text{ for } z \in U \setminus \{a\}\,.$$

13.5 Definition. If f has a pole at a and m is the smallest number for which the above holds, we say that f has a **pole of order m at a.**[1]

Suppose now that f has a pole of order m at $z = a$ and set

$$f(z) = \frac{g(z)}{(z-a)^m}$$

as in the definition of a pole of order m. Since g is analytic on $D(a, r)$ for some $r > 0$, we can write

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Hence

$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + g_1(t),$$

where $A_m \neq 0$ and g_1 is analytic on $D(a, r)$.

13.6 Definition. If f has a pole of order m at a and we then have an expansion as above, we call

$$\frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a}$$

the **singular part of f at** a.

Example (Rational Function)**.** Let

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials with no common factors (which implies that the poles of R are precisely the poles of Q. Also, the order of a pole of R at a is the order of the zero of Q at a.).

So suppose $Q(a) = 0$ and let $S(z)$ be the singular part of R at a. Then

$$R_1(z) = R(z) - S(z)$$

is another rational function whose poles are poles of R but which doesn't have a pole at a.

[1] Note that this implies that $g(a) \neq 0$.

Continuing by induction, we find that if $a_1, \ldots, a_n$ are the poles of R and $S_1, \ldots, S_n$ are the corresponding singular parts, we can write

$$R = \sum_{k=1}^{\infty} S_k(z) + \widetilde{P}(z),$$

where $\widetilde{P}$ is a rational function without any poles. But a rational function without poles is a polynomial and so we have obtained the partial fraction expansion of R.

13.7 Definition. If $\{z_n\}_{n=-\infty}^{\infty}$ is a doubly infinite sequence, we say $\sum\limits_{n=-\infty}^{\infty} z_n$ is absolutely convergent if

$$\sum_{n=0}^{\infty} z_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} z_{-n}$$

are both absolutely convergent. In this case, we get

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}.$$

If $\{u_n\}_{n=-\infty}^{\infty}$ is a doubly infinite sequence of functions defined on some set S and $\sum\limits_{n=-\infty}^{\infty} u_n$ is absolutely convergent at each point of S, we say $\sum\limits_{n=-\infty}^{\infty} u_n$ is uniformly convergent if

$$\sum_{n=0}^{\infty} u_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} u_{-n}$$

are both uniformly convergent on S.

Notation. $A(a, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - a| < r_2\}$ for $0 \leqslant r_1 < r_2 \leqslant \infty$.

13.9 Laurent Series Development. *Let f be analytic on $A(a, R_1, R_2)$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the convergence is uniform and absolute over any $A(a, r_1, r_2)$ where $R_1 < r_1 < r_2 < R_2$. Also the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z, \tag{8}$$

where $R_1 < r < R_2$ (any such r will do). Moreover, the series is unique.

Proof. If $R_1 < r_1 < r_2 < R_2$, let $\gamma_1 = C(a, r_1)$ and $\gamma_2 = C(a, r_2)$. Then $\gamma_1 \approx \gamma_2$ in $A := A(a, R_1, R_2)$. By Homological Cauchy, if g is analytic on A, then

$$\int_{\gamma_1} g(z)\,\mathrm{d}z = \int_{\gamma_2} g(z)\,\mathrm{d}z.$$

In particular, the integrals in (**??**) do not depend on r so that for each $n \in \mathbb{Z}$, a_n is a constant. Moreover, $f_2 : D(a, R_2) \to \mathbb{C}$ given by

$$f_2(z) = \frac{1}{2\pi i} \int\limits_{C(a,r_2)} \frac{f(w)}{w - z}\,\mathrm{d}w,$$

where $|z - a| < r_2$, $R_1 < r_2 < R_2$ is a well-defined function. By Lemma 11.4, f_2 is analytic in $D(a, r_2)$.

Similarly, if

$$G = \{z \in \mathbb{C} : |z - a| > R_1\} = A(a, R_1, \infty),$$

then $f_1 : G \to \mathbb{C}$ defined by

$$f_1(z) = -\frac{1}{2\pi i} \int\limits_{C(a,r_1)} \frac{f(w)}{w - z}\,\mathrm{d}w,$$

where $|z - a| > r_1$ and $R_1 < r_1 < R_2$ is well-defined and analytic on G. ■

If $z \in A = (A, R_1, R_2)$, choose $r_1, r_2 > 0$ such that $R_1 < r_1 < |z - a| < r_2 < R_2$ and let $\gamma_1 = C(a, r_1)$ and $\gamma_2 = C(a, r_2)$. Then $\Gamma = \gamma_2 - \gamma_1 \approx 0$ in A, and so by Cauchy's Homological Integral Formula, 11.5, that since $n(\Gamma, z) = 1$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\,\mathrm{d}w - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\,\mathrm{d}w \\ &= f_2(z) - f_1(z). \end{aligned}$$

The plan is to expand f_1 and f_2 as power series in $(z - a)$ (with f_1 having negative powers).

Since f_2 is analytic on $D(a, R_2)$, we have a Taylor series expansion by Taylor's Theorem with,

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

where

$$a_n = \frac{f_2^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int\limits_{C(a,r_2)} \frac{f(w)}{(w - a)^{n+1}}\,\mathrm{d}w$$

which matches the form of the statement.

Now define $g(\zeta)$ for $0 < |\zeta| < 1/R_1$ by

$$g(\zeta) = f_1\left(a + \frac{1}{\zeta}\right)$$

so that $\zeta = 0$ is an isolated singularity.

Claim. $\zeta = 0$ is removeable.

In fact if $r > R_1$ and we let $p(z) = \operatorname{dist}(z, C(a,r))$ and $M = \max\limits_{w \in C(a,r)} |f(w)|$, then for $|z - a| > r$,

$$|f_1(z)| \leqslant \frac{1}{2\pi} \frac{M 2\pi r}{p(z)} = \frac{Mr}{p(z)}.$$

But $\lim\limits_{z \to \infty} p(z) = \infty$, so

$$\lim_{\zeta \to 0} g(\zeta) = \lim_{\zeta \to 0} f_1\left(a + \frac{1}{\zeta}\right) = 0.$$

Hence if we define $g(0) = 0$, then g is analytic in $D(a, 1/R_1)$.

Using Taylor's Theorem, let

$$g(\zeta) = \sum_{n=1}^{\infty} B_n z^n$$

be its continuous power series expansion about 0. Since $z = a + 1/\zeta$, $\zeta = 1/(z-a)$ and thus gives

$$f_1(z) = g\left(\frac{1}{z-a}\right) = \sum_{n=1}^{\infty} B_n \left(\frac{1}{(z-a)^n}\right).$$

This gives

$$f_1(z) = \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

where

$$\begin{aligned}
a_{-n} = B_n &= \frac{1}{2\pi i} \int\limits_{C(0,1/r_1)} \frac{g(\zeta)}{\zeta^{n+1}} \,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int\limits_{C(0,1/r_1)} \frac{f_1(a+\frac{1}{\zeta})}{\zeta^{n+1}} \\
&= \frac{1}{2\pi i} \int\limits_{-C(0,r_1)} \frac{f_1(z)}{(z-a)^{-(n+1)}} \cdot \frac{-\mathrm{d}z}{(z-a)^2} \\
&= \frac{1}{2\pi i} \int\limits_{C(a,r_1)} \frac{f_1(z)}{(z-a)^{-n+1}} \,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int\limits_{C(a,r_1)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{-n+1}} - \frac{1}{2\pi i} \int\limits_{C(a,r_1)} f(z)(z-a)^{n-1}\,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int\limits_{C(a,r_1)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{-n+1}}
\end{aligned}$$

which gives us the formula for the remaining a_n in the statement.

Also, by the convergence properties of the series f_1 and f_2, the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

converges uniformly and absolutely on proper subannuli $A(a, r_1, r_2)$ with $R_1 < r_1 < r_2 < R_2$.

For uniqueness, suppose

$$\sum_{n=-\infty}^{\infty} b_n(z-a)^n$$

is another Laurent series expansion for f which converges uniformly on proper subannuli of some $A(a, R_1', R_2')$ where

$$R_1 \leqslant R_1' \leqslant R_2' \leqslant R_2.$$

Then if $R_1' < r < R_2'$,

$$\begin{aligned}
\frac{1}{2\pi i} \int\limits_{C(a,r)} \frac{\sum\limits_{m=-\infty}^{\infty} b_m(z-a)^m}{(z-a)^{n+1}} \,\mathrm{d}z &= \sum_{m=-\infty}^{\infty} \frac{b_m}{2\pi i} \int\limits_{C(a,r)} (z-a)^{m+n-1}\,\mathrm{d}z \\
a_n = \frac{1}{2\pi i} \int\limits_{C(a,r)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z &= \frac{b_n}{2\pi i} \int\limits_{C(a,r)} \frac{\mathrm{d}z}{z-a} = b_n.
\end{aligned}$$

So, $a_n = b_n$ for all n and the series is unique.

14 Using the Laurent Series Expansion to Classify Isolated Singularities

14.1 Corollary. *Let $z = a$ be an isolated singularity of f and let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

be its Laurent series expansion in $A(a, 0, R)$ for some $R > 0$. Then

(a) $z = a$ is a removeable singularity if and only if $a_n = 0$ for $n \leqslant -1$.

(b) $z = a$ is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$.

(c) $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative n.

Proof. (a) If $a_n = 0$ for $n \leqslant -1$, let $g(z)$ be defined in $D(a, R)$ by

$$g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Then g is analytic on $D(a, R)$ and agrees with f on $A(a, 0, R)$. The converse is equally easy (uses uniqueness of Laurent series expansion).

(b) Suppose $a_n = 0$ for $n \leqslant m-1$ and $a_{-m} \neq 0$. $(z-a)^m f(z)$ has a Laurent series expansion with no negative power of $f(z)$. By (a), $(z-a)^m f(z)$ has a removeable singularity at $z = a$. Then f has a removeable singularity $z = a$ and f has a pole of order m at a. The converse is obtained by reversing the argument.

(c) Follows directly from (a) and (b). ■

We already know that

$$\lim_{z \to a} |f(a)|$$

fails to exist in the neighborhood of an essential singularity. Can we say more?

14.2 Casorati-Weierstrass Theorem. *If f has an essential singularity at $z = a$, then for all $\delta > 0$,*

$$\overline{f(A(a, 0, \delta))} = \mathbb{C};$$

that is, the image of any punctured disk about a is dense in all of $\mathbb{C}$.

Proof. Let $R > 0$ be such that f is analytic on $A(a, 0, R)$. We need to show that for all $c \in \mathbb{C}$ and all $\varepsilon > 0$, for all $\delta > 0$, there exists $z \in A(a, 0, \delta)$ for which $|f(z) - c| < \varepsilon$. Assume this is false:

$$\exists c \in \mathbb{C},\ \exists \varepsilon > 0,\ \exists \delta > 0,$$

such that for all $z \in A(a, 0, \delta)$,

$$|f(z) - c| \geqslant \varepsilon.$$

Thus

$$\lim_{z \to a} \frac{|f(z) - c|}{|z - a|} = \infty,$$

and so $f(z) - c/z - a$ has a pole of some order m at $z = a$. Hence

$$\lim_{z \to a} |z - a|^m \, |f(z) - c| = 0.$$

Hence since

$$|z - a|^m |f(z)| \leqslant |z - a|^m |f(z) - c| + |z - a|^m |c|,$$

we see that

$$\lim_{z \to a} |(z - a)^m||f(z)| = 0.$$

Hence by an earlier result (derivation of the singular part for poles), f has a pole of order less than m or a removeable singularity at $z = a$. This is impossible as we had assumed that f had an essential singularity at $z = a$. ■

One can actually prove a stronger result. The proof of which is readily accessible in Conway.

14.3 Great Picard Theorem. *Suppose an analytic function f has an essential singularity at $z = a$. Then in every neighborhood of a, f assumes each complex number with at most one possible exception infinitely many times.*

15 Residues

15.1 Definition. Let f have an isolated singularity at $z = a$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

be its Laurent series expansion. The **residue** of f at a is the coefficient a_{-1}. Write $\operatorname{Res}(f; a) = a_{-1}$.

15.2 The Residue Theorem. *Let f be analytic on $U \subset \mathbb{C}$ open except for isolated singularities at points $a_1, \ldots, a_m$. If Γ is a cycle in U which does not pass through any of the points a_k and $\Gamma \approx 0$ in U, then*

$$\frac{1}{2\pi i} \int_\Gamma f(z) \, \mathrm{d}z = \sum_{k=1}^{m} n(\Gamma, a_k) \operatorname{Res}(f; a_k).$$

Proof. Let $U' = U \setminus \{a_1, \ldots, a_m\}$ and let $m_k = n(\Gamma, a_k)$ for each k. Now choose positive radii r_k such that the closed disks $\overline{D}(a_k, r_k)$ are pairwise disjoint and contained in U'.

Now let Γ' be the cycle in U' given by

$$\Gamma' = \Gamma - \sum_{k=1}^{m} m_k C(a_k, r_k).$$

Then $\Gamma \approx 0$ in U' and f is analytic in U' as we have removed all singularities. Hence by Cauchy, 11.6,

$$\int_{\Gamma'} f(z)\,\mathrm{d}z = 0;$$

i.e.

$$\int_{\Gamma} f(z)\,\mathrm{d}z = \sum_{k=1}^{m} m_k \int\limits_{C(a_k,r_k)} f(z)\,\mathrm{d}z.$$

By our choice of the $r_k's$, for each k we can find $R_k > r_k$ such that f is analytic on $A(a, 0, R_k)$ (by the compactness of $\overline{D}(a_k, r_k)$). For a fixed k, suppose the Laurent series expansion on f on $A(a_k, 0, R_k)$ is given by

$$\sum_{n=-\infty}^{\infty} a_{n,k}(z - a_k)^n.$$

This series converges uniformly and absolutely on proper subannuli by the theorem about Laurent series development and so by the theorem of line integrals of uniformly convergent sequences of functions, 5.14,

$$\int\limits_{C(a_k,r_k)} f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_{n,k} \int\limits_{C(a_k,r_k)} (z-a)^n\,\mathrm{d}z.$$

Since

$$\int\limits_{C(a_k,r_k)} f(z)\,\mathrm{d}z = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1, \end{cases}$$

and the result follows. ■

Calculating Residues

Suppose f has a pole of order $m \geqslant 1$ at $z = a$. Then we have

$$f(z) = \frac{b_{-m}}{(z-a)^m} + \cdots + \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + \cdots$$

on some annulus centered about a, where $b_{-m} \neq 0$. Also

$$\begin{aligned} g(z) &:= (z-a)^m f(z) \\ &= b_{-m} + b_{-m+1}(z-a) + \cdots + b_{-1}(z-a)^{m-1} + b_0(z-a)^m + \cdots \end{aligned}$$

is analytic on a disk about a. Hence by Taylor

$$b_{-1} = \frac{1}{(m+1)!}\frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}[(z-a)^m f(z)]|_{z=a}.$$

15.3 Proposition. *If f has a pole of order m at $z = a$, then*

$$\mathrm{Res}\,(f;a) = \frac{1}{(m+1)!}\frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}[(z-a)^m f(z)]|_{z=a}.$$

A Neat Trick: Suppose $f(z) = g(z)/h(z)$ where g and h are analytic on a disk about a. Suppose also that $g'(a) \neq 0$ and h has a simple zero at $z = a$. Let

$$\begin{aligned} g(z) &= a_0 + a_1(z-a) + \cdots \qquad a_0 \neq 0 \\ h(z) &= b_1(z-a) + b_2(z-a)^2 + \cdots \qquad b_1 \neq 0. \end{aligned}$$

Then f has a simple pose at $z = a$ and by 15.3,

$$\begin{aligned} \mathrm{Res}\,(f;a) &= \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)\frac{g(z)}{h(z)} \\ &= \lim_{z\to a}\frac{a_0(z-a) + a_1(z-a)^2 + \cdots}{b_1(z-a) + b_2(z-a)^2 + \cdots} \\ &= \lim_{z\to a}\frac{a_0 + a_1(z-a) + \cdots}{b_1 + b_2(z-a) + \cdots} \\ &= \frac{a_0}{b_1} \end{aligned}$$

by continuity of power series. But this is just $g(a)/h'(a)$ by Taylor. Hence we have:

15.4 Proposition. *Let $f(z) = g(z)/h(z)$ where g and h are analytic on some neighborhood of some point a. If $g(a) \neq 0$ and h has a simple zero at $z = a$, then f has a simple pole at a and*

$$\mathrm{Res}\,(f;a) = \frac{g(a)}{h'(a)}.$$

16 Using the Residue Theorem to do Integrals - Contour Integration

1. Integrals of the form

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x$$

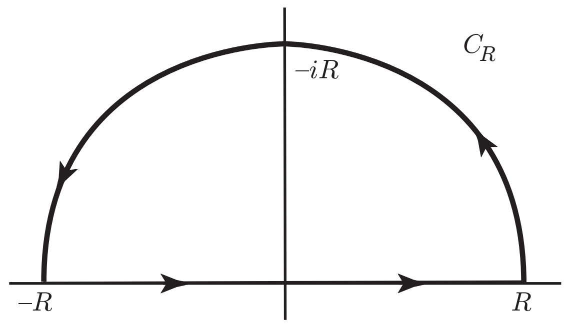
where $R(x) = P(x)/Q(x)$ is a rational function. Usually $P, Q \in \mathbb{R}[x]$. Standard estimates show that this improper integral will converge if and only if

1. $Q(x) \neq 0$ on $\mathbb{R}$.

2. $\deg Q \geqslant \deg P + 2$.

So we will assume from now on that these conditions are satisfied.

Let C_R be the contour consisting of this line segment which runs from $-R$ to R on the real axis and traverses the upper half of the circle $C(0, R)$ from R to $-R$.



By the Residue theorem, if R is sufficiently large,

$$\int_{C_R} R(z)\,\mathrm{d}z = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right)$$

where z_k are the zeros of Q lying in the upper half plane $\mathbb{H}$. [Note: It is very easy to see (using homotopy) that $n(C_R, a) = 1$ for all points lying inside the semicircle.]

Thus if we let Γ_R denote the curved part of C_R, then

$$\int_{-R}^{R} R(x)\,\mathrm{d}x + \int_{\Gamma_R} R(z)\,\mathrm{d}z = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right).$$

However, since $\deg Q \geqslant 2 + \deg P$, it follows from ML-formula that

$$\left|\int_{\Gamma_R} R(z)\,\mathrm{d}z\right| \leqslant \pi R \cdot \frac{A}{R^2}$$

for some constant A. Hence

$$\lim_{R\to\infty} \int_{\Gamma_R} R(z)\,\mathrm{d}z = 0.$$

And letting $R \to \infty$,

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_k \operatorname{Res}\left(\frac{P}{Q}; z_k\right).$$

Example. The function $1/(1+z^4)$ has simple poles at $e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}$, and $e^{7i\pi/4}$. Of these, only $e^{i\pi/4}, e^{3i\pi/4} \in \mathbb{H}$. Using our trick for simple poles,

$$\begin{aligned}\operatorname{Res}\left(\frac{1}{1+z^4}; e^{i\pi/4}\right) &= \frac{1}{4z^3}\Big|_{z=e^{i\pi/4}} = \frac{1}{4e^{3i\pi/4}} \\ &= \frac{1}{4\sqrt{2}(-1+i)} \\ &= \frac{1}{4\cdot\frac{1}{\sqrt{2}}\cdot 2}(-1-i) \\ &= \frac{1}{4\sqrt{2}}(1+i).\end{aligned}$$

Similarly,

$$\operatorname{Res}\left(\frac{1}{1+z^4}; e^{3i\pi/4}\right) = \frac{1}{4\sqrt{2}}(1-i).$$

Hence, since $\deg Q = \deg P + 4$ and has no singularities on $\mathbb{R}$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+z^4}\,\mathrm{d}z &= 2\pi i\left(\frac{1}{4\sqrt{2}}(1+i) + \frac{1}{4\sqrt{2}}(1-i)\right) \\ &= \frac{2\pi i(-2i)}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

2. Integrals of the form

$$\int_{-\infty}^{\infty} R(x)\cos x\,\mathrm{d}x \text{ or } \int_{-\infty}^{\infty} R(x)\sin x\,\mathrm{d}x$$

where $R(x) = P(x)/Q(x)$ and P, Q are polynomials. Hence the improper integrals converge if

1. $Q \neq 0$ on $\mathbb{R}$.
2. $\deg Q \geqslant \deg P + 1$.

However, we cannot use the same contour as before and do

$$\lim_{R\to\infty}\int_{C_R} R(z)\cos z\,\mathrm{d}z$$

(or similarly with $\sin x$). Instead we do

$$\lim_{R\to\infty}\int_{C_R} R(z)e^{iz}\,\mathrm{d}z$$

and then

$$\int_{-\infty}^{\infty} R(x)\cos x\,\mathrm{d}x \qquad \text{and} \qquad \int_{-\infty}^{\infty} R(x)\sin x\,\mathrm{d}x$$

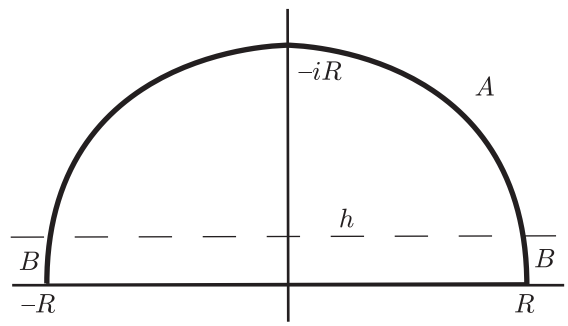
are just the real and imaginary parts of this limit (provided $P, Q \in \mathbb{R}[x]$) as long as the integral along $\Gamma_R \to 0$ as $R \to \infty$.

To show that

$$\int_{\Gamma_R} R(z)e^{iz}\,\mathrm{d}z \to 0 \text{ as } R \to \infty,$$

let $h > 0$ and split Γ_R into two subsets:

$$A = \{z \in \Gamma_R : \operatorname{Im} z \geqslant h\} \qquad \text{and} \qquad B = \{z \in \Gamma_R : \operatorname{Im} z < h\}.$$



For now, we leave h arbitrary, but we will make a choice later.

Now $|e^z| = e^{\operatorname{Re} z}$, and for $|z|$ large,

$$\deg Q \geqslant \deg P + 1 \text{ implies } |R(z)| \leqslant \frac{K}{|z|}$$

for some $K > 0$. Then

$$\begin{aligned}
\left|\int_A R(z)e^{iz}\,\mathrm{d}z\right| &\leqslant \int_A \left|R(z)e^{iz}\right| |\mathrm{d}z| \\
&= \int_A |R(z)| \left|e^{iz}\right| |\mathrm{d}z| \\
&= \int_A |R(z)|\, e^{-\operatorname{Im} z}\, |\mathrm{d}z| \\
&\leqslant \int_A |R(z)|\, e^{-h}\, |\mathrm{d}z| \\
&\leqslant \int_A \frac{K}{|z|} e^{-h}\, |\mathrm{d}z| \\
&\leqslant \pi R \cdot \frac{K}{|z|} e^{-h} \\
&= c_1 e^{-h}
\end{aligned}$$

for some constant c_1. Similarly,

$$\begin{aligned}
\left|\int_B R(z)e^{iz}\,\mathrm{d}z\right| &\leqslant \int_B |R(z)|\,e^{-\operatorname{Im} z}\,|\mathrm{d}z| \\
&\leqslant \int_B |R(z)|\cdot 1\cdot|\mathrm{d}z| \\
&\leqslant \int_B \frac{K}{|z|}\,|\mathrm{d}z| \\
&= \frac{K}{R}\cdot \ell(B) \\
&\leqslant \frac{K}{R}\cdot 2\cdot 2h
\end{aligned}$$

(since each piece of B is nearly a line segment of length h)

$$= \frac{4Kh}{R} = \frac{c_2 h}{R}$$

for some constant c_2.

So

$$\left|\int_{\Gamma_R} R(z)e^{iz}\,\mathrm{d}z\right| \leqslant c_1 e^{-h} + c_2\frac{h}{R}.$$

Now choose $h = \sqrt{R}$ so that

$$\left|\int_{\Gamma_R} R(z)e^{iz}\,\mathrm{d}z\right| \leqslant c_1 e^{-\sqrt{R}} + \frac{c_2}{\sqrt{R}}.$$

And so as $R \to \infty$,

$$\left|\int_{\Gamma_R} R(z)e^{iz}\,\mathrm{d}z\right| \to 0.$$

Hence

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_k \operatorname{Res}\big(R(z)e^{iz}; z_k\big)$$

where z_k are the residues of $R(z)e^{iz}$ in $\mathbb{H}$. Now take real imaginary parts to get what you want:

$$\int_{-\infty}^{\infty} R(x)\cos x\,\mathrm{d}x \text{ or } \int_{-\infty}^{\infty} R(x)\sin x\,\mathrm{d}x.$$

Example. Consider

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

We cannot simply say

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x$$

because $x = 0$ at $0 \in \mathbb{R}$ and e^{iz}/z has a pole at 0. Very clever trick:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\,\mathrm{d}x.$$

Now

$$\int_{C_R} \frac{e^{iz}-1}{z}\,\mathrm{d}z = \int_{-R}^{R} \frac{e^{ix}-1}{x}\,\mathrm{d}x + \int_{\Gamma_R} \frac{e^{iz}-1}{z}\,\mathrm{d}z = 0$$

by Cauchy as $(e^{iz}-1)/z$ has no poles anywhere, it has removeable singularities at $z = 0$. Thus

$$\begin{aligned} \int_{-R}^{R} \frac{e^{ix}-1}{x}\,\mathrm{d}x &= \int_{\Gamma_R} \frac{1-e^{iz}}{z}\,\mathrm{d}z \\ &= \int_{\Gamma_R} \frac{\mathrm{d}z}{z} - \int_{\Gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z \\ &\to \pi i \text{ as } R \to \infty. \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{iz}-1}{x}\,\mathrm{d}x = \pi i,$$

and so

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im} \pi i = \pi.$$

[Note that we didn't need residues at all to do this integral!]

Summing Series

Suppose we want to sum a series of the form

$$\sum_{n=-\infty}^{\infty} f(n).$$

We need a function g such that

$$\operatorname{Res}(g; n) = f(n) \text{ for } n \in \mathbb{Z}.$$

In view of our trick for calculating residues at simple poles, such a function is given by

$$\varphi(z) = \frac{\pi \cos \pi z}{\sin \pi z}.$$

Since if n is not a pole of f, setting $g(z) = \varphi(z) f(z)$,

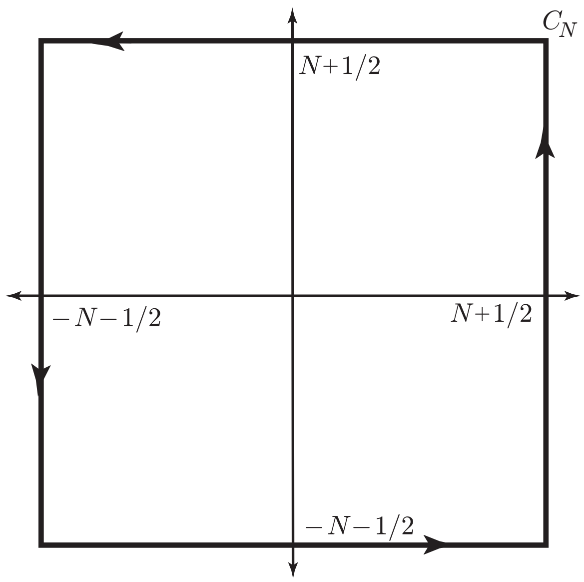
$$\operatorname{Res}(f(z)\pi \cot \pi z; n) = f(n) \frac{\pi \cos \pi z}{\pi \cos \pi z} = f(n).$$

Now apply the Residue theorem to

$$\int_{C_N} f(z) \pi \cot \pi z\,\mathrm{d}z$$

where C_N is a contour that contains (only) the integers $\{-N, \ldots, 0, \ldots, N\}$. Let

$$C_N = \partial\left(\left[-N - \frac{1}{2}, N + \frac{1}{2}\right]^2\right).$$



This contour avoids the poles of $\cot \pi z$ which are precisely $\mathbb{Z}$. In fact, $|\cot \pi z| < 2$ on C_N. Then

$$\begin{aligned}
|\cot \pi z| &= \left|\frac{\cos \pi z}{\sin \pi z}\right| = \left|\frac{\frac{e^{\pi z}+e^{-i\pi z}}{2}}{\frac{e^{\pi z}-e^{-i\pi z}}{2i}}\right| \\
&= \left|\frac{i(e^{2i\pi z}+1)}{e^{2\pi iz}-1}\right| = \left|\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1}\right| \\
&= \left|\frac{e^{2\pi i(N+1/2+iy)}+1}{e^{2\pi i(N+1/2+iy)}-1}\right| \\
&= \left|\frac{e^{2\pi iN+\pi i+2\pi y}+1}{e^{2\pi iN+\pi i+2\pi y}-1}\right| \\
&= \left|\frac{e^{\pi i-2\pi y}+1}{e^{\pi i-2\pi y}-1}\right| \text{ as } e^{2\pi iN} = 1 \\
&= \left|\frac{-e^{2\pi y}+1}{-e^{2\pi y}-1}\right| \text{ as } e^{\pi i} = -1 \\
&< 1.
\end{aligned}$$

Similarly, we ahve that $|\cot \pi z| < 1$ if $\operatorname{Re} z = -N - 1/2$ and $-N-1/2 \leqslant \operatorname{Im} z \leqslant$

$N+1/2$. If $\operatorname{Im} z = y = N+1/2$ and $-N-1/2 \leqslant x \leqslant N+1/2$, then

$$\begin{aligned}
|\cot \pi z| &= \left|\frac{e^{2\pi i}+1}{e^{2\pi i}-1}\right| \\
&= \left|\frac{e^{2\pi i(x+i(N+1/2))}+1}{e^{2\pi i(x+i(N+1/2))}-1}\right| \\
&= \left|\frac{\varepsilon^{2\pi i x-\pi(2N+1)}+1}{e^{2\pi i x-\pi(2N+1)}-1}\right| \\
&\leqslant \left|\frac{1+e^{-\pi(2N+1)}}{e^{-\pi(2N+1)}-1}\right| \\
&= \frac{1+e^{-\pi(2N+1)}}{1-e^{-\pi(2N+1)}} \\
&\leqslant \frac{1+e^{-\pi}}{1-e^{-\pi}} < 2.
\end{aligned}$$

Similarly, $|\cot \pi z| < 2$ for $\operatorname{Im} z = y = -N-1/2$ and $-N-1/2 \leqslant \operatorname{Re} z \leqslant N+1/2$.

Hence $|\cot \pi z| < 2$ on C_N and so

$$\begin{aligned}
\left|\int_{C_N} f(z)\pi \cot \pi z \, \mathrm{d}z\right| &\leqslant (8N+4)\cdot 2\cdot \pi \cdot \max_{z\in C_N} |f(z)| \\
&\leqslant A \max_{z\in C_N} |zf(z)| \qquad (9)
\end{aligned}$$

for some constant A since $|z|$ is comparable to N on C_N. By the Residue theorem,

$$\int_{C_N} f(z)\pi\cot\pi z\,\mathrm{d}z = 2\pi i\left[\sum_{\substack{n=-N\\ n\neq z_k}}^{N} f(n) + \sum_k \operatorname{Res}\left(f(z)\pi\cot\pi z; z_k\right)\right]$$

where z_k are the poles of f inside C_N. Assume that we have $|f(z)| \leqslant B$ for $|z|$ large for some constant B. This guarantees that $\sum f(n)$ converges except for the possibility of the poles in $\mathbb{Z}$, and also that

$$\max_{z\in C_N} |zf(z)| \to 0 \text{ as } N\to\infty.$$

So then by (**??**),

$$\left|\int_{C_N} f(z)\pi\cot\pi z\,\mathrm{d}z\right| \to 0$$

as $N\to\infty$. So letting $N\to\infty$, we have

$$\sum_{\substack{n=-N\\ n\neq z_k}}^{N} f(n) = -\sum_k \operatorname{Res}\left(f(z)\pi\cot\pi z; z_k\right)$$

where z_k are the poles of f.

Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2}.$$

So let $f(z) = 1/z^2$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right).$$

$\cot z$ has Laurent series expansion

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{1}{45} z^2 + \cdots$$

on $A(0, 0, \pi)$. So

$$\frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3} \frac{1}{z} - \frac{\pi^4 z}{45} + \cdots$$

on $A(0, 0, \pi)$. So

$$\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = \frac{-\pi^3}{3}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

For free, we also get that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. Integrals of the form

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) \, \mathrm{d}\theta$$

where R is a rational function. Natural substitution with $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta}$, and

$$\mathrm{d}\theta = -i \frac{\mathrm{d}z}{z}.$$

We get

$$-i \int\limits_{C(0,1)} R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \frac{\mathrm{d}z}{z}$$

as

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

and
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Now all we need to do is find residues of this new rational function at each of its poles inside $\mathbb{D}$.

Example. Consider
$$\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta} \text{ for } a > 1,$$

and
$$\begin{aligned}\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta} &= \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} \\ &= -i \int\limits_{C(0,1)} \frac{\mathrm{d}z}{2z(a+\frac{1}{2}(z+\frac{1}{z}))} \\ &= -i \int\limits_{C(0,1)} \frac{\mathrm{d}z}{z^2+2az+1}.\end{aligned}$$

Then $z^2+2az+1 = (z-\alpha)(z-\beta)$ for $\alpha = -a+\sqrt{a^2-1}$ and $\beta = -a-\sqrt{a^2-1}$ where $\sqrt{}$ is the ordinary square root function on $\mathbb{R}$. Clearly $\alpha\beta = 1$ and $|\alpha| < |\beta|$. Hence $|\alpha| < 1$ and $|\beta| > 1$. Residue at $z = a$ is

$$\lim_{z\to\alpha} \frac{(z-\alpha)}{z^2+2az+1} = \lim_{z\to\alpha}\frac{1}{z-\beta} = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{a^2-1}}.$$

Hence by the Residue theorem

$$\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta} = 2\pi i\cdot -i\cdot \frac{1}{2\sqrt{a^2-1}} = \frac{\pi}{\sqrt{a^2-1}}$$

17 The Argument Principle and Rouché's Theorem

Suppose f is analytic in $U \subset \mathbb{C}$ open, $a \in U$ is such that f has a zero of order m at a. Then $f(z) = (z-a)^m g(z)$ where g is analytic on U and $g(a) \neq 0$. Hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

for $z \in U$, $z \neq a$, and $g(z) \neq 0$. Now suppose f is analytic on U except at $z = a$, where is has a pole of order m. Then

$$f(z) = \frac{h(z)}{(z-a)^m}$$

where h is analytic on U with $h(a) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{h'(z)}{h(z)}$$

for $z \in U$, $z \neq a$, $h(z) \neq 0$.

17.1 The Argument Principle. *Let f be meromorphic on a domain $\Omega \subset \mathbb{C}$ with poles $p_1, \ldots, p_m$ and zeros $z_1, \ldots, z_n$ counted according to multiplicity. If $\Gamma \approx 0$ in U and doesn't pass through any zeros or poles of f, then*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{i=1}^{n} n(\Gamma, z_i) - \sum_{j=1}^{m} n(\Gamma, p_j).$$

Proof. By repeated applications of the earlier arguments,

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i} - \sum_{j=1}^{m} \frac{1}{z - p_j} + \frac{g'(z)}{g(z)} \tag{10}$$

where g is analytic on Ω and has no zeros or poles. Now integrate both sides of (**??**) over Γ and use the fact that

$$\int_\Gamma \frac{g'(z)}{g(z)} \, \mathrm{d}z = 0$$

by Cauchy's theorem as g'/g is analytic on U. ∎

Remark: Can think of f'/f as the derivative of $\log f$ if such a branch exists. However, if $\log f$ does not exist, then

$$\int_\Gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = 0$$

because it is a primitive of f'/f.
A straightforward generalization.

17.2 Corollary. *Let f, U, Γ be as in 17.1. Let g be another function analytic on U. Then*

$$\frac{1}{2\pi i} \int_\Gamma g(z) \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{i=1}^{n} g(z_i) n(\Gamma, z_i) - \sum_{j=1}^{m} g(p_j) n(\Gamma, p_j)$$

Proof.

$$\frac{g(z) f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{g(z)}{z - z_i} - \sum_{j=1}^{m} \frac{g(z)}{z - p_j} + g(z) \frac{h'(z)}{h(z)}.$$

Now we integrate both sides over Γ

$$\frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z - z_i} \, \mathrm{d}z = n(\Gamma, z_i) g(z_i)$$

by Cauchy's integral formula. Also

$$\int_\Gamma g(z)\frac{h'(z)}{h(z)}\,\mathrm{d}z = 0$$

by Cauchy's Theorem again as $g(z)h'(z)/h(z)$ is analytic on U. ■

Suppose $R > 0$ and f is one-to-one and analytic on a open set containing $D(a, R)$. Let $\Omega = f(D(a, R))$. If $|z_0 - a| < R$ and $\zeta_0 = f(z_0) \in \Omega$, then $f(z_0) - \zeta_0$ has one and only one zero in $D(a, R)$. If we then choose $g(z) = z$ on $D(a, R)$, then

$$\frac{1}{2\pi i}\int_{C(a,R)} \frac{zf'(z)}{f(z)-\zeta_0}\,\mathrm{d}z = z_0 n(C(a, R), z_0) = z_0.$$

We have shown:

17.3 Theorem. *Let f be analytic and one-to-one on an open set containing the closed disk $\overline{D}(a, R)$. If Ω is the domain $f(D(a, R))$, then $f^{-1}(\omega)$ is defined for each $\omega \in \Omega$ by the formula*

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_{C(a,R)} \frac{zf'(z)}{f(z)-\omega}\,\mathrm{d}z.$$

Note that we can relax the hypotheses slightly. Instead we can require just that f is analytic and one-to-one on $D(a, R)$ [really what we are saying is that f and f' extend continuously to $\overline{D}(a, R)$].

17.4 Rouché's Theorem. *Suppose f and g are mermorphic on an open neighborhood of $\overline{D}(a, R)$ with no zeros or poles on $C(a, R)$. If Z_f, Z_g and P_f, P_g denote the number of zeros and poles, respectively, of f and g, respectively, and if*

$$|f + g| < |f| + |g| \text{ on } C(a, R),$$

then

$$Z_f - P_f = Z_g - P_g.$$

Proof. From the hypothesis,

$$\left|\frac{f}{g} + 1\right| < \left|\frac{f}{g}\right| + 1 \tag{11}$$

on $C(a, R)$. If $\lambda(z) = f(z)/g(z)$ and $\lambda > 0$, then (**??**) would imply that

$$\lambda + 1 < \lambda + 1$$

which is impossible. Hence f/g maps $C(a, R)$ into $\Omega := \mathbb{C}\setminus[0, \infty)$ [then $f/g \neq 0$ on $C(a, R)$]. If ℓ is a branch of $\log z$ on Ω, then $\ell(f/g)$ is well-defined on an open neighborhood of $C(a, R)$ and is a primitive for

$$\frac{(f/g)'}{(f/g)}.$$

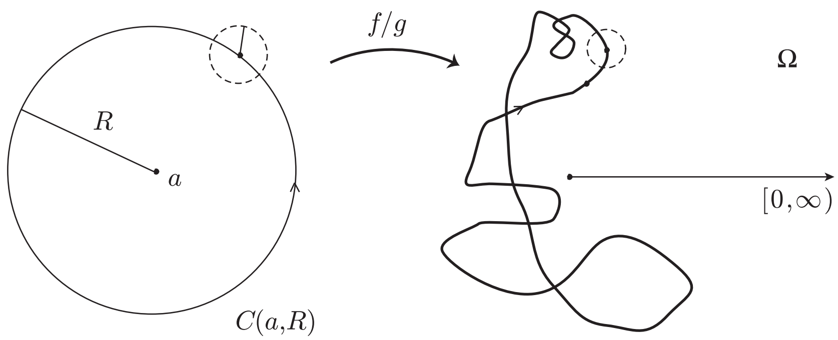


Figure 1: Construction of required neighborhood on $C(a, R)$ on which f/g is analytic and avoids $[0, \infty)$ by a simple compactness argument.

Thus

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{C(a,R)} \frac{(f/g)'}{(f/g)} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{C(a,R)} \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right) \mathrm{d}z \\ &= (Z_f - P_f) - (Z_g - P_g). \end{aligned}$$

Hence $Z_f - P_f = Z_g - P_g$. ■

Riemann Mapping Theorem. *Let Ω be a simply-connected domain and let $z_0 \in \Omega$. Then there exists a unique biholomorphic map of Ω onto $\mathbb{D}$ satisfying*

$$f(z_0) = 0 \qquad \textit{and} \qquad f'(z_0) > 0.$$