Trigonometric Integrals

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1. Trig Integrals

Our goal is to evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx \text{ and } \int \tan^m x \sec^n x \, dx$$

The relevant identities are

$\sin^2 x + \cos^2 x = 1$

and $\tan^2 x + 1 = \sec^2 x$

2. Example 1.

Use the identity $\sin^2 x = 1 - \cos^2 x$ to evaluate

$\int \sin^3 x \cos^4 x \, dx$ and

Our substitution is

$u = \cos x$

3. Sine–cosine integrals

To integrate $\int \sin^m x \cos^n x \, dx$:

If m is odd, substitute $u = \cos x$ and replace $\sin^2 x$ with $1 - \cos^2 x$.

If n is odd, substitute $u = \sin x$ and replace $\cos^2 x$ with $1 - \sin^2 x$.

What if both n and m are even?

4. Example 2

Integrate $\int_0^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$.

5. 3 types of TRIG

There are three types of trig:

Type I. Surveyor Trig. We use formulas such as $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ to find elevations of buildings.

Type II. EE Trig. We use trig to model periodic behavior such as alternating current as a wave.

Type III. Trig Identities. We use trig symbolically to prove mathematical equations involving the trig functions.

Over 90% of all trig used in Calculus is of Type III.

6. The hard case

When m and n are both even, we can use the following trig identities:

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

Letting $A = B = x$ gives the double angle formula

$\cos(2x) = \cos(x + x) = \cos x \cos x - \sin x \sin x$ or

$\cos(2x) = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$ Why?

$= 1 - 2\sin^2 x$ Why?

So $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

7. Example 3

Integrate $\displaystyle\int \cos^4 x \, dx$

8. Tangent–secant integrals

Integrate $\displaystyle\int \tan^m x \sec^n x \, dx$.

If the power of the secant, $n > 0$, is even, substitute $u = \tan x$ and save out a secant-squared factor.

If the power of the tangent, $m > 0$, is odd, substitute $u = \sec x$ and save out a secant-tangent factor.

9. Puzzler

Integrate $\displaystyle\int \tan x \sec^2 x \, dx$.

One way: Substitute $u = \tan x$

Second way: Substitute $u = \sec x$

Do these two methods give the same answer?

If not, "what up with that?"

10. Example 4

Integrate $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx.$

Substitute $u = \sec x$

Trick: Rewrite the integrand as

$\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\frac{\sec x}{\sec x}\,dx.$

11. Example 5

Integrate $\displaystyle\int \tan^3 7x \sec^4 7x\,dx.$

12. Example 6

Integrate $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx.$

Trick: Convert to sine and cosines.

13. Example 7

Integrate $\displaystyle\int_0^{\pi/8} \tan^4 2x\,dx.$

Trick: Write $\tan^4 2x = \tan^2 2x \tan^2 2x$

$= \tan^2 2x(\sec^2 2x - 1)$

14. Example 8 - The killer

Integrate $\displaystyle\int \sec^3 x\,dx.$

Trick: Integration by parts:

$u = \sec x$ and $dv = \sec^2 x\, dx$

15. When the angle changes

How do you integrate products of trig functions when the angle changes?

For example, $\int \cos 7x \cos 5x$

The secret is to combine the and difference formulas:

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Adding these two equations gives

$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$

or $\cos A \cos B = \frac{1}{2}\Big(\cos(A + B) + \cos(A - B)\Big)$

Plug in $A = 7x$:

$\cos 7x \cos 5x = \frac{1}{2}\big(\cos(7x + 5x) + \cos(7x - 5x)\big)$

$= \frac{1}{2}\big(\cos(12x) + \cos(2x)\big)$

16. Example completed

Thus, $\int \cos 7x \cos 5x$

$= \frac{1}{2}\int \big(\cos(12x) + \cos(2x)\big)$

$= \frac{1}{2}\Big(\frac{1}{12}\sin(12x) + \frac{1}{2}\sin(2x)\Big) + C$

$= \frac{1}{24}\sin(12x) + \frac{1}{4}\sin(2x) + C$

17. Variations

To integrate $\int \sin ax \sin bx$,

subtract the expansions for $\cos(A+B)$ and $\cos(A-B)$

To integrate $\int \sin ax \cos bx$,

add the expansions for $\sin(A+B)$ and $\sin(A-B)$

Integrate $\int \sin \frac{7}{2}x \cos \frac{3}{2}x \, dx$