David Lyon

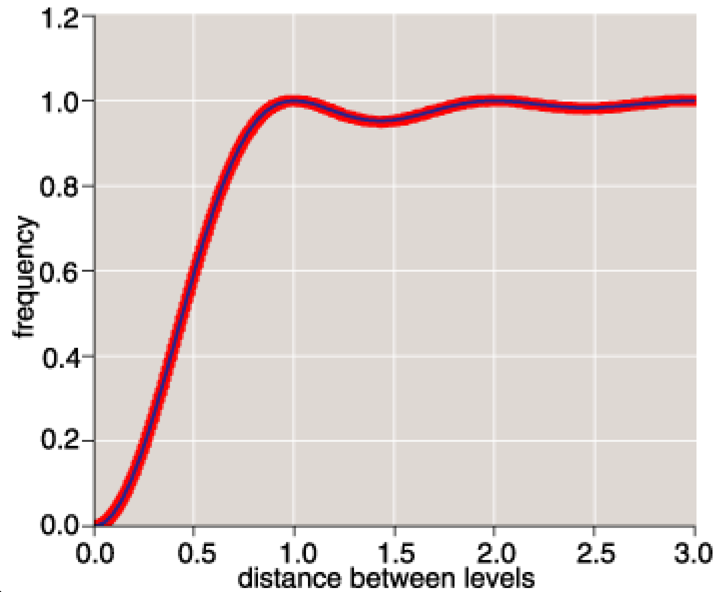
The Physics of the Riemann Zeta function

Abstract: One of the Clay Institute's Millennium Prize Problems is the Riemann Hypothesis. Interest in this problem has led to collaboration between mathematicians and physicists to study the Riemann Zeta function and related classes of functions called Zeta functions and L-functions. A unified picture of these functions is emerging which combines insights from mathematics with those from many areas of physics such as thermodynamics, quantum mechanics, chaos and random matrices. In this paper, I will give an overview of the connections between the Riemann Hypothesis and Physics.

The deceptively simple Riemann Zeta function $\zeta(s)$ is defined as follows, for complex s with real part > 1. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ In 1859, Bernhard Riemann published a paper showing how to analytically continue this function to the entire complex plane, giving a holomorphic function everywhere except for a simple pole at s=1. The paper also contained his famous hypothesis about the Riemann Zeta function, stating that the only zeroes are at the negative even integers and along the critical line Re(s) = 1/2. To prove this was problem number 8 on David Hilbert's famous list of 23 unsolved mathematical problems given at the International Congress of Mathematicians in 1900. Hilbert's list was a beacon to guide 20th century mathematicians in their explorations, leading to many important results and individual honors. In 2000, 100 years after Hilbert's list, the Clay Mathematics Institute proposed a list of 7 Millennium Problems. Each of the six remaining unsolved problems, including three in mathematics, two in physics, and one in computer science, carries a $1,000,000 bounty. One wonders, however, in this era of the declining value of the dollar, whether such a prize will attract interest for long. Among the three mathematical problems is to prove the Riemann Hypothesis. However, because the proof has eluded mathematicians for so long, they feel no shame in cooperating with the physics and computer science communities on this problem.

The Riemann Hypothesis had always been regarded as a problem in number theory until 1914, when David Hilbert and George Polya independently speculated that the zeroes of the form $\frac{1}{2} + it$ corresponded to some real eigenvalues t of a hermitian "Riemann Operator," so that these zeroes were spectral. There was little evidence for this until 1956, when the Selberg trace formula appeared. The Selberg trace formula is an extension of Fourier analysis to non-commutative operators on a compact Riemann surface. The idea is that the closed geodisics on the surface play the role of prime numbers. The spectrum is then not in frequency but rather in length. Many trace formulae for non-commutative topological groups were subsequently found, and the area of non-commutative harmonic analysis has been absorbed into the Langlands program, a large effort, begun in 1967, to channel the wisdom of number theory into knowledge about group representations. In the 21st century, the Langlands program has matured into the Geometric Langlands program, and many tantalizing connections between the Langlands program and modern physics have surfaced. The Riemann Zeta function's physical realization is merely one of the most humble of these connections, and thus one of the first to appear and possibly the key to unlocking the rest of the treasures we seek.

This connection first appeared in 1972, when Hugh Montgomery was studying the pair correlation function of the zeroes and found that the probability for two zeroes to be closer than a certain normalized distance together was very small, going as $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. The following graph shows this function plotted in blue, with the pair correlation function for a billion zeroes



near the 10^{23}rd zero plotted in red.

The importance of computer calculations in mathematics has increased constantly, as mathematicians have learned to perform numerical experiments by computer simulation to gain insight into complex problems. Of course, experimental data doesn't constitute a proof, so Montgomery's data was merely a hint as to the true nature of the Riemann Zeta function. In a famous chance meeting with Freeman Dyson at Princeton in 1972, Hugh Montgomery wrote down his pair correlation formula for large primes and Freeman Dyson immediately recognized the pair-correlation function for the eigenvalues of a special random matrix called the Gaussian Unitary Ensemble, which he had been working with since 1962 while attempting to model the energy resonances of heavy nuclei that appeared in neutron scattering experiments.

What is the the Gaussian Unitary Ensemble? The GUE for a dimension N is the space of all N by N Hermitian matrices along with a probability density P(N,H) over that space such that for a given matrix H,

$$P(N,H)= C_N exp(-\frac{N}{2}tr(H^2)) \quad where \quad C_N = (2\pi)^{-\frac{N(N+1)}{2}} N^{N^2/2}$$

The probability density is Gaussian in the trace, so that "lighter" matrices are favored. An equivalent definition of the Gaussian Unitary Ensemble is to use N^2 independent normal variables with mean 0 and variance 1 to populate an upper triangular NxN matrix. Then, multiply the diagonal by the square root of 2 and mirror the elements above the diagonal about the diagonal to construct a Hermitian matrix. The result is the Gaussian Unitary Ensemble, so called because its statistics remain unchanged by unitary transformations. The GUE is a multi-variable generalization of Gaussian statistics, assuming that instead of being independent, the variables are correlated binarily. This web of correlations leads to chaotic systems. A closely related matrix is the Gaussian Orthogonal Ensemble. We will see later that the relationship between the GOE and GUE has to do with time-reversibility.

The bible of the application of random matrix physics to nuclear spectra is "Random-matrix physics: Spectrum and strength fluctuations," published in 1981. [Brody] In this 96 page paper, the authors compared the statistics of many kinds of random matrices with spectral data from all types of nuclei. They wrote down a sum over single body non-interacting Hamiltonians and showed that the statistics will be Poisson, and thus couldn't be the Hamiltonian of heavy nuclei.

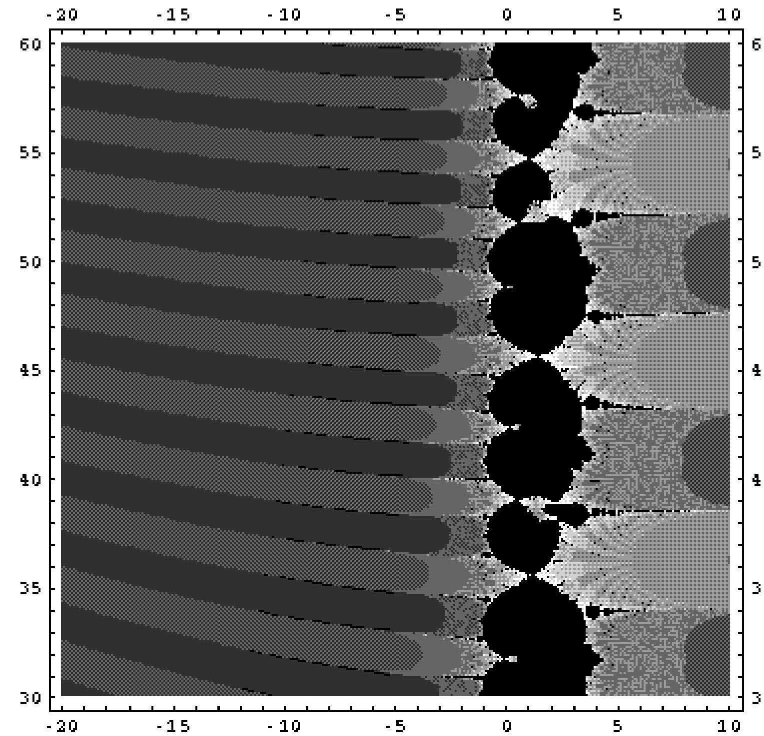
Then they proposed the next simplest model, a binary interaction model, and derived the Gaussian Orthogonal Ensemble, which fit the statistics of nuclei well. When many-particle interactions were included, the spectrum went from Gaussian to a semicircular form called the famous Wigner semicircle. But they noted that experiments showed Gaussian spectral distributions with very few hints of semicircular spectral distributions, and concluded that many particle interactions weren't very important. They then re-derived Dyson's 1962 result for the pair correlations between spectral lines, reproducing exactly Montgomery's formula. In the paper, they do most calculations using the GOE, and then at the end they do an analysis, designed by Wigner, that can separate the time-reversible and time-irreversible parts of the spectrum. What they find is that the time-reversible part goes as $d^{1/2}$, meaning that as nuclei become heavier, there is a crossover to GUE statistics, which becomes sharper as the size of the nucleus increases. Also, they find that the GUE is simpler to calculate with and produces results of similar accuracy as the GOE. They then mention that the level repulsion of the spectrum means that only limited information can be extracted from spectra. The rigidity of the spectrum wipes out whatever information it would have contained absent this level interaction. The spectral density of nuclei increases like $e^{sqrt(E)}$, so statistics were normalized against this exponential function in their theories to produce scale invariance in the spectra. Also, spectra from different angular momentum values had to be separated because only resonances with the same J showed repulsion. In other words, peaks with different J were uncorrelated while those with the same J were correlated. In fact, they confidently used the close approach of two energy levels to separate s states from p states in the data. The paper also included data from fissile nuclei, which behaved differently than stable nuclei. They concluded that random matrix theory was unable to reproduce the statistics from fissile nuclei and simply said that further study was needed. Light nuclei from Carbon to Magnesium obeyed mostly Poisson statistics because of symmetry in the nucleus. As the nuclei became heavier, the data showed more and more deviation from Poisson statistics and the nuclei became less symmetric. However, a combination of Poisson and random matrix statistics could fit the spectral line distributions. For very heavy nuclei like ^{238}U, with hundreds of spectral lines, pure random matrix statistics fit the spectral statistics almost exactly. However, spectra from different angular momentum values had to be separated, meaning that more than one matrix had to be used to model the full spectrum. In light nuclei, there is much less chaos, and Poisson statistics still applied. Nuclear physicists knew that large nuclei are far too complex to be modeled exactly, since in the confined space of the nucleus, each nucleon can interact with each other nucleon. Instead, they were agnostic about the interactions and simply created large model systems of fermions with random interactions. What they found is that the nucleus contains a mixture of integrable and chaotic orbits, and that by using random matrices, the energy level spacings of the matrix could be matched to the experimental energy level spacings of heavy nuclei quite precisely, showing exactly the level repulsion seen in experiment and in Montgomery's pair correlation function. The experimental data in favor of chaos in the nucleus was inarguable, as was the effectiveness of the GOE in matching the data.

Sir Michael Berry, in 1986, further investigated the Hilbert-Polya conjecture by assuming it to be true and analyzing the consequences. [Berry] He found that if the critical zeroes of ζ_R are treated as eigenvalues of a hermitian operator H, and this operator is the Hamiltonian of a quantum-mechanical system, then this system has the classical limit of chaotic orbits without time reversal symmetry. He gives references showing that only orbits that are chaotic without time-reversal symmetry have the statistics of the GUE. Orbits that are integrable have Poisson statistics and orbits that are chaotic but time reversible have GOE (Gaussian Orthogonal

Ensemble) statistics rather than GUE. This matches the conclusions of the earlier Brody paper, that separated time-reversible and time-irreversible parts by their GOE and GUE statistics. He derives an asymptotic large energy expression for the oscillations in the spectral density of chaotic orbits around the mean density and finds that it's the same as the asymptotic oscillations in the distribution of ζ_R zeroes. He notes that in the special case of the Selberg trace formula the expression is exact rather than asymptotic. These trace formulae equate a sum over lengths of orbits on one side (the prime numbers) to the sum over the energies on the other side (the ζ_R zeroes). In physics, we say that quantum interference cancels the energy contribution of open orbits, except for those very near closed orbits. Berry has characterized quantum-mechanical systems by their classical limits and showed which type gives the proper asymptotic statistics; however, despite an extensive search of systems of this type, mathematicians have been unable to find the exact Hamiltonian of the classical Riemann system that predicts the correct microscopic zeroes and thus the exact prime numbers.

Bernard Julia, in his 1990 paper, also makes the argument that we can better understand many number theoretical functions by viewing them as physical systems. The Riemann Zeta function can be seen as the partition function of a quantum gas, called the Bose Riemann Gas with the prime numbers labeling the eigenstates and with temperature s at very high chemical potential. State p has energy log p. $H|p\rangle = E_p|p\rangle$ and $E_p = E_o \log p$. One beauty of this transformation is that the multi-particle states are uniquely labeled by natural numbers since each number has a unique prime factorization. $|n\rangle = |p_1, p_2, p_3, \cdots\rangle = |p_1\rangle|p_2\rangle|p_3\rangle\cdots$ The Bose-Einstein condensate state for N particles is then $|2^N\rangle$, for example. In the Fermi Riemann gas, only states labeled by integers with distinct prime factors are allowed, while the Bose-Einstein Riemann gas allows all integer states. Julia notes that using the Euler product notation for $\zeta(s)$, which he calls ζ_R, the inverse of the Riemann Zeta function can be written using the Mobius function. $\frac{1}{\zeta_R} = Z_M(s) = \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ The value of the Mobius function $\mu(n)$ is $(-1)^N$ for N-particle fermionic states and zero for Bosonic states. Thus the partition function of the Fermi Riemann gas is almost the inverse of the partition function for the Bose Riemann gas, except one must take the supertrace instead of the trace because of the negative values of μ. He concludes that there is also something Fermionic about the prime numbers. Julia then studies the thermodynamics of the Bose Riemann gas, and concludes that the pole at s=1 reflects a breakdown of the grand canonical ensemble at the Hagedorn temperature of the system at β=1. This divergence happens because the distribution of energy levels becomes exponentially dense at high energies, which is related to the logarithmic decline in density of prime numbers at high values.

Although only real values of the fugacity are physically apparent, to see the full analytic behavior of thermodynamic functions one must extend ones view to the complex plane. Yang and Lee's classic paper [Yang] shows that in the limit as volume goes to infinity and at a fixed temperature, the zeroes of the pressure as a function of fugacity for a system of particles with hard-core repulsion approach the real line at exactly the phase transitions. At these points, the pressure is constant but the first derivative is discontinuous. Yang and Lee thus reduce the study of equations of state and phase transitions to the study of the distributions of roots of the grand partition function in the complex plane. In this case, the center of the critical strip corresponds to a bulk steady state of the system. Here is the Julia set of $\zeta(s)$, showing the basins of attraction



near each zero.

The real axis is along the bottom while the imaginary axis is along the side. Non-black areas are attracted to infinity. Julia proposed a program of study for the two variable Riemann Zeta function, which would correspond to a Riemann gas with finite chemical potential. This more generalized function would allow wider exploration of the phase space of number theoretical gases. In retrospect, it's not surprising that the exact Hamiltonian of any Zeta function has never appeared. As we have learned, Hamiltonians with any number of coupling constants will all flow towards the behavior of one with only binary interactions if that is the only relevant eigenvector. This means that all of the Hamiltonians in a particular universality class look exactly the same at the fixed points. However, working backwards to find any exact Hamiltonian by looking at behavior near the fixed point is impossible because of the semi-group property of RG flows. Presumably $\zeta(s)$ is completely general and has interactions of all orders in its associated Hamiltonian. However, all but the binary interaction disappear when we look at phase transitions or critical lines.

In 1991, I. Bakas and M.J. Bowick were motivated to study the Riemann gas as a toy model of gases with exponential densities of state, i.e. $p(E) \sim e^E$. [Bakas] They note that the standard Riemann gas is bosonic, but in a more general case we can use any Zeta function as a partition function. A Zeta function $L(s, \chi)$ looks just like $\zeta_R(s)$, except it has an arbitrary multiplicative arithmetic function in the numerator.

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

A multiplicative arithmetic function has the simple property that $\chi(m)\chi(n) = \chi(mn)$. For ζ_R, $\chi(n)=1$ for all n. They make the case that we can produce interactions between particles by twisted convolutions of their partition functions, a tool from non-commutative harmonic analysis, and give a few simple examples. They propose a more generalized research program, using Zeta functions of all types, convoluted together to give interactions between several kinds of particles. Investigating the thermodynamics of all sorts of L-functions seems more likely to enlighten than focusing on only the simplest one. The twisted convolutions seem an unnecessary complication at this stage, though.

Fields Medalist Alain Connes, inspired by Julia's paper, investigated in 1995 a physical system whose partition function is the Riemann Zeta function. [Connes] He finds that it shows a phase transition with spontaneous symmetry breaking at β=1. He then shows that the Galois group G = Gal(Q^{cycl}/Q), all permutations of the possible embeddings of the roots of unity in C*, is the symmetry group of this system. At high temperature, there is only a single disordered state. This is the case along the critical strip at β = ½, which is another indication that it represents a bulk phase. At low temperature (β>1), that symmetry is broken and the states are parameterized by the members of G = Gal(Q^{cycl}/Q). To prove this, he postulated a self-adjoint operator whose spectrum is the prime numbers. The case for natural numbers was solved in 1927 by Dirac, giving us the familiar creation and annihilation operators which opened the subject of quantum field theory. Connes used the creation and annihilation operators from bosonic quantum field theory and showed how they could be tensored together to cover the whole state space. It's essentially the bosonic gas construction of Julia using more sophisticated mathematical machinery. However, these creation and annihilation operators gave no information about the Hamiltonian of the exact Riemann system.

Knauf in 1998, seeking to improve upon Connes' result and realizing that using any Zeta function would work from Bakas' result, connected the empirical observation that Z(s) = ζ(*s-1*) / ζ(*s*) strongly resembles a spin chain in the critical strip for inverse temperature *s* between 0 and 1 to the Riemann Hypothesis, finding a logical sequence that would prove the RH if the thermodynamic analogy could be made exact. [Knauf] He used Euler's totient function, also known as the Euler phi function, as his multiplicative arithmetic function. This partition function is equivalent to that of a special infinite spin chain called the Number-Theoretical Spin Chain, which has exactly one phase transition, at s=2. He shows that the spin chain is ferromagnetic, and that by the Lee-Yang Theorem it must have a zero-free half plane. This means that a convergence proof for Z(s) for s > 3/2 would imply the RH. However, he can't prove convergence below s = 2 but only offer supporting evidence in the form of simplified transition matrices. In the end, he has to prove that three graphs have the Ramanujan property which means that they are optimal, but is unable to prove so. The appealing thing about this paper is his willingness to use a different Zeta functions than simply ζ(*s*) and therefore to arrive at an intuitive picture of the physical system. In fact, this spin chain model nearly provided a breakthrough since he was able to simplify the transition matrices to make the calculation more tractable. In the end, the Riemann Hypothesis was victorious as another attempt to prove it went down in flames, as we've come to expect.

By 2003, the picture had begun to sharpen. P. Lebeouf and A.G. Monastra investigated the thermodynamics of the Fermi Riemann gas and equated them with those of an atomic nucleus with chaotic classical dynamics. [Lboeuf] They discovered that there are two distinct types of thermodynamic fluctuations. The first type is local fluctuations, which occur on the order of the spacing between energy levels and are universally described by the GUE. The second type is long range correlations that are on an energy scale associated with the size of the system, i.e. with the energy of the lowest few energy levels, and depend on details of each system. They see both types manifested in nuclear phenomena. For example, widths of neutron resonances are described by the GUE while shell effects on the overall nuclear mass are clearly related to global correlations. This corresponds very well with what Berry said earlier, that the GUE is associated asymptotically with a whole class of Hamiltonians, but that finding a specific one that reproduces the exact low energy behavior of a system requires more information.

The body of the paper studied the properties of a mythical element called *Riemannium*. In pursuit of the RH, model physical systems have been constructed to attempt to gain insight into ζ_R and ζ_R has been used to give insight into real physical systems. In this paper, the latter relationship held. First, they separated the partition function into average and fluctuating parts so that the energy and entropy would likewise separate. When they studied the energy fluctuations in Riemannium at high chemical potential, they found them to be non-universal at all temperatures and dominated by the shortest orbits, which correspond to the smallest prime numbers. As they lowered the chemical potential, universal corrections appeared to the fluctuations from the long orbits, related to the GUE. Likewise, when they studied the entropy fluctuations, they found 3 regimes. For $T \ll T_\delta$, with $T_\delta = 1/(\pi \ln(\mu/2\pi))$ associated with the inverse of the (assumed large) chemical potential, the variance of the entropy fluctuations increased linearly. For $T_\delta < T < T_c$, with $T_c = 1/(\pi \ln(2))$ associated with the smallest orbit, the variance saturated to a constant value. These two regimes are consistent with universal random matrix theory. For temperatures above the critical temperature, the statistical properties of the gas were no longer universal, and the size of the fluctuations showed an exponential decay. This effect looks just like a finite size induced crossover, where the finite size effects appear when the correlation length reaches the size of the system. In this case, the parameter was the time scale. When the time scale became long enough for interactions to cross the entire system, the finite size of the system could no longer be ignored. Looking back to [Brody] from 1981, we see the same situation. Across all non-fissile large nuclei, the data showed universality and the statistics of random matrix theory. As the nuclei become small, the finite size of the system took over and the statistics become uncorrelated and dominated by the smallest orbits. In that case, detailed study of the properties of each nucleus was required within a shell model.

Finding the exact Hamiltonian which reproduces the Riemann Zeta function through direct physical analogy now seems unlikely, as does proving the Riemann Hypothesis thusly. As the papers I reviewed showed again and again, ζ_R and its permutations display the exact asymptotic statistical behavior of nonlinear physical systems. Berry's demonstration that different chaotic systems have different statistics is very interesting, and we now have new chaotic universality classes. However, we know that although the behavior of all members of a universality class flow towards the same fixed points, we can't simply reverse that flow to find the original system because a multitude of systems all show the same behavior near a fixed point and their irrelevant variables have vanished. Julia's suggestion for further research seems to be the most promising, to study multivariate zeta functions and use them as models for physical systems with several parameters to get a better idea of the whole phase space. The connection between L-functions and Gaussian statistics of correlated variables means that even if we can't find the physical systems that correspond to simple number-theoretical functions, we can certainly use number-theoretical functions as easily solvable models, giving the correct asymptotic statistics, of systems whose microscopic dynamics we don't understand. The fact that such old mathematics is so useful in studying such new physics is fascinating. Furthermore, Voronin's universality theorem, first published in 1975, says that in the critical strip, the Riemann Zeta function somewhere approximates to arbitrary accuracy the behavior of any smooth function on a finite disc. The theorem seems to say that it encodes all of the behavior that a chaotic quantum system can exhibit, so we need look no further for models of quantum chaos than Zeta functions. This result seems to be good news for physicists, who have already been using Zeta functions and random matrices as a tool for decades. There seems to be less news for mathematicians and the people at the Clay Mathematics Institute. But one gets the feeling that when this problem is finally solved as a consequence we will have a more unified picture of number theory and topology. Countless results in mathematics already assume the RH, so its proof will remove a mathematical logjam that has been piling up since 1859.

[Berry] Berry, M V, 1986, 'Riemann's zeta function: a model for quantum chaos?' in *Quantum chaos and statistical nuclear physics,* eds. T H Seligman and
H Nishioka, Springer Lecture Notes in Physics No. 263, 1-17.

[Connes] J.-B. Bost and Alain Connes, "Hecke Algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory", Selecta Math. (New Series), 1 (1995) 411-457.

[Julia] B. Julia. "Statistical theory of numbers", Number Theory and Physics, Les Houches Winter School, J.-M. Luck, P. Moussa et M. Waldschmidt eds., Springer-Verlag, 1990.

[Knauf] A. Knauf, *The number-theoretical spin chain and the Riemann zeroes*, Comm. Math. Phys.
196 (1998), no. 3, 703-731.

[Lebeouf] P. Leboeuf, A. G. Monastra,Quantum thermodynamic fluctuations of a chaotic Fermi-gas model, Nucl.Phys. A724 (2003) 69-84

[Bakas]I. Bakas and M.J. Bowick, "Curiosities of arithmetic gases", Journal of Mathematical Physics 32 (7) (1991) 1881-1884

[Yang] Statistical theory of equations of state and phase transitions. I. Theory of condensation Yang, C.N., Lee, T.D. 1952 *Physical Review* 87 (3), pp. 404-409

[Brody] Brody, T.A., Flores, J., French, J.B., Mello, P.A., Pandey, A., Wong, S.S.M. “Random-matrix physics: Spectrum and strength fluctuations” 1981 *Reviews of Modern Physics* 53 (3), pp. 385-479