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TRACKING AND KALMAN FILTERING MADE EASY

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To Larry and Vera,
Richard and Connie,
and Daniel, the little miracle

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PREFACE

At last a book that hopefully will take the mystery and drudgery out of the g–h, α–β, g–h–k, α–β–γ and Kalman filters and makes them a joy. Many books written in the past on this subject have been either geared to the tracking filter specialist or difficult to read. This book covers these filters from very simple physical and geometric approaches. Extensive, simple and useful design equations, procedures, and curves are presented. These should permit the reader to very quickly and simply design tracking filters and determine their performance with even just a pocket calculator. Many examples are presented to give the reader insight into the design and performance of these filters. Extensive homework problems and their solutions are given. These problems form an integral instructional part of the book through extensive numerical design examples and through the derivation of very key results stated without proof in the text, such as the derivation of the equations for the estimation of the accuracies of the various filters [see Note (1) on page 388]. Covered also in simple terms is the least-squares filtering problem and the orthonormal transformation procedures for doing least-squares filtering.

The book is intended for those not familiar with tracking at all as well as for those familiar with certain areas who could benefit from the physical insight derived from learning how the various filters are related, and for those who are specialists in one area of filtering but not familiar with other areas covered. For example, the book covers in extremely simple physical and geometric terms the Gram–Schmidt, Givens, and Householder orthonormal transformation procedures for doing the filtering and least-square estimation problem. How these procedures reduce sensitivity to computer round-off errors is presented. A simple explanation of both the classical and modified Gram–Schmidt procedures is given. Why the latter is less sensitive to round-off errors is explained in

physical terms. For the first time the discrete-time orthogonal Legendre polynomial (DOLP) procedure is related to the voltage-processing procedures.

Important real-world issues such as how to cope with clutter returns, elimination of redundant target detections (observation-merging or clustering), editing for inconsistent data, track-start and track-drop rules, and data association (e.g., the nearest-neighbor approach and track before detection) are covered in clear terms. The problem of tracking with the very commonly used chirp waveform (a linear-frequency-modulated waveform) is explained simply with useful design curves given. Also explained is the important moving-target detector (MTD) technique for canceling clutter.

The Appendix gives a comparison of the Kalman filter (1960) with the Swerling filter (1959). This Appendix is written by Peter Swerling. It is time for him to receive due credit for his contribution to the ''Kalman–Swerling'' filter.

The book is intended for home study by the practicing engineer as well as for use in a course on the subject. The author has successfully taught such a course using the notes that led to this book. The book is also intended as a design reference book on tracking and estimation due to its extensive design curves, tables, and useful equations.

It is hoped that engineers, scientists, and mathematicians from a broad range of disciplines will find the book very useful. In addition to covering and relating the g–h, α–β, g–h–k, α–β–γ, Kalman filters, and the voltage-processing methods for filtering and least-squares estimation, the use of the voltage-processing methods for sidelobe canceling and adaptive-array processing are explained and shown to be the same mathematically as the tracking and estimated problems. The massively parallel systolic array sidelobe canceler processor is explained in simple terms. Those engineers, scientists, and mathematicians who come from a mathematical background should get a good feel for how the least-squares estimation techniques apply to practical systems like radars. Explained to them are matched filtering, chirp waveforms, methods for dealing with clutter, the issue of data association, and the MTD clutter rejection technique. Those with an understanding from the radar point of view should find the explanation of the usually very mathematical Gram–Schmidt, Givens, and Householder voltage-processing (also called square-root) techniques very easy to understand. Introduced to them are the important concepts of ill-conditioning and computational accuracy issues. The classical Gram–Schmidt and modified Gram–Schmidt procedures are covered also, as well as why one gives much more accurate results. Hopefully those engineers, scientists, and mathematicians who like to read things for their beauty will find it in the results and relationships given here. The book is primarily intended to be light reading and to be enjoyed. It is a book for those who need or want to learn about filtering and estimation but prefer not to plow through difficult esoteric material and who would rather enjoy the experience. We could have called it ''The Joy of Filtering.''

The first part of the text develops the g–h, g–h–k, α–β, α–β–γ, and Kalman filters. Chapter 1 starts with a very easy heuristic development of g–h

filters for a simple constant-velocity target in "lineland" (one-dimensional space, in contrast to the more complicated two-dimensional "flatland"). Section 1.2.5 gives the *g–h* filter, which minimizes the transient error resulting from a step change in the target velocity. This is the well-known Benedict–Bordner filter. Section 1.2.6 develops the *g–h* filter from a completely different, common-sense, physical point of view, that of least-squares fitting a straight line to a set of range measurements. This leads to the critically damped (also called discounted least-squares and fading-memory) filter. Next, several example designs are given. The author believes that the best way to learn a subject is through examples, and so numerous examples are given in Section 1.2.7 and in the homework problems at the end of the book.

Section 1.2.9 gives the conditions (on *g* and *h*) for a *g–h* filter to be stable (these conditions are derived in problem 1.2.9-1). How to initiate tracking with a *g–h* filter is covered in Section 1.2.10. A filter (the *g–h–k* filter) for tracking a target having a constant acceleration is covered in Section 1.3. Coordinate selection is covered in Section 1.5.

The Kalman filter is introduced in Chapter 2 and related to the Benedict–Bordner filter, whose equations are derived from the Kalman filter in Problem 2.4-1. Reasons for using the Kalman filter are discussed in Section 2.2, while Section 2.3 gives a physical feel for how the Kalman filter works in an optimum way on the data to give us a best estimate. The Kalman filter is put in matrix form in Section 2.4, not to impress, but because in this form the Kalman filter applies way beyond lineland—to multidimensional space.

Section 2.6 gives a very simple derivation of the Kalman filter. It requires differentiation of a matrix equation. But even if you have never done differentiation of a matrix equation, you will be able to follow this derivation. In fact, you will learn how to do matrix differentiation in the process! If you had this derivation back in 1958 and told the world, it would be your name filter instead of the Kalman filter. You would have gotten the IEEE Medal of Honor and $20,000 tax-free and the $340,000 Kyoto Prize, equivalent to the Nobel Prize but also given to engineers. You would be world famous.

In Section 2.9 the Singer *g–h–k* Kalman filter is explained and derived. Extremely useful *g–h–k* filter design curves are presented in Section 2.10 together with an example in the text and many more in Problems 2.10-1 through 2.10-17. The issues of the selection of the type of *g–h* filter is covered in Section 2.11.

Chapter 3 covers the real-world problem of tracking in clutter. The use of the track-before-detect retrospective detector is described (Section 3.1.1). Also covered is the important MTD clutter suppression technique (Section 3.1.2.1). Issues of eliminating redundant detections by observation merging or clustering are covered (Section 3.1.2.2) as well as techniques for editing out inconsistent data (Section 3.1.3), combining clutter suppression with track initiation (Section 3.1.4), track-start and track-drop rules (Section 3.2), data association (Section 3.3), and track-while-scan systems (Section 3.4).

In Section 3.5 a tutorial is given on matched filtering and the very commonly used chirp waveform. This is followed by a discussion of the range bias error problem associated with using this waveform and how this bias can be used to advantage by choosing a chirp waveform that predicts the future—a fortune-telling radar.

The second part of the book covers least-squares filtering, its power and voltage-processing approaches. Also, the solution of the least-squares filtering problem via the use of the DOLP technique is covered and related to voltage-processing approaches. Another simple derivation of the Kalman filter is presented and additional properties of the Kalman filter given. Finally, how to handle nonlinear measurement equations and nonlinear equations of motion are discussed (the extended Kalman filter).

Chapter 4 starts with a simple formulation of the least-squares estimation problem and gives its power method solution, which is derived both by simple differentiation (Section 4.1) and by simple geometry considerations (Section 4.2). This is followed by a very simple explanation of the Gram–Schmidt voltage-processing (square-root) method for solving the least-squares problem (Section 4.3). The voltage-processing approach has the advantage of being much less sensitive to computer round-off errors, with about half as many bits being required to achieve the same accuracy. The voltage-processing approach has the advantage of not requiring a matrix inverse, as does the power method.

In Section 4.4, it is shown that the mathematics for the solution of the tracking least-squares problem is identical to that for the radar and communications sidelobe canceling and adaptive nulling problems. Furthermore, it is shown how the Gram–Schmidt voltage-processing approach can be used for the sidelobe canceling and adaptive nulling problem.

Often the accuracy of the measurements of a tracker varies from one time to another. For this case, in fitting a trajectory to the measurements, one would like to make the trajectory fit closer to the accurate data. The minimum-variance least-squares estimate procedure presented in Section 4.5 does this. The more accurate the measurement, the closer the curve fit is to the measurement.

The fixed-memory polynomial filter is covered in Chapter 5. In Section 5.3 the DOLP approach is applied to the tracking and least-squares problem for the important cases where the target trajectory or data points (of which there are a fixed number $L+1$) are approximated by a polynomial fit of some degree m. This method also has the advantage of not requiring a matrix inversion (as does the power method of Section 4.1). Also, its solution is much less sensitive to computer round-off errors, half as many bits being required by the computer.

The convenient and useful representation of the polynomial fit of degree m in terms of the target equation motion derivatives (first m derivatives) is given in Section 5.4. A useful general solution to the DOLP least-squares estimate for a polynomial fit that is easily solved on a computer is given in Section 5.5. Sections 5.6 through 5.10 present the variance and bias errors for the least-squares solution and discusses how to balance these errors. The important

method of trend removal to lower the variance and bias errors is discussed in Section 5.11.

In Chapter 5, the least-squares solution is based on the assumption of a fixed number $L + 1$ of measurements. In this case, when a new measurement is made, the oldest measurement is dropped in order to keep the number measurements on which the trajectory estimate is based equal to the fixed number $L + 1$. In Chapter 6 we consider the case when a new measurement is made, we no longer throw away the oldest data. Such a filter is called a growing-memory filter. Specifically, an mth-degree polynomial is fitted to the data set, which now grows with time, that is, L increases with time. This filter is shown to lead to the easy-to-use recursive growing-memory g–h filter used for track initiation in Section 1.2.10. The recursive g–h–k ($m = 2$) and g–h–k–l ($m = 3$) versions of this filter are also presented. The issues of stability, track initiation, root-mean-square (rms) error, and bias errors are discussed.

In Chapter 7 the least-squares polynomial fit to the data is given for the case where the error of the fit is allowed to grow the older the data. In effect, we pay less and less attention to the data the older it is. This type of filter is called a fading-memory filter or discounted least-squares filter. This filter is shown to lead to the useful recursive fading-memory g–h filter of Section 1.2.6 when the polynomial being fitted to is degree $m = 1$. Recursive versions of this filter that apply to the case when the polynomial being fitted has degree $m = 2, 3, 4$ are also given. The issues of stability, rms error, track initiation, and equivalence to the growing-memory filters are also covered.

In Chapter 8 the polynomial description of the target dynamics is given in terms of a linear vector differential equation. This equation is shown to be very useful for obtaining the transition matrix for the target dynamics by either numerical integration or a power series in terms of the matrix coefficient of the differential equation.

In Chapter 9 the Bayes filter is derived (Problem 9.4-1) and in turn from it the Kalman filter is again derived (Problem 9.3-1). In Chapters 10 through 14 the voltage least-squares algorithms are revisited. The issues of sensitivity to computer round-off error in obtaining the inverse of a matrix are elaborated in Section 10.1. Section 10.2 explains physically why the voltage least-squares algorithm (square-root processing) reduces the sensitivity to computer round-off errors. Chapter 11 describes the Givens orthonormal transformation voltage algorithm. The massively parallel systolic array implementation of the Givens algorithm is detailed in Section 11.3. This implementation makes use of the CORDIC algorithm used in the Hewlett-Packard hand calculators for trigonometric computations.

The Householder orthonormal transformation voltage algorithm is described in Chapter 12. The Gram–Schmidt orthonormal transformation voltage algorithm is revisited in Chapter 13, with classical and modified versions explained in simple terms. These different voltage least-squares algorithms are compared in Section 14.1 and to QR decomposition in Section 14.2. A recursive version is developed in Section 14.3. Section 14.4 relates these voltage-

processing orthonormal transformation methods to the DOLP approach used in Section 5.3 for obtaining a polynomial fit to data. The two methods are shown to be essentially identical. The square-root Kalman filter, which is less sensitive to round-off errors, is discussed in Section 14.5.

Up until now the deterministic part of the target model was assumed to be time invariant. For example, if a polynomial fit of degree m was used for the target dynamics, the coefficients of this polynomial fit are constant with time. Chapter 15 treats the case of time-varying target dynamics.

The Kalman and Bayes filters developed up until now depend on the observation scheme being linear. This is not always the situation. For example, if we are measuring the target range R and azimuth angle θ but keep track of the target using the east-north x, y coordinates of the target with a Kalman filter, then errors in the measurement of R and θ are not linearly related to the resulting error in x and y because

$$x = R \cos \theta \tag{1}$$

and

$$y = R \sin \theta \tag{2}$$

where θ is the target angle measured relative to the x axis. Section 16.2 shows how to simply handle this situation. Basically what is done is to linearize Eqs. (1) and (2) by using the first terms of a Taylor expansion of the inverse equations to (1) and (2) which are

$$R = \sqrt{x^2 + y^2} \tag{3}$$

$$\theta = \tan \frac{y}{x} \tag{4}$$

Similarly the equations of motion have to be linear to apply the Kalman–Bayes filters. Section 16.3 describes how a nonlinear equation of motion can be linearized, again by using the first term of a Taylor expansion of the nonlinear equations of motion. The important example of linearization of the nonlinear observation equations obtained when observing a target in spherical coordinates (R, θ, ϕ) while tracking it in rectangular (x, y, z) coordinates is given. The example of the linearization of the nonlinear target dynamics equations obtained when tracking a projectile in the atmosphere is detailed. Atmospheric drag on the projectile is factored in.

In Chapter 17 the technique for linearizing the nonlinear observation equations and dynamics target equations in order to apply the recursive Kalman and Bayes filters is detailed. The application of these linearizations to a nonlinear problem in order to handle the Kalman filter is called the extended Kalman filter. It is also the filter Swerling originally developed (without the

target process noise). The Chapter 16 application of the tracking of a ballistic projectile through the atmosphere is again used as an example.

The form of the Kalman filter given in Kalman's original paper is different from the forms given up until now. In Chapter 18 the form given until now is related to the form given by Kalman. In addition, some of the fundamental results given in Kalman's original paper are summarized here.

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E. B.

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