Southeast Asian Bulletin of Mathematics (2009) 33: 207–208

Southeast Asian
Bulletin of
Mathematics

A Note on Centrality in Infinite Groups

Alireza Abdollahi*
Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
Email: a.abdollahi@math.ui.ac.ir

Ali Tavakoli
Azad University of Majlesi, Majlesi, Isfahan, Iran

Received 27 March 2006
Accepted 26 April 2006

Communicated by Xiuyun Guo

AMS Mathematics Subject Classification(2000): 20F99

Abstract. In this short note, we prove that an element a of an infinite group is central if and only if $aX \cap Xa \neq \varnothing$ for all infinite subsets X.

Keywords: Centrality in groups; Infinite groups.

1. Introduction

The main result of [1] states that an infinite group G is abelian if and only if $XY \cap YX \neq \varnothing$ for all infinite subsets X and Y of G. This result can be interpreted as an criterion of commutativity in terms of infinite subsets for infinite groups. In this note, we prove a similar criterion for the centrality of infinite groups.

Theorem. *Let G be an infinite group and $a \in G$. Then $a \in Z(G)$ if and only if $aX \cap Xa \neq \varnothing$ for all infinite subsets X of G.*

As in [3], a subset X of a group G is called a Sidon subset of the second kind

*Supported by the Center of Excellence for Mathematics, University of Isfahan as well as partially supported by a grant from IPM (No. 87200118).

if for every 4-tuple (x, y, z, w) of elements of X with $|\{x, y, z, w\}| \geq 3$, $xy^{-1} \neq zw^{-1}$. In Proposition 8 of [3], it is proved that every infinite subset of a group contains an infinite Sidon subset of the second kind (see for some generalizations [2]). We use this combinatorial result in our proof of the Theorem.

2. Proof of the Theorem

We first prove that the centralizer

$$C = C_G(a) = \{x \in G \mid ax = xa\}$$

of a in G is infinite. Suppose, on the contrary, that C is finite. Then $T = \{a^g \mid g \in G\} \backslash C$ is infinite. Define inductively a sequence $x_1, x_2, \dots$ of elements of T as follows:

Choose x_1 to be an arbitrary element of T and then take x_{n+1} to be an arbitrary element of the set $M_n = T \backslash \{x_i^a, x_i^{a^{-1}} \mid i = 1, \dots, n\}$. Then, it follows that $aX \cap Xa = \varnothing$, where $X = \{x_1, x_2, \dots\}$. Thus C is infinite.

By Proposition 8 of [3], there exists an infinite Sidon subsets of the second kind such that $S \subset C$. Let b be any element of G. We have to prove that $ab = ba$. By hypothesis, we have $a(bS) \cap (bS)a \neq \varnothing$. Hence $abs = bta$ for some $s, t \in S$. Thus, as $t \in C$, $abs = bat$ and so $a^{-1}b^{-1}ab = ts^{-1}$. If $t = s$, we are done. So suppose, for a contradiction, that $t \neq s$ and consider the infinite subset $S' = S \backslash \{s, t\}$. Then by hypothesis, $a(bS') \cap (bS')a \neq \varnothing$. It follows that $a^{-1}b^{-1}ab = t's'^{-1}$ for some $t', s' \in S'$. Hence $ts^{-1} = t's'^{-1}$; this is a contradiction, since S is a Sidon subset of second kind and $|\{t, s, t', s'\}| \geq 3$. Thus the proof is complete. ■

References

[1] A. Abdollahi, A.M. Hassanabadi, A characterization of infinite abelian groups, *Bull. Iranian Math. Soc.* **24** (2) (1998) 41–48.

[2] A. Abdollahi, A.M. Hassanabadi, A permutability problem in infinite groups and Ramsey's theorem, *Bull. Austral. Math. Soc.* **64** (2001) 27–31.

[3] L. Babai, T.S. Sós, Sidon sets in groups and induced subgraphs of Cayley graphs, *European J. Combin.* **6** (1985) 101–114.