Scientia Magna
Vol. 4 (2008), No. 1, 79-82

The mean value of a new arithmetical function

Jin Zhang[†‡] and Pei Zhang [†]

[†]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
[‡] Department of Mathematics, Xi'an Normal School, Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary and the analytic methods to study the mean value properties of a Smarandache multiplicative function, and give two sharper asymptotic formulae for it.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, we call an arithmetical function $f(n)$ as the Smarandache multiplicative function if for any positive integers m and n with $(m, n) = 1$, we have $f(mn) = \max\{f(m), f(n)\}$. For example, the Smarandache function $S(n)$ and the Smarandache LCM function $SL(n)$ both are Smarandache multiplicative functions. Now we define a new Smarandache multiplicative function $f(n)$ as follows: $f(1) = 1$; If $n > 1$, then $f(n) = \max\limits_{1\leq i\leq k}\left\{\frac{1}{\alpha_i+1}\right\}$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into prime powers. The first few values of $f(n)$ are $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{2}$, $f(4) = \frac{1}{3}$, $f(5) = \frac{1}{2}$, $f(6) = \frac{1}{2}$, $f(7) = \frac{1}{2}$, $f(8) = \frac{1}{4}$, $f(9) = \frac{1}{3}$, $f(10) = \frac{1}{2}$, $f(11) = \frac{1}{2}$, $\cdots\cdots$.

Generally, for any prime p and positive integer α, we have $f(p^\alpha) = \frac{1}{1+\alpha}$. About the elementary properties of $f(n)$, it seems that none had studied it before. This function is interesting, because its value only depend on the power of primes. The main purpose of this paper is using the elementary and the analytic methods to study the mean value properties of $f(n)$, and give two sharper asymptotic formulas for it. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} f(n) = \frac{1}{2}x\ln\ln x + c\cdot x + O\left(\frac{x}{\ln x}\right),$$

where c is a computable constant.

Theorem 2. For any real number $x > 1$, we also have the asymptotic formula

$$\sum_{n\leq x}\left(f(n)-\frac{1}{2}\right)^2 = \frac{1}{36}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}\cdot\ln\ln x + d\cdot\sqrt{x} + O\left(x^{\frac{1}{3}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and d is a computable constant.

§2. Proof of the theorems

In this section, we shall using the elementary and the analytic methods to prove our Theorems. First we give following two simple lemmas:

Lemma 1. Let A denotes the set of all square-full numbers. Then we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\cdot x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2. Let B denotes the set of all cubic-full numbers. Then we have

$$\sum_{\substack{n\leq x\\ n\in B}} 1 = N\cdot x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}}\right),$$

where N is a computable constant.

Proof. The proof of these two Lemmas can be found in reference [3].

Now we use these two simple Lemmas to complete the proof of our Theorems. In fact, for any positive integer $n > 1$, we can write it as $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then from the definition of $f(n)$, we have

$$\sum_{n\leq x} f(n) = \sum_{\substack{n\leq x\\ n\in A}} f(n) + \sum_{\substack{n\leq x\\ n\in B}} f(n),$$

where A denotes the set of all square-full numbers. That is, $n > 1$, and for any prime p, if $p \mid n$, then $p^2 \mid n$. B denotes the set of all positive integers with $n \notin A$. Note that $f(n) \ll 1$, from the definition of A and Lemma 1 we have

$$\sum_{\substack{n\leq x\\ n\in A}} f(n) = O\left(x^{\frac{1}{2}}\right). \tag{1}$$

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in B}} f(n) &= \sum_{\substack{np\leq x\\ (n,\ p)=1}} f(n) = \sum_{p\leq x}\sum_{\substack{n\leq \frac{x}{p}\\ (n,\ p)=1}} \frac{1}{2}\\ &= \frac{1}{2}\sum_{p\leq x}\left(\frac{x}{p} - \frac{x}{p^2} + O(1)\right)\\ &= \frac{x}{2}\sum_{p\leq x}\frac{1}{p} - \frac{x}{2}\sum_{p\leq x}\frac{1}{p^2} + O\left(\frac{1}{2}\sum_{p\leq x} 1\right).\end{aligned} \tag{2}$$

Note that

$$\sum_{p\leq x}\frac{1}{p} = \ln\ln x + c + O\left(\frac{1}{\ln x}\right) \text{ (see Theorem 4.12 of reference [2]),}$$

$$\sum_{p\leq x}\frac{1}{p^2} = \sum_{p}\frac{1}{p^2} - \sum_{p>x}\frac{1}{p^2} = d + O\left(\frac{1}{x}\right),$$

where c and d are two computable constants.

And the Prime Theorem (see Theorem 3.2 of reference [3]):

$$\pi(x) = \sum_{p\le x} 1 = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

So from (2) we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in B}} f(n) &= \frac{x}{2}\left(\ln\ln x + c + O\left(\frac{1}{\ln x}\right)\right) - \frac{x}{2}\left(d + O\left(\frac{1}{x}\right)\right) + O\left(\frac{1}{2}\left(\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)\right)\right)\\ &= \frac{1}{2}x\ln\ln x + \frac{c}{2}x - \frac{d}{2}x + O\left(\frac{x}{\ln x}\right)\\ &= \frac{1}{2}x\ln\ln x + \lambda x + O\left(\frac{x}{\ln x}\right), \qquad (3)\end{aligned}$$

where λ is a computable constant.

Now combining (1) and (3) we may immediately get

$$\begin{aligned}\sum_{n\le x} f(n) &= 1 + \sum_{\substack{n\le x\\ n\in A}} f(n) + \sum_{\substack{n\le x\\ n\in B}} f(n)\\ &= 1 + O\left(x^{\frac{1}{2}}\right) + \frac{1}{2}x\ln\ln x + \lambda\cdot x + O\left(\frac{x}{\ln x}\right)\\ &= \frac{1}{2}x\ln\ln x + \lambda\cdot x + O\left(\frac{x}{\ln x}\right),\end{aligned}$$

where λ is a computable constant.

This proves Theorem 1.

Now we complete the proof of Theorem 2. From the definition of $f(n)$ and the properties of square-full numbers, we have

$$\begin{aligned}\sum_{n\le x}\left(f(n) - \frac{1}{2}\right)^2 &= \frac{1}{4} + \sum_{\substack{n\le x\\ n\in A}}\left(f(n) - \frac{1}{2}\right)^2 + \sum_{\substack{n\le x\\ n\notin A}}\left(f(n) - \frac{1}{2}\right)^2\\ &= \frac{1}{4} + \sum_{\substack{n\le x\\ n\in A}}\left(f(n) - \frac{1}{2}\right)^2.\end{aligned}$$

where A also denotes the set of all square-full numbers. Let C denotes the set of all cubic-full numbers. Then from the properties of square-full numbers, Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2 &= \sum_{\substack{np^2\le x\\ (n,\ p)=1,\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2+\sum_{\substack{n\le x\\ n\in C}}\left(f(n)-\frac{1}{2}\right)^2\\
&= \sum_{p^2\le x}\sum_{\substack{n\le \frac{x}{p^2}\\ (n,\ p)=1,\ n\in A}}\left(\frac{1}{3}-\frac{1}{2}\right)^2+O\left(\sum_{\substack{n\le x\\ n\in C}}1\right)\\
&= \sum_{p^2\le x}\left(\sum_{\substack{n\le \frac{x}{p^2}\\ n\in A}}\frac{1}{36}-\sum_{\substack{n\le \frac{x}{p^4}\\ n\in A}}\frac{1}{36}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{1}{36}\sum_{p^2\le x}\left(c\cdot\frac{x^{\frac{1}{2}}}{p}-c\cdot\frac{x^{\frac{1}{2}}}{p^2}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{c}{36}\cdot\sqrt{x}\cdot\sum_{p\le\sqrt{x}}\left(\frac{1}{p}-\frac{1}{p^2}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{1}{36}c\cdot\sqrt{x}\cdot\ln\ln x+d\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right).
\end{aligned}$$

where $c=\dfrac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}$, d is a computable constant.

So we have the asymptotic formula

$$\sum_{n\le x}\left(f(n)-\frac{1}{2}\right)^2=\frac{1}{36}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}\cdot\ln\ln x+d\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right).$$

This completes the proof of Theorem 2.

References

[1] F.Smarandache, Only problems, Not solutions, Xiquan Publishing House, Chicago, 1993.

[2] Tom M. Apostol, Introduction to Analytical Number Theory, Spring-Verlag, New York, 1976.

[3] Aleksandar Ivić, The Riemann Zeta-Function, Dover Publications, New York, 2003.

[4] Zhefeng Xu, On the k-full Number Sequences, Smarandache Notions Journal, **14**(2004), 159-163.

[5] Jinping Ma, The Smarandache multiplicative function, Scientia Magna, **1**(2005), No.1, 125-128.