Math 259: Introduction to Analytic Number Theory

The Riemann zeta function and its functional equation
(and a review of the Gamma function and Poisson summation)

Recall Euler's identity:

$$[\zeta(s) :=] \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \left(\sum_{c_p=1}^{\infty} p^{-c_p s} \right) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}. \tag{1}$$

We showed that this holds as an identity between absolutely convergent sums and products for real $s > 1$. Riemann's insight was to consider (1) as an identity between functions of a *complex* variable s. We follow the curious but nearly universal convention of writing the real and imaginary parts of s as σ and t, so

$$s = \sigma + it.$$

We already observed that for all real $n > 0$ we have $|n^{-s}| = n^{-\sigma}$, because

$$n^{-s} = \exp(-s \log n) = n^{-\sigma} e^{it \log n}$$

and $e^{it \log n}$ has absolute value 1; and that both sides of (1) converge absolutely in the half-plane $\sigma > 1$, and are equal there either by analytic continuation from the real ray $t = 0$ or by the same proof we used for the real case. Riemann showed that the function $\zeta(s)$ extends from that half-plane to a meromorphic function on all of $\mathbf{C}$ (the "Riemann zeta function"), analytic except for a simple pole at $s = 1$. The continuation to $\sigma > 0$ is readily obtained from our formula

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \left[n^{-s} - \int_n^{n+1} x^{-s}\, dx \right] = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\, dx,$$

since for $x \in [n, n+1]$ ($n \geq 1$) and $\sigma > 0$ we have

$$|n^{-s} - x^{-s}| = \left| s \int_n^x y^{-1-s}\, dy \right| \leq |s| n^{-1-\sigma}$$

so the formula for $\zeta(s) - (1/(s-1))$ is a sum of analytic functions converging absolutely in compact subsets of $\{\sigma + it : \sigma > 0\}$ and thus gives an analytic function there. (See also the first Exercise below.) Using the Euler-Maclaurin summation formula with remainder, we could proceed in this fashion, extending ζ to $\sigma > -1$, $\sigma > -2$, etc. However, once we have defined $\zeta(s)$ on $\sigma > 0$ we can obtain the entire analytic continuation at once from Riemann's *functional equation* relating $\zeta(s)$ with $\zeta(1-s)$. This equation is most nicely stated by introducing the meromorphic function $\xi(s)$ defined by[1]

$$\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

[1] Warning: occasionally one still sees $\xi(s)$ defined as what we would call $(s^2 - s)\xi(s)$ or $(s^2 - s)\xi(s)/2$, as in [GR 1980, 9.561]. The factor of $(s^2 - s)$ makes the function entire, and does not affect the functional equation since it is symmetric under $s \leftrightarrow 1 - s$. However, for most uses it turns out to be better to leave this factor out and tolerate the poles at $s = 0, 1$.

for $\sigma > 0$. Then we have:

Theorem (Riemann)**.** *The function ξ extends to a meromorphic function on* $\mathbf{C}$*, regular except for simple poles at $s = 0, 1$, which satisfies the functional equation*

$$\xi(s) = \xi(1 - s). \tag{2}$$

It follows that ζ also extends to a meromorphic function on $\mathbf{C}$*, which is regular except for a simple pole at $s = 1$, and that this analytic continuation of ζ has simple zeros at the negative even integers $-2, -4, -6, \ldots$, and no other zeros outside the closed* critical strip $0 \leq \sigma \leq 1$.

[The zeros $-2, -4, -6, \ldots$ of ζ outside the critical strip are called the *trivial zeros* of the Riemann zeta function.]

The *proof* has two ingredients: properties of $\Gamma(s)$ as a meromorphic function of $s \in \mathbf{C}$, and the *Poisson summation formula.* We next review these two topics.

The *Gamma function* was defined for real $s > 0$ by Euler[2] as the integral

$$\Gamma(s) := \int_0^\infty x^s e^{-x} \frac{dx}{x}. \tag{3}$$

We have $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$ and, integrating by parts,

$$s\Gamma(s) = \int_0^\infty e^{-x} d(x^s) = -\int_0^\infty x^s d(e^{-x}) = \Gamma(s+1) \qquad (s > 0),$$

so by induction $\Gamma(n) = (n-1)!$ for positive integers n. Since $|x^s| = x^\sigma$, the integral (3) defines an analytic function on $\sigma > 0$, which still satisfies the recursion $s\Gamma(s) = \Gamma(s+1)$ (proved either by repeating the integration by parts or by analytic continuation from the positive real axis). That recursion then extends Γ to a meromorphic function on $\mathbf{C}$, analytic except for simple poles at $0, -1, -2, -3, \ldots$. (What are the residues at those poles?) For s, s' in the right half-plane $\sigma > 0$ the *Beta function*[3] $\mathrm{B}(s, s')$, defined by the integral

$$\mathrm{B}(s, s') := \int_0^1 x^{s-1}(1-x)^{s'-1}\, dx,$$

is related with Γ by

$$\Gamma(s + s')\mathrm{B}(s, s') = \Gamma(s)\Gamma(s'). \tag{4}$$

(This is proved by Euler's trick of calculating $\int_0^\infty\int_0^\infty x^{s-1}y^{s'-1}e^{-(x+y)}\, dx\, dy$ in two different ways.) Since $\Gamma(s) > 0$ for real positive s, it readily follows that Γ has no zeros in $\sigma > 0$, and therefore none in the complex plane.

This is enough to derive the poles and trivial zeros of ζ from the functional equation (2). [Don't take my word for it — do it!] But where does the functional equation come from? There are several known ways to prove it; we give

[2] Actually Euler used $\Pi(s-1)$ for what we call $\Gamma(s)$; thus $\Pi(n) = n!$ for $n = 0, 1, 2, \ldots$.

[3] a.k.a. "Euler's first integral", with (3) being "Euler's second integral".

Riemann's original method, which generalizes to $L(s,\chi)$, and further to L-series associated to modular forms.

Riemann expresses $\xi(s)$ as a Mellin integral involving the *theta function*[4]

$$\theta(u) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u} = 1 + 2(e^{-\pi u} + e^{-4\pi u} + e^{-9\pi u} + \ldots),$$

the sum converging absolutely to an analytic function on the upper half-plane $\mathrm{Re}(u) > 0$. Integrating termwise we find:

$$2\xi(s) = \int_0^\infty (\theta(u) - 1) u^{s/2} \frac{du}{u} \qquad (\sigma > 0).$$

(That is, $\xi(-2s)$ is the Mellin transform of $(\theta(u)-1)/2$.) But we shall see:

Lemma. *The function $\theta(u)$ satisfies the identity*

$$\theta(1/u) = u^{1/2}\theta(u). \tag{5}$$

Assume this for the time being. We then rewrite our integral for $2\xi(s)$ as

$$\int_0^1 (\theta(u) - 1) u^{s/2} \frac{du}{u} + \int_1^\infty (\theta(u) - 1) u^{s/2} \frac{du}{u}$$
$$= -\frac{2}{s} + \int_0^1 \theta(u) u^{s/2} \frac{du}{u} + \int_1^\infty (\theta(u) - 1) u^{s/2} \frac{du}{u},$$

and use the change of variable $u \leftrightarrow 1/u$ to find

$$\int_0^1 \theta(u) u^{s/2} \frac{du}{u} = \int_1^\infty \theta(u^{-1}) u^{-s/2} \frac{du}{u}$$
$$= \int_1^\infty \theta(u) u^{(1-s)/2} \frac{du}{u} = \frac{2}{s-1} + \int_1^\infty (\theta(u) - 1) u^{(1-s)/2} \frac{du}{u}$$

if also $\sigma < 1$. Therefore

$$\xi(s) + \frac{1}{s} + \frac{1}{1-s} = \frac{1}{2} \int_1^\infty (\theta(u) - 1)(u^{s/2} + u^{(1-s)/2}) \frac{du}{u},$$

which is manifestly symmetrical under $s \leftrightarrow 1-s$, and analytic since $\theta(u)$ decreases exponentially as $u \to \infty$. This concludes the proof of the functional equation and analytic continuation of ξ, assuming our lemma (5).

This lemma, in turn, is the special case $f(x) = e^{-\pi u x^2}$ of the Poisson summation formula:

[4] Jacobi introduced four "theta functions" of two variables; in his notation, our $\theta(u)$ would be $\theta_3(0, e^{-\pi u})$. We can call this $\theta(u)$ because we shall not use $\theta_1, \theta_2, \theta_4$, nor $\theta_3(z, q)$ for $z \neq 0$.

Theorem. *Let* $f : \mathbf{R} \to \mathbf{C}$ *be a* $\mathcal{C}^2$ *function such that* $(|x|^r + 1)(|f(x)| + |f''(x)|)$ *is bounded for some* $r > 1$*, and let* $\hat{f}$ *be its Fourier transform*

$$\hat{f}(y) = \int_{-\infty}^{+\infty} e^{2\pi ixy} f(x)\, dx.$$

Then

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \hat{f}(n), \tag{6}$$

the sums converging absolutely.

[The hypotheses on f can be weakened, but this formulation of Poisson summation is more than enough for our purposes.]

Proof: Define $F : \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ by

$$F(x) := \sum_{m=-\infty}^{\infty} f(x+m),$$

the sum converging absolutely to a $\mathcal{C}^2$ function by the assumption on f. Thus the Fourier series of F converges absolutely to F, so in particular

$$F(0) = \sum_{n=-\infty}^{\infty} \int_0^1 e^{2\pi inx} F(x)\, dx.$$

But $F(0)$ is just the left-hand side of (6), and the integral is

$$\sum_{m \in \mathbf{Z}} \int_0^1 e^{2\pi inx} f(x+m)\, dx = \sum_{m \in \mathbf{Z}} \int_m^{m+1} e^{2\pi inx} f(x)\, dx = \int_{-\infty}^{\infty} e^{2\pi inx} f(x)\, dx$$

which is just $\hat{f}(n)$, so its sum over $n \in \mathbf{Z}$ yields the right-hand side of (6). $\square$

Now let $f(x) = e^{-\pi ux^2}$. The hypotheses are handily satisfied for any r, so (6) holds. The left-hand side is just $\theta(u)$. To evaluate the right-hand side, we need the Fourier transform of f, which is $u^{-1/2} e^{-\pi u^{-1} y^2}$. [Contour integration reduces this claim to $\int_{-\infty}^{\infty} e^{-\pi ux^2}\, dx = u^{-1/2}$, which is the well-known Gauss integral — see the Exercises.] Thus the right-hand side is $u^{-1/2}\theta(1/u)$. Multiplying both sides by $u^{1/2}$ we then obtain (5), and finally complete the proof of the analytic continuation and functional equation for $\xi(s)$.

Remarks. We noted already that to each number field K there corresponds a zeta function

$$\zeta_K(s) := \sum_I |I|^{-s} = \prod_{\wp} (1 - |\wp|^{-s})^{-1} \qquad (\sigma > 1),$$

in which $|I|$ is the norm of an ideal I, the sum and product extend respectively over ideals I and prime ideals $\wp$ of the ring of integers O_K, and their equality expresses unique factorization. As in our case of $K = \mathbf{Q}$, this zeta function extends to a meromorphic function on $\mathbf{C}$, regular except for a simple pole at $s = 1$. Moreover it satisfies a functional equation $\xi_K(s) = \xi_K(1-s)$, where

$$\xi_K(s) := \Gamma(s/2)^{r_1}\Gamma(s)^{r_2}(4^{-r_2}\pi^{-n}|d|)^{s/2}\zeta_K(s),$$

in which $n = r_1 + 2r_2 = [K : \mathbf{Q}]$, the exponents r_1, r_2 are the numbers of real and complex embeddings of K, and d is the discriminant of $K/\mathbf{Q}$. The factors $\Gamma(s/2)^{r_1}, \Gamma(s)^{r_2}$ may be regarded as factors corresponding to the "archimedean places" of K, as the factor $(1 - |\wp|^{-s})^{-1}$ corresponds to the finite place $\wp$. The functional equation can be obtained from generalized Poisson summation as in [Tate 1950]. Most of our results for $\zeta = \zeta_{\mathbf{Q}}$ carry over to these ζ_K, and yield a Prime Number Theorem for primes of K; L-series generalize too, though the proper generalization requires some thought when the class and unit groups need no longer be trivial and finite as they are for $\mathbf{Q}$. See for instance H.Heilbronn's "Zeta-Functions and L-Functions", Chapter VIII of [CF 1967].

Exercises

Concerning the analytic continuation of $\zeta(s)$:

1. Show that if $\alpha : \mathbf{Z} \to \mathbf{C}$ is a function such that $\sum_{m=1}^{n} \alpha(m) = O(1)$ (for instance, if α is a nontrivial Dirichlet character) then $\sum_{n=1}^{\infty} \alpha(n)n^{-s}$ converges uniformly, albeit not absolutely, in compact subsets of $\{\sigma + it : \sigma > 0\}$, and thus defines an analytic function on that half-plane. Apply this to

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + - \cdots$$

(with $\alpha(n) = (-1)^{n-1}$) and to $(1 - 3^{1-s})\zeta(s)$ to obtain a different proof of the analytic continuation of ζ to $\sigma > 0$.

2. Prove that the Bernoulli polynomials B_n $(n > 0)$ have the Fourier expansion

$$\mathrm{B}_n(x) = -n! \sum_k{}' \frac{e^{2k\pi i x}}{(2k\pi i)^n} \tag{7}$$

for $0 < x < 1$, in which $\sum_k'$ is the sum over nonzero integers k. Deduce that

$$\zeta(n) = \frac{1}{2}(2\pi)^n \frac{|B_n|}{n!} \quad (n = 2, 4, 6, 8, \ldots),$$

and thus that $\zeta(1-n) = -B_n/n$ for all integers $n > 1$. For example, $\zeta(-1) = -1/12$. What is $\zeta(0)$?

It is known that in general $\zeta_K(-m) \in \mathbf{Q}$ $(m = 0, 1, 2, \ldots)$ for any number field K. In fact the functional equation for ζ_K indicates that once $[K : \mathbf{Q}] > 1$ all the $\zeta_K(-m)$ vanish unless K is totally real and m is odd, in which case the rationality of $\zeta_K(-m)$ was obtained in [Siegel 1969].

A further application of (7):

3. Prove that $\sum\sum_{k,k'=0}^{\infty}(kk'(k+k'))^{-n}$ is a rational multiple of π^{3n} for each $n=2,4,6,8,\ldots$; for instance,

$$\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}\frac{1}{(kk'(k+k')^2)}=\frac{4\pi^6}{3}\int_0^1 \mathrm{B}_2(x)^3\,dx=\frac{\pi^6}{2835}.$$

Concerning the Gamma function:

4. If you've never seen it yet, or did it once but forgot, prove (4) by starting from the integral representation of the right-hand side as

$$\int_0^{\infty}\int_0^{\infty} x^{s-1}y^{s'-1}e^{-(x+y)}\,dx\,dy$$

and applying the change of variable $(x,y)=(uz,(1-u)z)$.

We will have little use for the Beta function in Math 259, but an analogous transformation will arise later in the formula relating Gauss and Jacobi sums.

5. Now take $s=s'=1/2$ to prove that $\Gamma(1/2)=\sqrt{\pi}$, and thus to obtain the Gauss integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}.$$

Then take $s'=s$ and use the change of variable $u=(1-2x)^2$ in the integral defining $\mathrm{B}(s,s)$ to obtain $\mathrm{B}(s,s)=2^{1-2s}\mathrm{B}(s,1/2)$, and thus the *duplication formula*

$$\Gamma(2s)=\pi^{-1/2}2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2}).$$

Concerning functional equations:

6. Use the duplication formula and the identity $\mathrm{B}(s,1-s)=\pi/\sin(\pi s)$ to write (2) in the equivalent form

$$\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}\zeta(s).$$

This asymmetrical formulation of the functional equation has the advantage of showing the trivial zeros of $\zeta(s)$ more clearly (given the fact that $\zeta(s)$ has a simple pole at $s=1$ and no other poles or zeros on the positive real axis).

7. Let χ_8 be the Dirichlet character mod 8 defined by $\chi_8(\pm 1)=1$, $\chi_8(\pm 3)=-1$. Show that if f is a function satisfying the hypotheses of Poisson summation then

$$\sum_{m=-\infty}^{\infty}\chi_8(m)f(m)=8^{-1/2}\sum_{n=-\infty}^{\infty}\chi_8(n)\hat{f}(n/8).$$

Letting $f(x)=e^{-\pi u x^2}$, obtain an identity analogous to (5), and deduce a functional equation for $L(s,\chi_8)$.

8. Now let χ_4 be the character mod 4 defined by $\chi_4(\pm 1) = \pm 1$. Show that, again under the Poisson hypotheses,

$$\sum_{m=-\infty}^{\infty} \chi_4(m) f(m) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \chi_4(n) \hat{f}(n/4).$$

This time, taking $f(x) = e^{-\pi u x^2}$ does not accomplish much! Use $f(x) = x e^{-\pi u x^2}$ instead to find a functional equation for $L(s, \chi_4)$.

We shall see that the L-function associated to any primitive Dirichlet character χ satisfies a similar functional equation, with the Gamma factor depending on whether $\chi(-1) = +1$ or $\chi(-1) = -1$.

9. For light relief after all this hard work, differentiate the identity (5) with respect to u, set $u = 1$, and conclude that $e^{\pi} > 8\pi - 2$. What is the approximate size of the difference?

Further applications of Poisson summation:

10. Use Poisson summation to evaluate $\sum_{n=1}^{\infty} 1/(n^2 + c^2)$ for $c > 0$. [The Fourier transform of $1/(x^2 + c^2)$ is a standard exercise in contour integration.] Verify that your answer approaches $\zeta(2) = \pi^2/6$ as $c \rightarrow 0$.

11. [Higher-dimensional Poisson, and more on zeta functions of quadratic forms] Let A be a real positive-definite symmetric matrix of order r, and $Q : \mathbf{R}^r \rightarrow \mathbf{R}$ the associated quadratic form $Q(x) = (x, Ax)$. The *theta function of* Q is

$$\theta_Q(u) := \sum_{n \in \mathbf{Z}^r} \exp(-\pi Q(n) u).$$

For instance, if $r = 1$ and $A = 1$ then $\theta_Q(u)$ is just $\theta(u)$. More generally, show that if A is the identity matrix I_r (so $Q(x) = \sum_{j=1}^{r} x_j^2$) then $\theta_Q(u) = \theta(u)^r$. Prove an r-dimensional generalization of the Poisson summation formula, and use it to obtain a generalization of (5) that relates $\theta_Q(u)$ with $\theta_{Q^*}(1/u)$, where Q^* is the quadratic form associated to A^{-1}. Using this formula, and a Mellin integral formula for

$$\zeta_Q(s) = \sum_{\substack{n \in \mathbf{Z}^r \\ n \neq 0}} \frac{1}{Q(n)^s},$$

conclude that ζ_Q extends to a meromorphic function on $\mathbf{C}$ that satisfies a functional equation relating ζ_Q with ζ_{Q^*}. Verify that when $r = 2$ and $A = I_2$ your functional equation is consistent with the identity $\zeta_Q(s) = 4\zeta(s) L(s, \chi_4)$ and the functional equations for $\zeta(s)$ and $L(s, \chi_4)$.

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