MA 3111 Complex Analysis I

Notes by Chan Heng Huat

Contents

References

The main references are "Complex Analysis" by L. Ahlfors, "Complex Analysis" by J. Bak and D.J. Newman, and "Complex variables and applications" by J.W. Brown and R.V. Churchill

1 Facts about complex numbers

1.1 Beyond real numbers

Let

$$\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$$

be the set of integers. Let x be a variable and let $\mathbf{Z}[x]$ denote the set of polynomials in x with coefficients in $\mathbf{Z}$. In other words, if

$$p(x) \in \mathbf{Z}[x],$$

then

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

with

$$a_j \in \mathbf{Z}, 0 \leq j \leq n \quad \text{and} \quad a_n \neq 0.$$

The integer n is called the degree of the polynomial $p(x)$.

New numbers are often discovered when we ask for solutions to the equation

$$p(x) = 0$$

for some polynomial $p(x)$. In the case when $b \neq 0$, the solution of the equation [1]

$$bx - a = 0$$

leads to rational number denoted by

$$\frac{a}{b}.$$

The set of rational numbers are numbers of the form ab^{-1} (more commonly written as $\frac{a}{b}$), with $a, b \in \mathbf{Z}, b \neq 0$, is denoted by $\mathbf{Q}$.

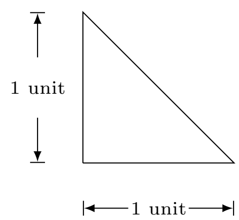
REMARK 1.1 Formal definitions of rational numbers involve constructing equivalence classes from (a, b), with $a, b \in \mathbf{Z}$.

[1] A formal construction of rational numbers are done using equivalence relations on the set $\mathbf{Z} \times \mathbf{Z}$.

Consider the equation

$$x^2 - 2 = 0.$$

This equation is motivated by the attempt to determine the hypotenuse of the following isosceles right angled triangle with 1 unit length.



By Pythagoras' Theorem, the hypotenuse must satisfy

$$x^2 = 1^2 + 1^2 = 2.$$

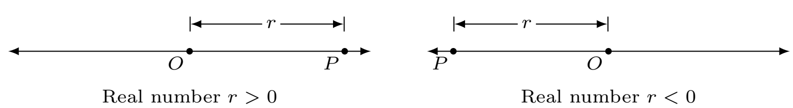
We denote the positive solution to the above equation by $\sqrt{2}$. It can be shown that $\sqrt{2}$ is not rational. We call such numbers irrational numbers.

A number a that satisfies

$$p(a) = 0$$

for some $p(x) \in \mathbf{Z}[x]$ is called an algebraic number. The number $\sqrt{2}$ is an algebraic number.

With the notion of rational numbers, we can construct real numbers using Cauchy sequences of rational numbers but we will not discuss its construction here. There are real numbers which are not algebraic. Examples of such numbers are e and π. Graphically, these numbers can be represented by points on a number line with a fixed origin O. If r is a positive real number, we represent it as a point P to the right of O on the number line with distance r. If r is a negative real number, we represent it as a point to the left of O with distance $|r|$.



The set of real numbers is denoted by $\mathbf{R}$. A real number r satisfies the property that

$$r^2 \geq 0.$$

In other words, if r is real, r cannot satisfy the equation

$$r^2 = -1 < 0$$

and therefore the solutions to the equation

$$x^2 + 1 = 0 \tag{1.1}$$

cannot be real. It is clear that the solutions of (1.1), if exist, leads to "new numbers". We call these numbers imaginary numbers. We denote one of the two solutions of (1.1) by i or $\sqrt{-1}$ and we usually write

$$i \cdot i = -1.$$

REMARK 1.2 Note that i, the solution of $x^2 = -1$ is an algebraic number that is not real.

1.2 Complex numbers

In the previous section, we have seen that it is possible to construct "numbers" that are not real. We now define these "numbers" formally.

DEFINITION 1.1 Let $\mathbf{R}$ be the set of real numbers. The set of complex numbers $\mathbf{C}$ is the set of ordered pairs of real numbers (a, b) with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

We also define a scalar product $\cdot$, namely, if $r \in \mathbf{R}$,

$$r \cdot (a, b) = (r \cdot a, r \cdot b).$$

Here $\cdot$ is the ordinary multiplication of real numbers.

Note that with the definition of scalar product, together with $+$, allow us to show that the set $\mathbf{C}$ is a vector space of dimension 2 over $\mathbf{R}$.

We now consider $\cdot$. By definition of $\cdot$, we find that

$$(0, 1) \cdot (0, 1) = (-1, 0) = (-1) \cdot (1, 0).$$

With the notation $1 = (1, 0)$ and $i = (0, 1)$,

$$i \cdot i = (-1) \cdot 1.$$

We have thus found a "solution" to

$$x \cdot x = -1.$$

With the notation $1 = (1, 0)$ and $i = (0, 1)$, we can now write a complex number as

$$a \cdot 1 + b \cdot i.$$

Ignoring the colors for the operators, we arrive at the definition of complex numbers given in many textbooks.

The sum and product of complex numbers can now be written as

$$a + ib + c + id = (a + c) + i(b + d)$$

and

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

respectively. Note that multiplication of complex numbers is motivated by treating $a + ib$ and $c + id$ like ordinary numbers with multiplication that distributes over addition.

DEFINITION 1.2 When a complex number is written as

$$z = a + ib,$$

we call a the Real part of z and b the Imaginary part of z.

1.3 Division of complex numbers

When we first construct rational number, we ask for solution x such that

$$bx = a,$$

with $b \neq 0$. And we define $x = ab^{-1}$ and this leads to division of a by b. In a similar way, we ask for solution

$$(c + id)x = (a + ib).$$

We let

$$x = u + iv.$$

Then we must find u, v from comparing the real part and imaginary part of the numbers of both sides of the equation

$$cu - vd + i(cv + du) = a + ib.$$

The number u and v can be found using the relation

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

which gives

$$u + iv = \frac{ca + db}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

REMARK 1.3 From

$$cu - vd + i(cv + du) = a + ib,$$

we have

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} -v \\ u \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

This implies that we can identify $s + it$ with

$$\begin{pmatrix} s & -t \\ t & s \end{pmatrix}.$$

REMARK 1.4 Division usually done using "rationalization" by "multiplying" numerator and denominator by $c - id$ if we write

$$x = \frac{a + ib}{c + id}.$$

DEFINITION 1.3 Let z be a non-zero complex number. We define the multiplicative inverse of z, denoted by z^{-1}, as the complex number w satisfying the equation

$$wz = 1.$$

Note that by the division of complex numbers, we find that

$$(a + ib)^{-1} = \frac{1}{a^2 + b^2}(a - ib).$$

In mathematics, a group is a nonempty set G together with a binary operation $\circ : G \times G \to G$ that satisfies the following conditions:

1. There exists an element $e \in G$ such that $g \circ e = e \circ g = g$ for all $g \in G$,
2. For every $g \in G$, there exists $g' \in G$ such that $g \circ g' = g' \circ g = e$,
3. For all $g, h, k \in G$, $g \circ (h \circ k) = (g \circ h) \circ k$.

In the process of showing that $\mathbf{C}$ is a vector space of dimension 2 over $\mathbf{R}$, we would have shown that $(\mathbf{C}, +)$ is a group. With the multiplicative inverse defined, we can also show that $(\mathbf{C}\backslash\{0\}, \cdot)$ is a group provided that $\cdot$ is associative, which we leave as an exercise.

EXAMPLE 1.1
Show that if z, w and u are complex numbers then $z\cdot(u\cdot w) = (z\cdot u)\cdot w$ and $z\cdot u = u\cdot z$.

The addition and multiplication also satisfy the distributive law as can be verified in the following exercise.

EXAMPLE 1.2
Show that for complex numbers z, w and u,

$$z\cdot(w+u) = z\cdot w + z\cdot u.$$

The facts that $(\mathbf{C}, +)$ and $(\mathbf{C}\backslash\{0\}, \cdot)$ are abelian groups [2] and that $\cdot$ distributes over $+$ show that $(\mathbf{C}, +, \cdot)$ is a field.

1.4 Conjugate and modulus of z

There is a recurring appearance of the number $a^2 + b^2$ (in the inverse of z) and as determinant of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

DEFINITION 1.4 The number

$$|z| = \sqrt{a^2 + b^2}$$

is called the modulus of z.

Note that $a^2 + b^2 = |z|^2$.

DEFINITION 1.5 The conjugate of z, denoted by $\overline{z}$, is defined by $a - ib$.

Note that

$$z\overline{z} = |z|^2 + i0. \tag{1.2}$$

In many textbooks, the above is written as

$$z\overline{z} = |z|^2. \tag{1.3}$$

This is not accurate since by (1.2), we know that

$$|z|^2 = \text{Re}(z\overline{z}).$$

However, when we are familiar with the complex numbers, we usually do not distinguish $z\cdot\overline{z}$ from $|z|^2$ and use (1.3) instead.

[2] Abelian groups are groups with binary operation having the additional property that $g \circ g' = g' \circ g$.

REMARK 1.5 Note that $|\cdot|$ is consistent with absolute value $|\cdot|_{\mathbf{R}}$, we encountered in the case of real numbers. For, we have

$$|a + i \cdot 0| = \sqrt{a^2} = |a|_{\mathbf{R}}.$$

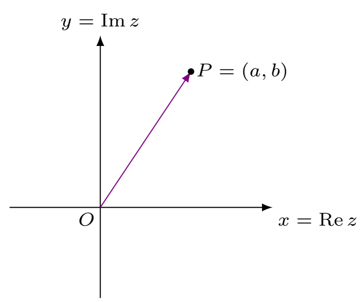
EXAMPLE 1.3
Establish the following facts:

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}), \operatorname{Im} z = \frac{1}{2i}(z - \overline{z}), \overline{(z + w)} = \overline{z} + \overline{w},$$
$$\overline{zw} = \overline{z} \cdot \overline{w}, |zw| = |z||w|, |z/w| = |z|/|w|, |\overline{z}| = |z|.$$

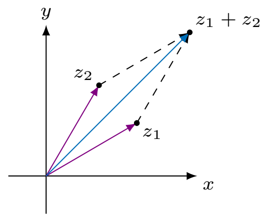
EXAMPLE 1.4
Show that if $z, w \in \mathbf{C}$, then $\overline{zw} = \overline{z} \cdot \overline{w}$ and $|zw| = |z||w|$. Deduce that if A and B are integers that can be written as a sum of two squares then AB is a sum of two squares.

1.5 The complex plane

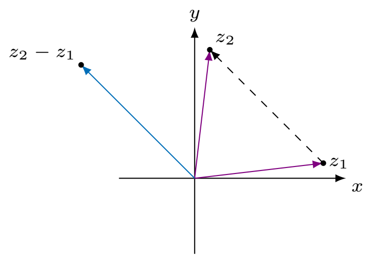
As seen in previous sections, a complex number is defined as a number of the form (a, b). Just as a real number can be represented graphically by a point on a number line, a complex number can be represented by a point on a plane. This is illustrated in the following diagram:



Addition of complex numbers then corresponds to addition of “vectors” as shown in the following diagram:



The following diagram illustrates the difference of two complex numbers:



It is known that if $|r|_{\mathbf{R}}$ is the absolute value of a real number r then

$$|r + s|_{\mathbf{R}} \leq |r|_{\mathbf{R}} + |s|_{\mathbf{R}}.$$

The same is true for the modulus of complex numbers. More precisely, we have the following triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.4}$$

To see (1.4) geometrically, we first observe that the modulus $|z|$ is the length of the vector (a, b) that represents z. Now, note that z_1, z_2 and $z_1 + z_2$ form the vertices of a triangle. Since $|z_1 + z_2|$ is the length of a side of the triangle, it must be less than or equal to the sum of the lengths of the other two sides and this gives (1.4). We now give an algebraic proof of (1.4).

Proof of (1.4)
First, we observe that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2}).$$

Hence,

$$|z_1 + z_2|^2 = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

But

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\text{Re}(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}| = 2|z_1| \cdot |z_2|.$$

Therefore,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

□

COROLLARY 1.6 Let z_1 and z_2 be complex numbers. Then

$$|z_1| - |z_2| \le |z_1 - z_2|.$$

Proof
We have $|z+w| \le |z| + |w|$. Take $z = z_2, w = z_1 - z_2$. Then

$$|z_1| \le |z_2| + |z_1 - z_2|,$$

and this completes the proof. □

EXAMPLE 1.5
A function $d : X \times X \to \mathbf{R} \backslash \mathbf{R}^-$ is called a metric on X if

$$d(x, z) \le d(x, y) + d(y, z),$$

$$d(x, y) = 0 \quad \text{if and only if } x = y,$$

$$d(x, y) = d(y, x).$$

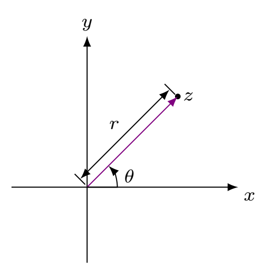
Define $d(u, v) = |u - v|$ when $X = \mathbf{C}$. Use the triangle inequality to show that d is a metric on $\mathbf{C}$.

1.6 The polar representation of complex numbers

Given a point in the plane which is not the origin, we may represent the point using polar coordinate system rather than the usual rectangular coordinate system. Let (x, y) be a point in the first quadrant. Then the complex number $z = x + iy$ can be written as

$$z = r(\cos\theta + i\sin\theta).$$

Note that $r = |z|$ and θ represents the angle, measured in radians, that z makes with the positive real axis.



When z is not in the first quadrant, we use the more general definitions of sine and cosine for obtuse angles and negative angles. This allows us to write in general,

$$z = r(\cos\theta + i\sin\theta).$$

Note that θ is not unique since $\sin x$ and $\cos x$ are both periodic with period 2π. $\Theta \in (-\pi, \pi]$ then $\Theta = \theta$.

We denote θ by arg z, called the argument of z, the principal value of arg z is denoted by $\operatorname{Arg} z$ and it refers to $\arg z$ when $\arg z \in (-\pi, \pi]$. Our angle θ is indeed $\operatorname{Arg} z$ when z is on the upper half plane, with $\operatorname{Re} z > 0$.

EXAMPLE 1.6

Let $z = i$. Determine $|z|$, arg z, Arg z and express z in polar coordinates.

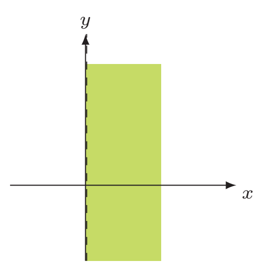
Solutions. The answers are $|z| = 1$, $\arg z = \pi/2 + 2k\pi, k \in \mathbf{Z}$, $\operatorname{Arg} z = \pi/2$, and $i = (\cos\pi/2 + i\sin\pi/2)$.

EXAMPLE 1.7

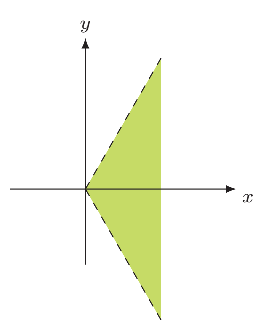
Sketch the following regions representing the following sets in a complex plane:

(a) $\{z|\operatorname{Re} z > 0\}$
(b) $\{z| -\pi/3 < \operatorname{Arg} z < \pi/3\}$
(c) $\{z||z+1| < 1\}$

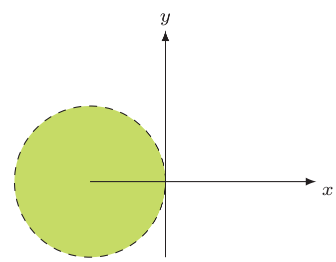
Solutions. (a)



(b)



(c)



THEOREM 1.7 Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\psi + i\sin\psi)$. Then

$$zw = rs(\cos(\theta+\psi) + i\sin(\theta+\psi)),$$

that is,

$$|zw| = |z||w| \quad \text{and} \quad \arg zw = \arg z + \arg w.$$

Proof
We have

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= rs(\cos\theta\cos\psi - \sin\theta\sin\psi + i(\cos\theta\sin\psi + \sin\theta\cos\psi)) \\ &= rs(\cos(\theta+\psi) + i\sin(\theta+\psi)), \end{aligned}$$

where we have used the standard sine and cosine formula for addition of angles, namely,

$$\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

and

$$\sin(\theta+\psi) = \cos\theta\sin\psi + \sin\theta\cos\psi.$$

□

Using induction, we deduce that

COROLLARY 1.8 Let $z_1, z_2, \cdots, z_n \in \mathbf{C}$. Then

$$|z_1 z_2 \cdots z_n| = |z_1| \cdot |z_2| \cdots |z_n|,$$

and

$$\arg z_1 z_2 \cdots z_n = \arg z_1 + \cdots + \arg z_n.$$

In Corollary 1.8, we let $z_1 = z_2 = \cdots = z$ and conclude that

COROLLARY 1.9 Let n be an integer. Then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

REMARK 1.6 The relation in Theorem 1.9 holds only for integers. It is not true for rational numbers as one would encounter the notion of multi-valued function such as $(\cos x + i\sin x)^{1/n}$, $n \in \mathbf{Z}$.

EXAMPLE 1.8
Find all complex numbers z satisfying $z^3 = 1$.

Solutions. Write $1 = \cos\theta + i\sin\theta$ where $\theta = 2k\pi, k \in \mathbf{Z}$. Write $z = r(\cos\psi + i\sin\psi)$. By Corollary 1.9,

$$z^3 = r^3\left(\cos 3\psi + i\sin 3\psi\right).$$

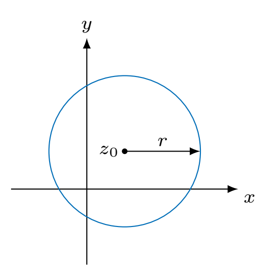
Since $z^3 = 1$, we conclude that $r = 1$ and $3\psi = 2k\pi, k \in \mathbf{Z}$. Therefore, $\psi = 2k\pi/3, k \in \mathbf{Z}$. This gives rise to three distinct solutions of ψ in $(-\pi, \pi]$ and they are $-2\pi/3, 0$, and $2\pi/3$.

1.7 Sets in the complex plane

When we study the real numbers, we define objects such as open intervals, closed intervals and bounded sets. In this section, we give definitions to analogous objects for the complex numbers. Most of these definitions look intimidating at first sight but they define very natural objects.

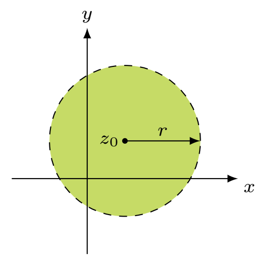
1. A circle of radius $r > 0$ centered at z_0 is the set

$$C(z_0; r) := \{z \mid |z - z_0| = r\}.$$



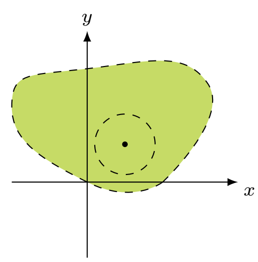
2. An open ball of radius r and center z_0 is the set

$$B(z_0; r) := \{z \mid |z - z_0| < r\}.$$



The dotted line is used to indicate that the circle $|z - z_0| = r$ is not in the set $B(z_0; r)$. The set $B(z_0; r)$ is called a neighbourhood of z_0. For example, $B(i; 1)$ is a neighbourhood of i and it is the disc centered at i with radius 1. [3]

3. A subset S of $\mathbf{C}$ is said to be open in $\mathbf{C}$ if for any $z \in S$ there exist a $\delta > 0$ such that $B(z; \delta) \subset S$. We also say that S is an open set.



An open ball is an open set.

The set

$$S = \{z| - \frac{\pi}{4} < arg\, z < \frac{\pi}{4}\}$$

is open. In general, one may visualize open sets in $\mathbf{C}$ as shaded regions with dotted boundaries.

There are sets which are not open. Examples of such sets are $C(z_0; r)$ and $\{z|\mathrm{Re}\, z \geq 1\}$.

REMARK 1.7 By defining open sets, we have essentially defined a "topology" on $\mathbf{C}$.

4. A subset S of $\mathbf{C}$ is said to be closed if the complement of S in $\mathbf{C}$, denoted by S^c, is open. Examples of closed sets are $C(z_0; r)$ and $\{z|\mathrm{Re}\, z \geq 1\}$.

REMARK 1.8 A set S that is not open is not necessarily closed. The set $\{0 \leq \mathrm{Re}\, z < 1\}$ is one such example.

[3] We will not use neighbourhood in this course but just open ball.

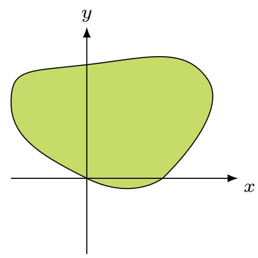
5. For any set S, let

$$S^c = \mathbf{C} \backslash S = \mathbf{C} - S$$

be the complement of S in $\mathbf{C}$, that is,

$$S^c = \{z \in \mathbf{C} | z \notin S\}.$$

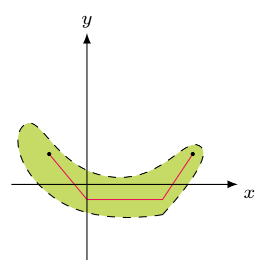
A set is said to be closed if S^c is open. Therefore S^c is closed if $S = B(z;r)$. Closed sets may be visualized as sets with solid lines as boundaries. The following is an example of a closed set:



6. Let S be a subset of $\mathbf{C}$. The set of points B with the property that every open ball of the form $B(z_0;r)$, with $z_0 \in B$, has non-empty intersection with S and S^c. The set B is called the boundary of S and the notation for this set is ∂S.
7. A set S is bounded if it is contained in a ball $B(0;M)$ for some $M > 0$.
8. Let $[z_1, z_2]$ denotes the line segment with endpoints z_1 and z_2. A polygonal line is a finite union of line segments of the form

$$[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n].$$

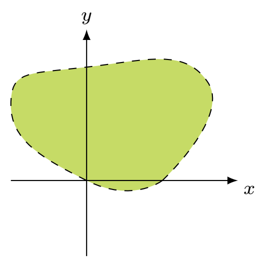
If any two points of S can be connected by a polygonal line contained in S, S is said to be polygonally connected. The following set is an example of a polygonally connected set.



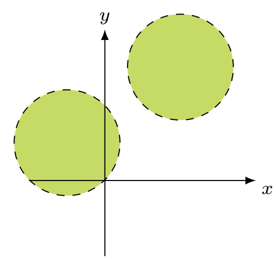
9. A nonempty open polygonally connected set in $\mathbf{C}$ will be called a region.

In this course, we will study functions defined on a region.

10. A set S in $\mathbf{C}$ is said to be disconnected if there exist two disjoint open sets A and B in $\mathbf{C}$ such that $S = (S \cap A) \cup (S \cap B)$ and that neither A nor B above contains S. If S is not disconnected it is said to be connected (see the following diagrams).



Connected set



Disconnected set

REMARK 1.9 It can be shown that in $\mathbf{C}$, *an open set is polygonally connected if and only if it is connected.* For more details see [p. 54, Ahlfors]. The usual definition of a region is a non-empty open connected set.

1.8 Stereographic projection

When dealing with real numbers, we frequently use the concept of infinity and speak of $+\infty$ and $-\infty$. In dealing with complex numbers, we also speak of infinity, which we call "the complex number infinity". It is designated by ∞. It does not have an argument and its modulus is larger than any preassigned real number. In order to visualize ∞, we introduce the stereographic projection.

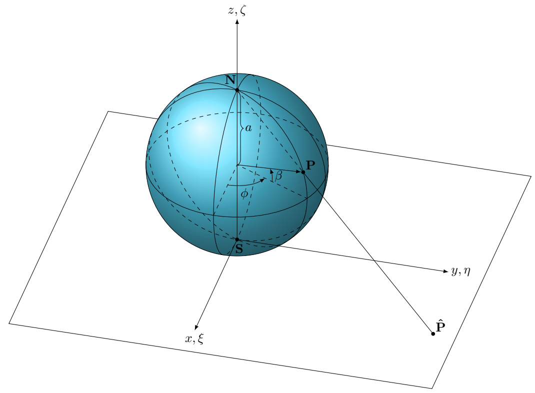
Consider the sphere

$$\Sigma = \left\{ (\xi, \eta, \zeta) \,|\, \xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}.$$

To each (ξ, η, ζ), we assign a complex number to be the point of intersection of the xy-plane and the ray from $(0,0,1)$ passing through (ξ, η, ζ).

Conversely if $z = x + iy$, then we obtain a point on the sphere as the point of intersection of the sphere and the line joining $(0,0,1)$ and $(x, y, 0)$. For more details, refer to the following diagram: [4]

[4] Diagram from http://www.texample.net/tikz/examples/tag/3d/



Explicitly, given (ξ, η, ζ), we project the line joining $(0, 0, 1)$, (ξ, η, ζ) and $(x, y, 0)$ to the yz-plane and obtain a line joining $(0, 0, 1)$, $(0, \eta, \zeta)$ and $(0, y, 0)$. Computing the slope of this line, we obtain

$$y = \frac{\eta}{1 - \zeta}. \tag{1.5}$$

Similarly, we find that

$$x = \frac{\xi}{1 - \zeta}. \tag{1.6}$$

Now, substituting (1.6) and (1.5) into

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4},$$

we find that

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}. \tag{1.7}$$

Substituting (1.7) into (1.6) and (1.5), we conclude that

$$\xi = \frac{x}{x^2 + y^2 + 1}, \quad \text{and} \quad \eta = \frac{y}{x^2 + y^2 + 1}.$$

The sphere is called the Riemann sphere. Note that with this one to one correspondence between the points on the sphere and the complex plane illustrates that the further away z is from the origin, the closer is the corresponding point on the sphere to the point $(0, 0, 1)$. In other words, the "north pole" $(0, 0, 1)$ corresponds to the point ∞. The z-plane with point at infinity is called the extended z-plane. When ∞ is not included, we will just say the z-plane or the finite z-plane.

1.9 Appendix-Complex numbers as topological space

I hope students will read this section and try to understand the contents here.

DEFINITION 1.10 A topological space $T = \{A, \mathcal{T}\}$ consists of a non-empty set A together with a fixed collection $\mathcal{T}$ of subsets satisfying

(T1) $A, \phi \in \mathcal{T}$
(T2) the intersection of any two sets in $\mathcal{T}$ is again in $\mathcal{T}$
(T3) the union of any collection of sets in $\mathcal{T}$ is again in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a topology for A and the members of $\mathcal{T}$ are called the open sets of T.

EXAMPLE 1.9
Let $A = \{a, b\}$ and $\mathcal{T} = \{A, \phi, \{a\}\}$. Then $(A, \mathcal{T})$ is a topological space.

As one can see from the example, topological space can be very strange.

In the case of complex numbers, we declare set U to be "open in $\mathbf{C}$" if for each $z \in U$, then there exists an $\epsilon > 0$ such that $B(z; \epsilon) \subset U$. (Note that we are NOT saying that all examples of open sets must arise from this definition. See the above example.)

We let $\mathcal{T}$ to be the collection of "open sets U in $\mathbf{C}$". We now check that $(\mathbf{C}, \mathcal{T})$ is a topological space. Note that (T1) is clearly satisfied.

For (T2), let U_1 and U_2 be open set in $\mathbf{C}$. Let $z \in U_1 \cap U_2$. Then $z \in U_1$ and $z \in U_2$. Since U_1 is open in $\mathbf{C}$, there exists an $\epsilon_1 > 0$ such that $B(z; \epsilon_1) \subset U_1$. Similarly, since U_2 is open in $\mathbf{C}$, there exists an $\epsilon_2 > 0$ such that $B(z; \epsilon_2) \subset U_2$. Choose $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $B(z; \epsilon) \subset U_1 \cap U_2$ and therefore, $U_1 \cap U_2$ is open.

For condition (T3), if z is in a union of open sets, say A, where

$$A = \bigcup_i U_i,$$

then $z \in U_1$, say. Since U_1 is open in $\mathbf{C}$, there exists an $\epsilon > 0$ such that

$$B(z; \epsilon) \subset U_1 \subset A.$$

Hence, A is open and

$$\bigcup_i U_i \in \mathcal{T}.$$

In conclusion, we have $(\mathbf{C}, \mathcal{T})$ is a topological space.

2 Analytic Functions

2.1 Functions of a complex variable

When we first encounter real numbers, we define functions on $S \subset \mathbf{R}$ as a rule that assigns to each $x \in S$, a unique real number $y \in \mathbf{R}$. For example the rule $f(x) = x^2$ sends a real number to its square.

In a similar manner, we define a function on a set $S \subset \mathbf{C}$ to be a rule that assigns $z \in S$, a complex number $w \in \mathbf{C}$. The number w is called the value of f at z and is denoted by

$$w = f(z).$$

The set S is called the domain of definition of f. Note that it is possible to have a rule which assigns a complex number z to multiple values. We referred to such a rule as a multi-valued function. An example of a multi-valued function is $\arg(z)$. Another example of a multi-valued function is $\sqrt{z}$, where $\sqrt{z}$ is defined to u satisfying

$$u^2 = z.$$

In the following few chapters, we will only concentrate on rules that assign z to a unique complex number. We call these rules single-valued functions or simply functions.

To define a function, both the domain of definition and the rule must be given. The function

$$f(z) = \frac{1}{z}$$

is defined on $\mathbf{C}\backslash\{0\}$ since $f(z)$ is not defined at 0.

EXAMPLE 2.1

Let $f(z) = z^2$. Note that in terms of x and y coordinate,

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Suppose $w = u + iv$ is the value of a function of $f(z)$ at $z = x + iy$. Then u and v depend on the real variables x and y and we find that

$$f(z) = w = u(x, y) + iv(x, y).$$

EXAMPLE 2.2

When $f(z) = z^2$,

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

This implies that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

EXAMPLE 2.3

Let $f(z) = z^3$. Then

$$f(x + iy) = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

This implies that

$$u(x, y) = x^3 - 3xy^2 \quad \text{and} \quad v(x, y) = 3x^2y - y^3.$$

EXAMPLE 2.4

Let $f(z) = \dfrac{1}{1 - |z|^2} = \dfrac{1}{1 - x^2 - y^2}$. Then

$$u(x, y) = \frac{1}{1 - x^2 - y^2} \quad \text{and} \quad v(x, y) = 0.$$

EXAMPLE 2.5

Let

$$f(z) = \frac{z}{z + \bar{z}} = \frac{x + iy}{2x}.$$

Then

$$u(x, y) = \frac{1}{2} \quad \text{and} \quad v(x, y) = \frac{y}{2x}.$$

2.2 Limits

In the case of a real variable, we say that

$$\lim_{x\to x_0} f(x) = w_0$$

if for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$|f(x) - w_0|_{\mathbf{R}} < \epsilon \quad \text{whenever} \quad 0 < |x - x_0|_{\mathbf{R}} < \delta_\epsilon.$$

In the case of complex variable, the definition is similar.

DEFINITION 2.1 We say that

$$\lim_{z\to z_0} f(z) = w_0$$

if and only if for every $\epsilon > 0$, there exist a $\delta_\epsilon > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_\epsilon.$$

Sometimes, we write

$$(z) \to w_0 \quad \text{for} \quad z \to z_0$$

to represent

$$\lim_{z\to z_0} f(z) = w_0.$$

If w_0 is ∞, we write $f(z) \to \infty$ for $z \to z_0$ if for every $M > 0$, there exist $\delta_M > 0$ such that

$$|f(z)| > M \quad \text{whenever} \quad 0 < |z - z_0| < \delta_M.$$

EXAMPLE 2.6
Let $f(z) = 1/z$. Then

$$\lim_{z\to 0} f(z) = \infty.$$

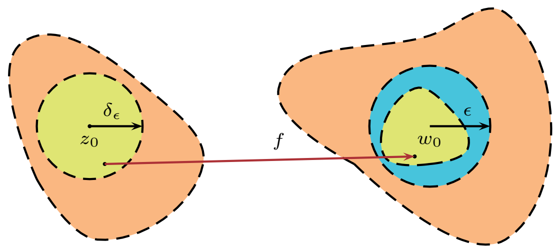
We say that $f(z)$ has a pole at $z = 0$.

The concept of limit for $w_0 \neq \infty$ is illustrated in the following diagram. Note that the yellow region on the right hand side denote the image of $B(z_0; \delta_\epsilon)$ under f. It also shows that we can replace the definition of limit by the following:

DEFINITION 2.2 For every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B'(z_0, \delta_\epsilon)) \subset B(w_0; \epsilon).$$

Here B' denote that z_0 is excluded from the open ball.



EXAMPLE 2.7

Let $f(z) = \dfrac{iz}{2}$. Show that

$$\lim_{z \to 1} f(z) = \frac{i}{2},$$

using the definition of limit.

To show that the above holds, we need to show that for each $\epsilon > 0$, there exist a $\delta_\epsilon > 0$ such that

$$\left| \frac{iz}{2} - \frac{i}{2} \right| < \epsilon \quad \text{whenever} \quad 0 < |z - 1| < \delta_\epsilon.$$

We usually work "backwards" to obtain our δ_ϵ. Let $\epsilon > 0$ be arbitrarily chosen. Now,

$$\left| \frac{iz}{2} - \frac{i}{2} \right| < \epsilon$$

if and only if

$$\left| \frac{i}{2} \right| |z - 1| < \epsilon$$

if and only if

$$0 < |z - 1| < 2\epsilon$$

since $|i| = 1$. Hence, we may take our δ_ϵ to be 2ϵ. Therefore, $f(z) \to \dfrac{i}{2}$ when $z \to 1$.

EXAMPLE 2.8

Show that the limit of the function $f(z) = \dfrac{\overline{z}}{z}$ as $z \to 0$ does not exist.

Solutions. Let $z = 0 + it$ and we see that $\overline{z}/z = -it/it$ tends to -1 as z tends to 0 along the imaginary axis. Let $z = t$ and $\overline{z}/z$ tends to 1 as z tends to 0 along the real axis. Hence, the limit of $\overline{z}/z$ does not exist as z tends to 0.

2.3 Theorems on Limits

Most of the results in this section are familiar as they also appear in the same form in Calculus and Real Analysis.

If

$$\lim_{z \to z_0} f(z) = A \quad \text{and} \quad \lim_{z \to z_0} g(z) = B,$$

then

$$\lim_{z \to z_0} (f(z) + g(z)) = A + B. \tag{2.1}$$

The proof of this is similar to that for real variable.

Suppose $f(z) \to w_0$ as $z \to z_0$, then

$$\lim_{z \to z_0} \operatorname{Re} f(z) = u_0$$

and

$$\lim_{z \to z_0} \operatorname{Im} f(z) = v_0,$$

where $w_0 = u_0 + iv_0$. This is because for $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_\epsilon.$$

Now, observe that

$$|f(z) - w_0| = |\overline{f(z)} - \overline{w_0}|$$

and hence, we find that $\overline{f(z)} \to \overline{w_0}$. Now,

$$\operatorname{Re}(f(z)) = \frac{f(z) + \overline{f(z)}}{2}.$$

Therefore, by (2.1)

$$\lim_{z \to z_0} \operatorname{Re}(f(z)) = \frac{w_0 + \overline{w_0}}{2} = \operatorname{Re}(w_0) = u_0.$$

Similarly,

$$\lim_{z \to z_0} \operatorname{Im}(f(z)) = \operatorname{Im}(w_0) = v_0.$$

Conversely, if $f(z) = u(x,y) + iv(x,y)$ and $F(z) := u(x,y) \to u_0$ and $G(z) = v(x,y) \to v_0$ [1] as $z \to z_0$, then by (2.1), we deduce that $f(z) = F(z) + iG(z) \to u_0 + iv_0$ as $z \to z_0$. We have thus proved the following theorem :

THEOREM 2.3 Suppose $f(z) = u(x,y) + iv(x,y), z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$. Then

$$\lim_{z \to z_0} f(z) = w_0,$$

[1] Note that $2x = z + \overline{z}$ and $2iy = z - \overline{z}$ and so, $u(x,y)$ and $v(x,y)$ are functions of z.

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

THEOREM 2.4 Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

(a) $\lim_{z\to z_0} f(z)g(z) = AB$;

(b) If $B \neq 0$, then $\lim_{z\to z_0} f(z)/g(z) = A/B$.

2.4 Continuity

DEFINITION 2.5 A function $f(z)$ is said to be continuous at z_0 if

(a) $\lim_{z\to z_0} f(z)$ exists,

(b) $f(z_0)$ exists, and

(c) $\lim_{z\to z_0} f(z) = f(z_0)$.

Statement (c) says that for every $\epsilon > 0$ there exist a δ_ϵ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_\epsilon.$$

In other words, $f(z)$ is continuous at z_0 if we obtain the limit $\lim_{z\to z_0} f(z)$ by simply substituting z_0 into $f(z)$.

DEFINITION 2.6 A function $f(z)$ is said to be continuous in a domain D if it is continuous at every point in D.

Theorem 2.3 says that $f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if the corresponding real and imaginary parts are continuous at (x_0, y_0), i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u(x_0,y_0), \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v(x_0,y_0).$$

Hence we may decide if a complex valued function $f(z)$ is continuous at any point z_0 by using known results in Calculus of two variables.

EXAMPLE 2.9

The function $f(z) = xy^2 + i(2x - y)$ is continuous at every point (x,y) since the corresponding real and imaginary parts are continuous functions of two variables.

As in the case of real variables, we have the fact that if $f(z)$ and $g(z)$ are continuous, then the functions $f(z)+g(z)$, $f(z)g(z)$, $f(z)/g(z)$ and $f(g(z))$ are continuous in the domain for which the functions are defined.

EXAMPLE 2.10
The polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is continuous for all $z \in \mathbf{C}$.

Solutions. z is continuous implies that z^2 is continuous since product of continuous functions is continuous. By induction, z^k is continuous for all positive integers k. Now, sum of continuous functions is continuous, hence every polynomial in z is continuous.

EXAMPLE 2.11
Show that if $f(z)$ is continuous at z_0 then $|f(z)|$ is continuous at z_0.

Solutions. By triangle inequality,

$$|f(z)| - |f(z_0)| < |f(z) - f(z_0)|$$

and

$$|f(z_0)| - |f(z)| < |f(z) - f(z_0)|.$$

Hence,

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|.$$

The result follows since

$$|f(z) - f(z_0)| < \epsilon$$

implies that

$$|f(z)| - |f(z_0)| < \epsilon.$$

REMARK 2.1 Strictly speaking, $|f|$ is a function from $\mathbf{C}$ to $\mathbf{R}$. However, we may identify this function with $F = |f| + i \cdot 0$ and conclude that F is continuous on $\mathbf{C}$.

REMARK 2.2 The notion of continuity can be generalized to arbitrary topological spaces other than $\mathbf{C}$. In order to achieve that, one has to define continuous function based on open sets instead of open balls. For more details, see the Appendix.

2.5 Derivative

The derivative of a function f at a (in the real variable case), denoted by $f'(a)$, is defined as the limit (if it exists)

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

For the case of complex variable, the definition is similar :

DEFINITION 2.7 The derivative of a function f at z, denoted by $f'(z)$, is defined as the limit (if it exists)

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a}.$$

We may also write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

EXAMPLE 2.12
Show that the function

$$f(z) = \overline{z}$$

is not differentiable at any point z.

Solutions. Observe that

$$\frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

By Example 2.8, we find that

$$\lim_{h\to 0} \frac{\overline{h}}{h}$$

does not exist and so $f(z)$ is not differentiable at any point z.

EXAMPLE 2.13
For a given $z_0 \in \mathbf{C}$, discuss the differentiability of $f(z) = |z|^2$ at $z = z_0$.

Solutions. Observe that

$$\begin{aligned}\frac{|z+h|^2-|z|^2}{h} &= \frac{(z+h)\overline{(z+h)}-z\overline{z}}{h}\\ &= \frac{h\overline{z}+\overline{h}z+h\overline{h}}{h}\\ &= \overline{z}+z\frac{\overline{h}}{h}+\overline{h}.\end{aligned}$$

We have already shown that the limit of $\overline{h}/h$ does not exist as $h \to 0$ and hence the function $|z|^2$ is differentiable only at $z = 0$. The derivative $f'(0) = 0$ in this case.

Note that both functions $\overline{z}$ and $|z|^2$ are continuous functions on $\mathbf{C}$. But $f(z) = \overline{z}$ is not differentiable everywhere while $f(z) = |z|^2$ is differentiable only at $z = 0$. It turns out that as in the case of real variable, differentiable functions are continuous. The proof is exactly the same as that for the real variable case.

The rules for differentiating a complex valued functions are the same as those for real variables. This is because the definitions for $f'(x)$ and $f'(z)$ are similar. [2] As such, we leave it as an exercise for the readers to prove the following standard results for differentiation:

1. $\dfrac{d}{dz}c = 0$
2. $\dfrac{d}{dz}z^n = nz^{n-1}, n \in \mathbf{Z}^+$
3. $\dfrac{d}{dz}(f(z)+F(z)) = f'(z)+F'(z)$
4. $\dfrac{d}{dz}(f(z)F(z)) = f(z)F'(z)+f'(z)F(z)$
5. (Chain Rule) If $F(z) = g(f(z))$, then $\dfrac{d}{dz}F(z) = g'(f(z))f'(z)$.

2.6 Cauchy-Riemann equations

We first recall the definition of partial derivatives. If $F(x,y)$ is a function of two real variables then

$$\frac{\partial F}{\partial x} = \lim_{\delta x\to 0}\frac{F(x+\delta x,y)-F(x,y)}{\delta x}$$

whenever the limit exists. Similarly,

$$\frac{\partial F}{\partial y} = \lim_{\delta y\to 0}\frac{F(x,y+\delta y)-F(x,y)}{\delta y}.$$

[2] For example, to prove that $f'(z) = 2z$ when $f(z) = z^2$, the proof is similar to functions on real numbers.

We recall that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

If the above limit exists, then regardless of how h approaches 0, the resulting value would be the same. We now let h approaches 0 along the real axis. Write $h = \delta x$. Write $f = u(x, y) + iv(x, y)$. Let $h = \delta x$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x}.$$

Therefore,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Next, we let h approaches 0 along the imaginary axis. Write $h = i\delta y,$, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y}.$$

Therefore,

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Hence,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

and

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

We conclude that the Cauchy-Riemann equations hold, namely,

$$u_x = v_y \quad \text{and} \quad v_x = -u_y,$$

where $F_t = \dfrac{\partial F}{\partial t}$.

EXAMPLE 2.14

Verify that the function $f(z) = z^4$ satisfies the Cauchy-Riemann equations. (This is not surprising since all polynomials in z are differentiable and hence, Cauchy-Riemann equations are satisfied.)

Solutions. In this case, $u(x, y) = x^4 - 6x^2y^2 + y^4$ and $v = 4x^3y - 4y^3x$. One checks easily that u and v satisfy the Cauchy-Riemann equations.

EXAMPLE 2.15
Suppose $f(z) = e^{-y} \sin x + iv(x, y)$ is differentiable everywhere. Find $v(x, y)$.

Solutions. Let $u = e^{-y} \sin x$. Then $u_x = e^{-y} \cos x$. Since the function is differentiable everywhere, the Cauchy-Riemann equations must be satisfied. Hence,

$$v_y = u_x = e^{-y} \cos x.$$

We conclude that

$$v = -e^{-y} \cos x + C(x),$$

where $C(x)$ is a function of x.

Next, $u_y = -e^{-y} \sin x$. Hence

$$v_x = e^{-y} \sin x.$$

Therefore,

$$v = -e^{-y} \cos x + D(y),$$

where $D(y)$ is a function of y. But $C(x) = D(y)$ implies that $C(0) = D(y)$ for all $y \in \mathbf{R}$ and so, $D(y)$ is a constant. Therefore, $C(x) = D(y) = a$ with $a \in \mathbf{C}$. We therefore conclude that

$$f(z) = e^{-y} \sin x - ie^{-y} \cos x + ia.$$

We have seen that if $f = u + iv$ and f is differentiable at $z = z_0$, then u and v satisfies the Cauchy-Riemann equation at $z_0 = x_0 + iy_0$. We can use this to show that a function is NOT differentiable at $z = z_0$.

EXAMPLE 2.16
Show that if $z \neq 0$, then $f(z) = 2xy + i(x^2 - y^2)$ is not differentiable.

Solutions. Let $u = 2xy$ and $v = x^2 - y^2$. If $f(z)$ is differentiable at z, then $u_x = v_y$. But this implies that $y = 0$. Similarly, if $u_y = -v_x$ then $x = 0$. Since $z \neq 0$, we conclude that $f(z)$ is not differentiable at z.

EXAMPLE 2.17
Show that the function $f(z) = \overline{z}$ is not differentiable at any point z.

Solutions. Now, $f(z) = x - iy$. Hence, $u = x$ and $v = -y$. But $u_x = 1$ and $v_y = -1$ implies that $u_x \neq v_y$ and therefore, $f(z)$ is not nowhere differentiable.

We return to the following statement :

If $f = u + iv$ and f is differentiable at $z = z_0$, then u and v satisfies the Cauchy-Riemann equation at $z_0 = x_0 + iy_0$.

The converse, however, is false. In other words, if u and v satisfies the Cauchy-Riemann equation at $z = z_0$, it does not imply that f is differentiable at $z = z_0$.

EXAMPLE 2.18
Consider the following example : Let

$$f(z) = f(x,y) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

Solutions. We first show that $f(z)$ is not differentiable at $z = 0$. Note that

$$\frac{f(z)-f(0)}{z} = \left(\frac{xy(x+iy)}{x^2+y^2} - 0\right)\frac{1}{x+iy} = \frac{xy}{x^2+y^2}.$$

Letting $z = h + ih$, we find that

$$\frac{f(z)-f(0)}{z} = \frac{h^3}{2h^3} \to \frac{1}{2}$$

as $h \to 0$. Letting $z = h + i \cdot 0$, we find that

$$\frac{f(z)-f(0)}{z} \to 0$$

as $h \to 0$. Therefore

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

does not exist and f is not differentiable at $z = 0$.

We first note that

$$u = \frac{x^2y}{x^2+y^2} \quad \text{and} \quad v = \frac{xy^2}{x^2+y^2}.$$

This implies that $u(h,0) = 0$ and that

$$\frac{u(h,0)-u(0,0)}{h} = 0.$$

Similarly, $v(0,h) = 0$ and

$$\frac{v(0,h)-v(0,0)}{h} = 0.$$

Therefore, $u_x(0,0) = 0 = v_y(0,0)$. In a similar way, $v_x(0,0) = v_y(0,0) = 0$. Therefore, Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

THEOREM 2.8 Let $f(z) = u(x,y) + iv(x,y)$. Suppose u_x, u_y, v_x, v_y exist in a neighborhood of z. Then if u_x, u_y, v_x, v_y are continuous at z and the Cauchy-Riemann equations hold, i.e.,

$$u_x = v_y, u_y = -v_x,$$

then f is differentiable at z.

Proof

The proof of this theorem depends on mean value theorem. We recall that if F is a continuous function of a real variable, then

$$F(x+h) - F(x) = F'(\xi)h,$$

where $|\xi - x| < h$.

We next write

$$f(z) = u(x,y) + iv(x,y).$$

Then

$$f(z+\delta z) = f(x+\delta x + i(y+\delta y)) = u(x+\delta x, y+\delta y).$$

This implies that

$$\begin{aligned} &\frac{f(z+\delta z) - f(z)}{\delta z} \\ &\quad = \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}. \end{aligned} \tag{2.2}$$

We now apply mean value theorem to the real part of the left hand side of (2.2) and deduce that

$$\begin{aligned} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} &= u_x(x+\theta_1, y+\delta y)\left(\frac{\delta x}{\delta x + i\delta y}\right) \\ &\quad + u_y(x, y+\theta_2)\left(\frac{\delta y}{\delta x + i\delta y}\right), \end{aligned}$$

where $(\theta_1, \theta_2) \to (0,0)$ as $(\delta x, \delta y) \to (0,0)$.

Similarly, applying mean value theorem to the imaginary part of the left hand side of (2.2), we deduce that

$$\begin{aligned} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y} &= v_y(x, y+\theta_3)\left(\frac{\delta y}{\delta x + i\delta y}\right) \\ &\quad + v_x(x+\theta_4, y+\delta y)\left(\frac{\delta x}{\delta x + i\delta y}\right), \end{aligned}$$

where $(\theta_3, \theta_4) \to (0,0)$ as $(\delta x, \delta y) \to (0,0)$. Thus,

$$\begin{aligned} \frac{f(z+\delta z) - f(z)}{\delta z} &= \frac{\delta y}{\delta x + i\delta y}\left(u_y(x, y+\theta_2) + iv_y(x, y+\theta_3)\right) \\ &\quad + \frac{\delta x}{\delta x + i\delta y}\left(u_x(x+\theta_1, y+\delta y) + iv_x(x+\theta_4, y+\delta y)\right). \end{aligned}$$

Now, using $u_x = v_y$ and $u_y = -v_x$, we conclude that

$$\begin{aligned}
&\frac{f(z+\delta z)-f(z)}{\delta z} - (u_x + iv_x) \\
&= \frac{\delta y}{\delta x + i\delta y}\left(u_y(x, y+\theta_2) - u_y(x,y) + iv_y(x, y+\theta_3) - iv_y(x,y)\right) \\
&+ \frac{\delta x}{\delta x + i\delta y}\left(u_x(x+\theta_1, y+\delta y) - u_x(x,y) + iv_x(x+\theta_4, y+\delta y) - iv_x(x,y)\right).
\end{aligned}$$

Since

$$\left|\frac{\delta x}{\delta x + i\delta y}\right| \le 1 \quad \text{and} \quad \left|\frac{\delta y}{\delta x + i\delta y}\right| \le 1,$$

we have

$$\begin{aligned}
&\left|\frac{f(z+\delta z)-f(z)}{\delta z} - (u_x + iv_x)\right| \\
&\le |(u_y(x, y+\theta_2) - u_y(x,y)| + |v_y(x, y+\theta_3) - v_y(x,y)| \\
&\qquad + |u_x(x+\theta_1, y+\delta y) - u_x(x,y)| + |v_x(x+\theta_4, y+\delta y) - v_x(x,y)|,
\end{aligned}$$

which tends to 0 as $(\delta x, \delta y) \to 0$, since $(\theta_1, \theta_2, \theta_3, \theta_4) \to (0,0,0,0)$ as $(\delta x, \delta y) \to (0,0)$ and u_x, u_y, v_x, v_y are all continuous. Hence,

$$\lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z}$$

exists and equals to $u_x + iv_x$. □

EXAMPLE 2.19
Show that $g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is differentiable for every $z \in \mathbf{C}$. Write $g(z)$ in terms of z.

Solutions. By the above theorem, it suffices to verify that $u_x = v_y$ and $v_x = -u_y$ and that these are continuous functions. To express $g(z)$ as a function of z, set $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$. The final answer is $g(z) = 3z^2 + 2z - 1$. Note that since $g(z)$ is a polynomial, we have another proof of the fact that $g(z)$ is differentiable for every $z \in \mathbf{C}$.

2.7 Analytic functions

DEFINITION 2.9 A function f is analytic at z if f is differentiable in a neighborhood of z. A function f is analytic on a set S if f is analytic at every $z \in S$.

EXAMPLE 2.20
Polynomials are analytic on $\mathbf{C}$ since their derivatives exist at every $z \in \mathbf{C}$.

EXAMPLE 2.21
The function $f(z) = |z|^2$ is differentiable only at $z = 0$ and hence, it is not analytic at $z = 0$.

EXAMPLE 2.22
In calculus, we know that if $f'(x) = 0$ on (a, b) then $f(x)$ is a constant on (a, b). Is this true for analytic functions in $\mathbf{C}$? In other words, if $f'(z) = 0$, is $f(z)$ necessarily a constant?

Solutions. The result is true. Let $f(z) = u + iv$. Suppose

$$f'(z) = u_x + iv_x = v_y - iu_y = 0.$$

This implies that $u_x = u_y = 0$ or $u = g(y) = h(x)$ for some functions g and h. Since x and y are independent variables, we find that $u = g(y) = h(x) = a$, where a is a constant in $\mathbf{R}$. Similarly, $v = b$, where b is a constant in $\mathbf{R}$. Therefore, $f = u + iv = a + ib$ is a constant in $\mathbf{C}$.

EXAMPLE 2.23
If $f = u + iv$ is analytic in a region D and u is constant, show that f is a constant.

Solutions. If u is a constant, then $u_x = u_y = 0$. By Cauchy Riemann equations, $v_x = v_y = 0$. This implies that $v = s(x) = t(y)$ and hence v must also be a constant. Therefore f is a constant.

EXAMPLE 2.24
If f is analytic in a region and if $|f|$ is constant there, show that f is a constant.

Solutions. Suppose $|f|$ is a constant. If $|f| = 0$ then $u = 0$ and $v = 0$. Suppose $|f| \neq 0$. Then $|f|^2 = u^2 + v^2 = c$ where c is a constant. This implies that

$$uu_x + vv_x = 0 \tag{2.3}$$

and

$$uu_y + vv_y = 0. \tag{2.4}$$

Using the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$, we rewrite (2.3) and (2.3) and (2.4) as

$$uu_x - vu_y = 0 \tag{2.5}$$

and

$$uu_y + vu_x = 0. \tag{2.6}$$

Eliminating u_y using (2.5) and (2.6), we conclude that

$$u_x(u^2 + v^2) = 0.$$

Since $|f|^2$ is a non-zero constant, we find that $u_x = 0$. Solving u_y in a similar way, we conclude that $u_y = 0$. This implies that $u = s(y) = t(x)$ and so, u is a constant. By Example 2.23, we conclude that f is a constant.

2.8 The exponential function

DEFINITION 2.10 A function which is analytic on the whole of **C** is said to be entire.

EXAMPLE 2.25

Find an entire function $f(z)$ which satisfies the conditions

$$f(z+w) = f(z)f(w) \tag{2.7}$$

and

$$f(x) = e^x, \quad \text{when } x \in \mathbf{R}. \tag{2.8}$$

Solutions. Let $z = x + iy$. Then by (2.7),

$$f(z) = f(x)f(iy) = e^x f(iy) = e^x(A(y) + iB(y))$$

for some functions $A(y)$ and $B(y)$.

Now, since f is analytic, Cauchy Riemann equations are satisfied and with $u = e^x A(y)$ and $v = e^x B(y)$, we have $u_x = v_y$ and $u_y = -v_x$ implies that

$$A(y) = B'(y) \quad \text{and} \quad A'(y) = -B(y)$$

respectively. This implies that

$$A''(y) + A(y) = 0.$$

It is known that the solutions to the above differential equation are [3]

$$A(y) = \alpha \cos y + \beta \sin y.$$

Therefore,

$$B(y) = -A'(y) = -\beta \cos y + \alpha \sin y.$$

Since $f(x) = e^x$, we conclude that $A(0) = \alpha = 1$. Also, $B(0) = 0 = -\beta$. Therefore,

$$A(y) = \cos y \quad \text{and} \quad B(y) = \sin y.$$

[3] A second order differential equations has solutions in the form $y = af + bg, a, b \in \mathbf{R}$ if f and g are independent solutions. For more details, see p. 58, Theorem 5 of E.A. Coddington, *An introduction to ordinary differential equations.*

Hence,

$$f(z) = e^x (\cos y + i \sin y).$$

The function f extends e^x and is entire. We write $f(z)$ as e^z.

Note that if we compare $e^z = e^{x+iy}$ with $e^x(\cos y + i \sin y)$, then we see that

$$e^{iy} = \cos y + i \sin y.$$

Therefore,

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

As in the case of e^x, we may define sine and cosine with variables $z \in \mathbf{C}$ as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

2.9 Harmonic functions

Let $f(z) = u + iv$ be analytic. We will show later that $f'(z)$ is analytic. Suppose for the moment, we assume that $f'(z)$ is analytic. We observe that

$$f'(z) = u_x + iv_x = v_y - iu_y. \tag{2.9}$$

Let $U = u_x$ and $V = v_x$, i.e., $f' = U + iV$. Using (2.9) with f replaced by f', we find that

$$f''(z) = U_x + iV_x = V_y - iU_y$$

or

$$u_{xx} + iv_{xx} = v_{xy} - iu_{xy}, \tag{2.10}$$

where

$$F_{xy} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x}.$$

Next, since $f'(z) = v_y - iu_y$, we may let $U^* = v_y$ and $V^* = -u_y$ and conclude using (2.9) (with f replaced by f') that

$$f''(z) = (U^*)_x + i(V^*)_x = -(V^*)_y - i(U^*)_y.$$

This implies that

$$v_{yx} - iu_{yx} = -u_{yy} - iv_{yy}. \tag{2.11}$$

Comparing (2.10) and (2.11), we conclude that

$$u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0, \tag{2.12}$$

$$u_{xy} = u_{yx} \quad \text{and} \quad v_{xy} = v_{yx}.$$

We note from (2.12) that u and v are functions satisfying

$$h_{xx} + h_{yy} = 0.$$

DEFINITION 2.11 A real-valued function $h(x, y)$ which is twice continuously differentiable and satisfies

$$h_{xx} + h_{yy} = 0$$

is called a harmonic function.

We sometimes use $\nabla^2 h$ or Δh to denote $h_{xx} + h_{yy}$.

We have seen in the beginning of this section that if f is analytic at z then u and v are harmonic functions.

DEFINITION 2.12 Let u be harmonic. If v is a harmonic function satisfying the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

then v is called the conjugate harmonic function of u.

EXAMPLE 2.26

Verify that $u = xy - x + y$ is a harmonic function and find its harmonic conjugate.

Solutions. The verification that u is a harmonic function is straightforward since $u_{xx} = u_{yy} = 0$. Now, $u_x = y - 1$ and $u_y = x + 1$. Using Cauchy-Riemann equations, we find that $v_y = y - 1$ or $v = y^2/2 - y + s(x)$. Also, $-v_x = x + 1$ implies that $v = -x^2/2 - x + t(y)$. Now, we must have

$$-\frac{x^2}{2} - x - s(x) = \frac{y^2}{2} - y - t(y) = C$$

where C is a constant. Hence,

$$t(y) = \frac{y^2}{2} - y + C$$

and

$$v = -\frac{x^2}{2} - x + \frac{y^2}{2} - y + C.$$

2.10 Appendix

We motivate the general definition of continuous functions by proving the following :

THEOREM 2.13 Let f be a surjective function from $\mathbf{C}$ to $\mathbf{C}$. The following are equivalent :

(a) f is continuous
(b) If V is an open set in $\mathbf{C}$, then $f^{-1}(V)$ is an open set in $\mathbf{C}$, where

$$f^{-1}(A) = \{z \in \mathbf{C} \,|\, f(z) \in A\}.$$

Proof
We first show (a) implies (b). We translate continuity of f in terms of open balls. Observe that f is continuous at z_0 if and only if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(z_0; \delta_\epsilon)) \subset B(f(z_0); \epsilon).$$

Now, let V be open in $\mathbf{C}$. We must show that $f^{-1}(V)$ is open in $\mathbf{C}$. Let $z_0 \in f^{-1}(V)$. Then $f(z_0) \in V$. Since V is open, there exists $\epsilon > 0$ such that

$$B(f(z_0); \epsilon) \subset V.$$

For this $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(z_0; \delta_\epsilon)) \subset B(f(z_0); \epsilon) \subset V,$$

by continuity of f. This means that

$$B(z_0; \delta_\epsilon) \subset f^{-1}(B(f(z_0); \epsilon)) \subset f^{-1}(V).$$

Hence $f^{-1}(V)$ is open.

To show that (b) implies (a). Let f be a function satisfying (b). Let $\epsilon > 0$ and since $B(f(z_0); \epsilon)$ is open, $f^{-1}(B(f(z_0); \epsilon))$ is open in the domain of f. Since $z_0 \in f^{-1}(B(f(z_0); \epsilon))$ and the set is open, there exists $\delta_\epsilon > 0$ such that

$$B(z_0; \delta_\epsilon) \subset f^{-1}(B(f(z_0); \epsilon)),$$

or

$$f(B(z_0; \delta_\epsilon)) \subset B(f(z_0); \epsilon),$$

and so f is continuous. □

The usual definition for continuous function from X to Y where X and Y are topological spaces is the following:

DEFINITION 2.14 A function f from topological spaces X to Y is continuous if for every open set V in Y, the set $f^{-1}(V)$ is open in X.

If X and Y are subsets of $\mathbf{C}$, we can form topological spaces from X and Y by declaring that the open sets in X and Y are sets of the form $O \cap X$ and $O \cap Y$ respectively, with O open in $\mathbf{C}$. We sometimes call such open sets relatively open sets. I prefer to call sets $O \cap X$ (or $O \cap Y$), O open in $\mathbf{C}$, as sets that are "open in X" (or "open in Y").

We now show the following:

THEOREM 2.15 If f is a continuous function from X to Y (in the usual sense), then for every V open in Y, $f^{-1}(V)$ is open in X.

Proof
The function f is continuous at u means that for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(u;\delta_\epsilon)) \subset B(f(u);\epsilon).$$

Suppose V is open in Y. Then $V = O \cap Y$ where O is an open set of $\mathbf{C}$. Now

$$f^{-1}(V) = f^{-1}(O) \cap X.$$

Let $u \in f^{-1}(V)$. Then $f(u) \in V$ and hence, $f(u) \in O$. This implies that there exists $\epsilon > 0$ such that

$$B(f(u);\epsilon) \cap Y \subset O \cap Y$$

since O is open. Since f is continuous at u, there exists $\delta(u) > 0$ such that

$$f(B(u;\delta(u)) \cap X) \subset B(f(u);\epsilon) \cap Y.$$

Hence

$$B(u;\delta) \cap X \subset f^{-1}(B(f(u);\epsilon)) \cap X \subset f^{-1}(V).$$

Therefore,

$$f^{-1}(V) = \bigcup_{u \in f^{-1}(V)} (B(u;\delta(u)) \cap X) = \left(\bigcup_{u \in f^{-1}(V)} B(u;\delta) \right) \cap X$$

and this implies that $f^{-1}(V)$ is open. □

3 Line Integrals

In real analysis, we encounter integrals defined by

$$\int_a^b f(x)dx$$

for some interval (a, b). In complex analysis, we have analogue of integrals as well. We will begin with the simplest type of complex integrals, the line integrals.

3.1 Properties of Line integrals

DEFINITION 3.1 Let $f(t) = u(t) + iv(t)$ be any continuous complex valued function of the real variable t, $a \leq t \leq b$. Then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Note that in general, our $f(z)$ depends on two variables x, y since $z = x + iy$. Here, we assume that z is of the form $x(t) + iy(t)$ and consequently, $f(z)$ is a function of t.

DEFINITION 3.2 If $x(t)$, $y(t)$ are continuous on $[a, b]$ and their derivatives $x'(t)$, $y'(t)$ are continuous on intervals $[a, x_1], [x_1, x_2], \cdots, [x_{n-1}, b]$, where

$$[a, b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}], x_0 = a, x_n = b,$$

then we say that the curve

$$z(t) = x(t) + iy(t)$$

is piecewise differentiable and we set

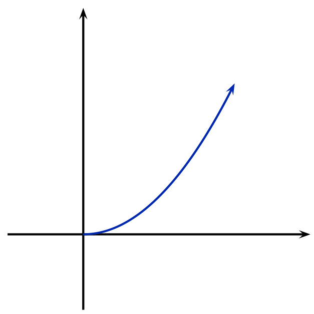
$$\dot{z}(t) = x'(t) + iy'(t).$$

DEFINITION 3.3 The curve is said to be smooth if $\dot{z}(t) \neq 0$ except at a finite number of points.

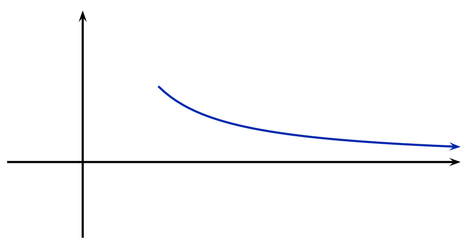
We will assume smoothness throughout our course unless otherwise stated.

EXAMPLE 3.1
Examples of smooth curves:

(i) $z(t) = t + it^2, 0 \leq t \leq 2$.



(ii) $z(t) = t + \dfrac{i}{t}, 1 \leq t \leq 5$.



DEFINITION 3.4 Let C be a smooth curve given by $z(t), a \leq t \leq b$, and suppose f is continuous at all points $z(t)$. Then the integral of f along C is defined by

$$\int_C f(z)dz := \int_a^b f(z(t))\dot{z}(t)dt.$$

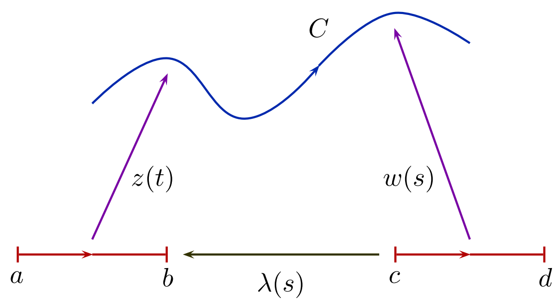
DEFINITION 3.5 The two curves

$$C_1 : z(t), a \leq t \leq b,$$

$$C_2 : w(s), c \leq s \leq d,$$

are smoothly equivalent if there exist a one to one function $\lambda : [c, d] \to [a, b]$ which has continuous first derivative such that $\lambda(c) = a$, $\lambda(d) = b$ and

$$w(s) = z(\lambda(s)).$$



EXAMPLE 3.2
The functions

$$z(t) = t + it^2, 0 \leq t \leq 3$$

and

$$w(s) = (s - 1) + i(s - 1)^2, 1 \leq s \leq 4$$

give the same curve. Here, $\lambda(s) = s - 1$.

THEOREM 3.6 If C_1 and C_2 are smoothly equivalent then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Proof
Suppose C_1 is given by $z(t)$, $t \in [a, b]$ and C_2 is given by $w(s)$, $s \in [c, d]$, with $\lambda(s)$ being a one to one function with continuous first derivative from $[c, d]$ to $[a, b]$ and

$$w(s) = z(\lambda(s)).$$

Now,

$$\begin{aligned}\int_{C_2} f(w)dw &= \int_c^d f(w(s))\dot{w}(s)ds \\ &= \int_c^d f(z(\lambda(s)))\dot{z}(\lambda(s))\lambda'(s)ds \\ &= \int_a^b f(z(\mu))\dot{z}(\mu)d\mu, \quad \text{with } \mu = \lambda(s) \\ &= \int_{C_1} f(z)dz.\end{aligned}$$

Hence the result. □

REMARK 3.1 The second equality above follows by splitting the second integral into real and complex part, apply integral by substitution, and collect the resulting integrals. We write the proof as above to show that the result follows from the substitution $\mu = \lambda(s)$.

DEFINITION 3.7 Suppose C is given by $z(t), a \leq t \leq b$. Then $-C$ is defined by $z(b+a-s), a \leq s \leq b$.

THEOREM 3.8 Let C be a smooth curve and f be a continuous complex-valued function. Then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Proof
Let $w(s) = z(b+a-s)$. Then

$$\begin{aligned}\int_{-C} f(w)\,dw &= \int_a^b f(w(s))\dot{w}(s)ds = \int_a^b f(z(b+a-s))\dot{z}(b+a-s)(b+a-s)'\,ds \\ &= \int_b^a f(z(b+a-s))\dot{z}(b+a-s)d(b+a-s) = \int_b^a f(z(t))\dot{z}(t)dt \\ &= -\int_a^b f(z(t))\dot{z}(t)dt = -\int_C f(z)dz.\end{aligned}$$

□

EXAMPLE 3.3
Set $C : z(t) = R\cos t + iR\sin t, 0 \leq t \leq 2\pi, R \neq 0$. Show that

$$\int_C \frac{1}{z}\,dz = 2\pi i.$$

Solutions. The curve C is parametrized by $z(t) = R\cos t + iR\sin t$. This implies that $\dot{z}(t) = -R\sin t + R\cos t$. Hence

$$\begin{aligned}\int_C \frac{1}{z}dz &= \int_0^{2\pi} \frac{1}{R(\cos t + i\sin t)}(R(-\sin t + i\cos t))\,dt \\ &= \int_0^{2\pi} \frac{(-\sin t + i\cos t)(\cos t - i\sin t)}{\sin^2 t + \cos^2 t}\,dt \\ &= \int_0^{2\pi} i\,dt = 2\pi i.\end{aligned}$$

Note that the above integral is independent of the radius R of the contour C.

EXAMPLE 3.4

Suppose $f(z) = x^2 + iy^2$, and Let $C : z(t) = t + it, 0 \le t \le 1$. Compute

$$\int_C x^2 + iy^2\,dt.$$

Solutions. The curve C is parametrized by $C : z(t) = t + it, 0 \le t \le 1$. Then $\dot{z}(t) = 1 + i$.

$$\int_C f(z)dz = \int_0^1 (t^2 + it^2)(1+i)dt = (1+i)^2 \int_0^1 t^2\,dt = \frac{2i}{3}.$$

The line integral of $f(z)$ over C has properties similar to Riemann integrals. We state two of them here and leave the proofs as exercises.

THEOREM 3.9 Let C be a smooth curve. Let f and g be continuous functions on C, and let α be any complex number. Then

(a) $\int_C f(z) + g(z)dz = \int_C f(z)dz + \int_C g(z)dz$,

(b) $\int_C \alpha f(z)dz = \alpha \int_C f(z)dz$.

3.2 An analogue of the Fundamental Theorem of Calculus

THEOREM 3.10 Suppose C is a smooth curve parametrized by $z(t) : a \le z \le b$ and suppose $F'(z) = f(z)$. Then

$$\int_C f(z)dz = F(z(b)) - F(z(a)).$$

In other words, the integral depends only on the end points. We call F a primitive of f.

Proof
Let $F(z) = U(x,y) + iV(x,y)$. Since $F'(z) = f(z)$, we conclude that

$$f(z) = F'(z) = U_x(x,y) + iV_x(x,y).$$

Hence,

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(z(t))\dot{z}(t)dt = \int_a^b (U_x + iV_x)(x'(t) + iy'(t))dt \\ &= \int_a^b (U_x)x'(t) - V_x y'(t)dt + i\int_a^b V_x x'(t) + U_x y'(t)dt \\ &= \int_a^b U_x x'dt + U_y y'dt + i\int_a^b V_x x'dt + V_y y'dt,\end{aligned}$$

where the last equality follows from the Cauchy-Riemann equations. Now,[1]

$$dU = U_x dx + U_y dy$$

and hence,

$$\frac{dU}{dt}dt = U_x x'dt + U_y y'dt,$$

and

$$\begin{aligned}\int_C f(z)dz &= \int_a^b \frac{dU}{dt}dt + i\int_a^b \frac{dV}{dt}dt \\ &= U(x(b),y(b)) - U(x(a),y(a)) + i(V(x(b),y(b)) - V(x(a),y(a))) \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

□

REMARK 3.2 Theorem 3.10 shows that if $f(z)$ has a primitive $F(z)$, then the integral of $f(z)$ from z_1 to z_2 is dependent only on the two end points z_1 and z_2 and is independent of the path taken to move from z_1 to z_2.

EXAMPLE 3.5
By finding a primitive for $f(z)$, evaluate

$$\int_{[i,i/2]} e^{\pi z}dz$$

where $[i, i/2]$ denote the straight path from i to $i/2$.

Solutions. It is known that if $F(z) = \dfrac{e^{\pi z}}{\pi}$, then (this is obtained by the integrating $e^{\pi x}$ when x is real)

$$F'(z) = e^{\pi z}.$$

[1] Stewart's Calculus, p. 932, Equation 10 and p. 937, Theorem 15.5.2.

Hence,

$$\int_{[i,i/2]} e^{\pi z} dz = \frac{1}{\pi}\left(e^{\pi i/2} - e^{\pi i}\right).$$

REMARK 3.3 It is possible to evaluate the integral

$$\int_{[i,i/2]} e^{\pi z} dz$$

by parametrizing the path using $z(t) = (1-t)i + ti/2, 0 \leq t \leq 1$. We leave this as an exercise.

3.3 The ML-formula

LEMMA 3.11 Suppose $G(t)$ is a continuous complex valued function of t. Then

$$\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|\, dt.$$

Proof

Let

$$\int_a^b G(t)dt = Re^{i\theta}, R \geq 0,$$

since the integral is a complex number. Therefore,

$$\int_a^b e^{-i\theta} G(t)dt = R.$$

Suppose

$$e^{-i\theta}G(t) = A(t) + iB(t),$$

where $A(t)$ and $B(t)$ are real valued functions of t. Then

$$R = \int_a^b A(t)dt + i\int_a^b B(t)dt.$$

But since R is real,

$$\int_a^b B(t)dt = 0$$

and

$$R = \int_a^b A(t)dt = \int_a^b \text{Re}\left(e^{-i\theta}G(t)\right) dt.$$

But $|\text{Re}\,(z)| < |z|$ and so,

$$R \leq \int_a^b |e^{-i\theta}G(t)|\,dt = \int_a^b |G(t)|\,dt$$

since $|e^{i\theta}| = 1$. Therefore,

$$\left|\int_a^b G(t)dt\right| = R|e^{i\theta}| = R \leq \int_a^b |G(t)|dt,$$

or

$$\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|dt.$$

□

We now establish the ML-formula.

THEOREM 3.12 Let C be a smooth curve of length L and f be continuous on C and $|f| \leq M$ on C. Then

$$\left|\int_C f(z)dz\right| \leq ML.$$

Proof
Let C be represented by $z(t) = x(t) + iy(t)$. Then by Lemma 3.11,

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\dot{z}(t)dt\right| \leq \int_a^b |f(z(t))| \cdot |\dot{z}(t)|\,dt.$$

Now,

$$\begin{aligned}\int_a^b |f(z(t))| \cdot |\dot{z}(t)|\,dt &\leq \int_a^b \max_{z\in C}(|f(z(t))|)|\dot{z}(t)|dt \\ &\leq M\int_a^b |\dot{z}(t)|dt.\end{aligned}$$

Next recall that if a curve $z(t) = x(t) + iy(t)$ then the length L of the curve is given by [2]

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b |\dot{z}(t)|dt.$$

Hence,

$$\left|\int_C f(z)dz\right| \leq ML.$$

□

[2] Stewart's Calculus, Theorem 9.1.2.

EXAMPLE 3.6

Let C be the contour $z = 3e^{i\theta}, 0 \leq \theta \leq \pi$. Show that

$$\left|\int_C \frac{z}{z^2+1}dz\right| \leq \frac{9\pi}{8}.$$

Solutions. From Theorem 3.12, we know that

$$\left|\int_C \frac{z}{z^2+1}dz\right| \leq ML,$$

where M is the bound of $|f|$ on C and L is the length of C. We now compute M. On C, $z = 3e^{i\theta}$ and so, $|z| = 3$ and $|z^2+1| \geq 9 - 1 = 8$. This implies that

$$\frac{1}{|z^2+1|} \leq \frac{1}{8}.$$

Hence,

$$\left|\frac{z}{z^2+1}\right| \leq \frac{3}{8},$$

on C. Therefore $M \leq \frac{3}{8}$. Next, the length L of the arc C is clearly 3π since this is the perimeter of the semi-circle with radius 3. Therefore, by the ML-formula, we deduce that

$$\left|\int_C \frac{z}{z^2+1}dz\right| \leq \frac{9\pi}{8}.$$

4 The Cauchy-Goursat Theorem

4.1 The Cauchy-Goursat Theorem

DEFINITION 4.1 A closed curve C is a curve where the initial point and terminal point meets. A simple closed curve C is a closed curve which has no other meeting points.

DEFINITION 4.2 A set S is star-shaped if it has a point s, called the star center, so that for each $z \in S$, the segment $[s, z]$ lies in S. Here $[s, z]$ denote the line joining s and z.

REMARK 4.1 Suppose $P(x, y)$ and $Q(x, y)$ together with their partial derivatives are continuous in the region bounded by a simple closed curve. Then according to Green's Theorem in advanced calculus

$$\int_C P dx + Q dy = \iint_R (Q_x - P_y) dx dy.$$

The first integral is over the contour and the second is over the region bounded by the contour.

Now consider

$$f(z) = u(x, y) + iv(x, y),$$

is analytic in R bounded by C. Then

$$\begin{aligned} \int_C f(z)\,dz &= \int_C (u + iv)(dx + idy) \\ &= \int_C udx - vdy + i(\int_C vdx + udy) \\ &= \iint_R (-v)_x - u_y dxdy + i(\iint_R u_x - v_y dxdy) = 0, \end{aligned}$$

since Cauchy-Riemann equations are satisfied, namely, $u_x = v_y$ and $u_y = -v_x$. Therefore, Therefore,

$$\int_C f(z) dz = 0.$$

In the above discussion, we have assumed that f' is continuous. Our aim will be to prove a result similar to the above without assuming the continuity of f'.

EXAMPLE 4.1
Let $f(z) = ze^{-z}$. Now ze^{-z} is analytic in $|z| \leq 1$. Therefore

$$\int_C f(z)dz = 0.$$

DEFINITION 4.3 Let $d(x, y) = |x - y|$ and if S is a set in $\mathbf{C}$, we let the diameter of S, denoted by diam S, to be

$$\operatorname{diam} S = \sup\{d(x, y)|x, y \in S\}.$$

LEMMA 4.4 Suppose $\{F_k\}$ is a collection of non-empty closed sets with

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

and diam $F_n \to 0$ as $n \to \infty$, then $\displaystyle\bigcap_{n=1}^{\infty} F_n$ consists of a single point.

Proof
We need the fact that $\mathbf{C}$ is complete. In other words, every Cauchy sequence [1] in $\mathbf{C}$ is convergent.

Let $z_k \in F_k$. Since diam $F_n \to 0$ as $n \to \infty$, we find that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$\operatorname{diam} F_n < \epsilon$$

whenever $n \geq N_\epsilon$. Let $n > m > N_\epsilon$. Then $z_n, z_m \in F_{N_\epsilon}$ and

$$|z_n - z_m| \leq \operatorname{diam} F_{N_\epsilon} < \epsilon.$$

This implies that $\{z_k\}$ is a Cauchy sequence and therefore it converges to a limit z_0. If

$$z_0 \in \bigcap_{n=1}^{\infty} F_n,$$

then we have proved that $\displaystyle\bigcap_{n=1}^{\infty} F_n$ is non-empty.

[1] A sequence $\{z_k\}$ is Cauchy if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that $|z_n - z_m| < \epsilon$ whenever $n > m > N_\epsilon$.

If

$$z_0 \notin \bigcap_{n=1}^{\infty} F_n,$$

then

$$z_0 \in \bigcup_{n=1}^{\infty} F_n^c.$$

This implies that $z_0 \in F_s^c$ for some integer $s \in \mathbf{Z}^+$. Since F_s is closed, F_s^c is open and $B(z_0;\delta) \subset F_s^c$ for some $\delta > 0$. This implies that

$$B(z_0;\delta) \cap F_s = \phi,$$

or

$$|w - z_0| \geq \delta \tag{4.1}$$

for all $w \in F_s$. Since $z_k \to z_0$, for $\delta > 0$, there exists $N_\delta \in \mathbf{Z}^+$ such that

$$|z_k - z_0| < \delta \tag{4.2}$$

whenever $k \geq N_\delta$. Now take $N' = \max(N_\delta, s)$. Then for $k \geq N'$, $z_k \in F_s$ and $z_k \in F_{N_\delta}$. But this means that both (4.1) and (4.2) are true and $\delta < \delta$, which is impossible. Hence,

$$z_0 \in \bigcap_{n=1}^{\infty} F_n.$$

Now, suppose $z_0, y \in \bigcap_{n=1}^{\infty} F_n$. Then

$$d(z_0, y) \leq \text{diam} F_n,$$

for $n \geq 1$. But diam $F_n \to 0$ as $n \to \infty$ implies that $d(z_0, y) = 0$. Hence $z_0 = y$. This implies that

$$\bigcap_{n=1}^{\infty} F_n = \{z_0\}.$$

□

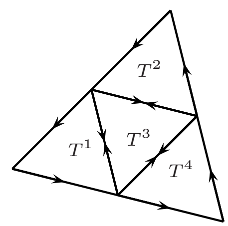
LEMMA 4.5 If f is analytic on an open set D and T is a closed triangular region with boundary ∂T that lies in D, then

$$\int_{\partial T} f(z)dz = 0.$$

Proof
We will prove the result by contradiction. Suppose

$$I := \int_{\partial T} f(z)dz \neq 0.$$

Take the mid-point of each side and subdivide T into four triangles T^1, T^2, T^3, and T^4 (See the following diagram).



Note that

$$\int_{\partial T} f(z)dz = \sum_{k=1}^{4} \int_{\partial T^k} f(z)dz.$$

Hence, by triangle inequality,

$$0 \neq \left| \int_{\partial T} f(z)dz \right| \leq \sum_{k=1}^{4} \left| \int_{\partial T^k} f(z)dz \right|.$$

Note that the above inequality implies that if for $1 \leq i \leq 4$,

$$\left| \int_{\partial T^i} f(z)dz \right| < \frac{1}{4} \left| \int_{\partial T} f(z)dz \right|,$$

then $|I| < |I|$, which is impossible. Therefore, there exists one T^i such that

$$\left| \int_{\partial T^i} f(z)dz \right| \geq \frac{1}{4} \left| \int_{\partial T} f(z)dz \right|.$$

Note that the diameter of T^i is half the diameter of T, i.e.,

$$\text{diam}(T^i) = \frac{\text{diam}(T)}{2},$$

and the length of ∂T^i is also half of the length of ∂T, or,

$$L(\partial T^i) = \frac{L(\partial T)}{2}.$$

Denote T by T_0 and let i_0 be such that the integral over $\partial T_0^{i_0}$ has modulus greater or equal to a quarter of $|I|$. Let $T_1 = T_0^{i_0}$. Repeat the above process with

T_0 replaced by T_1. We then obtain a sequence of triangles $T_1, T_2 = T_1^{i_1}, \cdots, T_n = T_{n-1}^{i_{n-1}}$ such that the diameter of T_n is $1/2^n$ times the diameter of T, namely,

$$\text{diam}(T_n) = \frac{\text{diam}(T)}{2^n},$$

the length of ∂T_n is $1/2^n$ times the length of T, or

$$L(\partial T_n) = \frac{L(\partial T)}{2^n}.$$

and that

$$\left| \int_{\partial T_n} f(z)dz \right| \geq \frac{1}{4^n} \left| \int_{\partial T} f(z)dz \right|.$$

Note that $\{T_n\}$ is nested, closed and so there must be a point

$$z_0 \in \cap T_n,$$

by Lemma 4.4.

Let

$$\epsilon = \frac{|I|}{2\text{diam}(T)L(\partial T)} > 0.$$

Since f is differentiable at z_0, there is a $\delta_\epsilon > 0$ so that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon,$$

whenever $0 < |z - z_0| < \delta_\epsilon$. In other words,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0|.$$

We next note that there is an index N such that $T_N \subset B(z_0; \delta_\epsilon)$. To see this, we note that since diam $T_n \to 0$ as $n \to \infty$, there exists $N \in Z^+$ such that

$$\text{diam}\, T_n < \frac{\delta_\epsilon}{2}$$

whenever $n \geq N$. For all $u \in T_N$, we find that

$$|u - z_0| < \text{diam}\, T_N < \frac{\delta_\epsilon}{2},$$

since $z_0 \in T_N$. This means that T_N is contained in $B(z_0; \delta_\epsilon)$.

Now, both $f'(z_0)(z - z_0)$ and $f(z_0)$ have primitives which are $f'(z_0)(z^2/2 - z_0 z)$ and $f(z_0)z$, respectively. Hence,

$$\int_{\partial T_N} f'(z_0)(z - z_0)dz = 0 \quad \text{and} \quad \int_{\partial T_N} f(z_0)dz = 0$$

and we deduce that

$$\int_{\partial T_N} (f(z) - f(z_0) - f'(z_0)(z - z_0))dz = \int_{\partial T_N} f(z)dz.$$

Hence, we conclude that

$$\begin{aligned}0 < |I| &= \left|\int_{\partial T} f(z)dz\right| \le 4^N \left|\int_{\partial T_N} f(z)dz\right| \\ &\le 4^N \int_{\partial T_N} |f(z) - f(z_0) - f'(z_0)(z - z_0)|dz \\ &< 4^N \epsilon (\text{diam} T_N) L(\partial T_N) \\ &\le 4^N \epsilon \frac{1}{2^N} \text{diam}(T) \frac{1}{2^N} L(\partial T) = \frac{|I|}{2}.\end{aligned}$$

This is clearly a contradiction and we conclude that $I = 0$. □

4.2 Existence of primitive

Recall in calculus that if f is continuous on (a, b), we can create a primitive by letting

$$F(x) = \int_{x_0}^{x} f(t)\, dt.$$

We then show that $F'(x) = f(x)$. As an application, we define

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

and note that

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

In order to create primitive for continuous function on star-shaped region, we will follow the above idea and set

$$F(z) = \int_{[s,z]} f(\zeta)\, d\zeta$$

where s is the star center. This is the starting point of the proof of the following theorem.

THEOREM 4.6 Let S be an open star-shaped region and f be continuous on S. Let T be a closed triangular region and ∂T be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z)dz = 0$$

for every T in S, then f has a primitive F on S.

Proof
Let s be a star center for S. For each $z \in S$, define

$$F(z) = \int_{[s,z]} f(\zeta)d\zeta.$$

We will show that

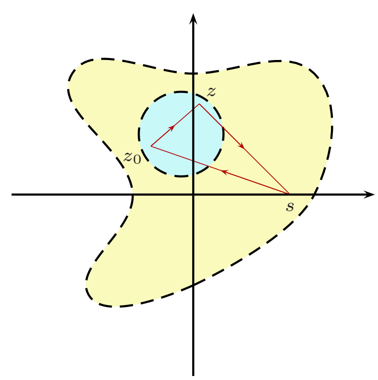
$$F'(z_0) = f(z_0)$$

for all $z_0 \in S$.

Let $z_0 \in S$. Since S is open, there exist a $\delta_1 > 0$ such that $B(z_0; \delta_1) \subset S$ (S is open).

Next, since f is continuous at z_0, we find that for $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(\zeta) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_2 \tag{4.3}$$



Let $\delta = \min(\delta_1, \delta_2)$. For $z \in B(z_0; \delta)$,

$$\int_{[z,z_0]} + \int_{[z_0,s]} + \int_{[s,z]} f(\zeta)d\zeta = 0,$$

since by hypothesis, the integral over any boundary of a triangle in S is 0. This implies that

$$\begin{aligned}
\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0}\left\{\int_{[s,z]} f(\zeta)d\zeta - \int_{[s,z_0]} f(\zeta)d\zeta - (z - z_0)f(z_0)\right\} \\
&= \frac{1}{z - z_0}\left\{\int_{[z_0,z]} f(\zeta)d\zeta\right\} - \frac{f(z_0)}{z - z_0}\int_{[z_0,z]} d\zeta \\
&= \frac{1}{z - z_0}\int_{[z,z_0]} f(z_0) - f(\zeta)d\zeta.
\end{aligned}$$

Therefore, for $0 < |z - z_0| < \delta$,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \frac{1}{|z - z_0|} \epsilon |z - z_0| = \epsilon,$$

where we have used (4.3) and the ML-formula. This implies that $F'(z_0) = f(z_0)$. □

We now state the Cauchy Goursat Theorem for star shaped region.

THEOREM 4.7 Suppose f is analytic on a star-shaped region S. Then for every simple closed path C in S traversed in the counterclockwise direction,

$$\int_C f(z)dz = 0.$$

Proof
By Lemma 4.5,

$$\int_{\partial T} f(z)dz = 0$$

for any closed triangle T. Therefore, by Theorem 4.6, there exist a function $F(z)$ such that

$$F'(z) = f(z).$$

But if f has a primitive, then by Theorem 3.10, $\int_C f(z)dz$ depends on the end points of C. But since C is a closed curve, its end points are the same. Hence,

$$\int_C f(z)dz = 0,$$

and this completes the proof of Theorem. □

EXAMPLE 4.2
Let $C := \{z \in \mathbf{C} : |z| = 1\}$. Show that

$$\int_C \frac{z^2}{z-3} dz = 0.$$

Solutions. In the region $|z| \leq 1$, the function $\dfrac{z^2}{z-3}$ is analytic. Hence, by the Cauchy Theorem,

$$\int_C \frac{z^2}{z-3} dz = 0.$$

REMARK 4.2 We summarize what we have done in order to prove Theorem 4.7. We first show that if f is analytic on a region D then the integral of f over any boundary of a triangle in D is 0. We use this fact to construct a primitive of f for function analytic on a star-shaped region. We now observe that if $S = \mathbf{C} - \{\mathrm{Re}\, z \leq 0\} = \{z | -\pi < \arg z < \pi\}$ then S is a star shaped region with star center 1 [2] Hence if we let

$$\mathrm{Ln}\, z = \int_1^z \frac{1}{\zeta} d\zeta,$$

we find that

$$\frac{d\mathrm{Ln}\ \mathrm{z}}{dz} = \frac{1}{z}.$$

Note that when z is real and positive,

$$\int_1^x \frac{1}{t} dt = \ln x.$$

It can be shown that

$$e^{\mathrm{Ln} z} = z$$

and hence $\mathrm{Ln} z$ is an inverse function for e^z. We use the word "an" because $\mathrm{Ln} z + 2\pi i k$ is also an inverse function for e^z for any integer k.

4.3 Extended Cauchy Goursat Theorem

In this section, we prove a slightly more general result than the Cauchy-Goursat Theorem.

THEOREM 4.8 Let f be continuous on star shaped region and analytic on $S - \{z_0\}$. Then f has a primitive on S and consequently,

$$\int_C f(z) dz = 0$$

for every simple closed curve in S traversed in the counterclockwise direction.

The proof of the above is exactly the same as Theorem 4.7. The only difference is that we need a different version of Lemma 4.5 which we now state.

LEMMA 4.9 Let f be continuous on S and analytic on $S - \{z_0\}$. If T is any triangle contained in S, then

$$\int_{\partial T} f(z) dz = 0.$$

[2] We can choose any positive real number to be the star center.

Proof
We split our proof into several cases.

Case 1. If the closed triangle T does not contain z_0, then the conclusion follows using Lemma 4.5.

Case 2. Suppose z_0 is a vertex of T and suppose that

$$|I| = \left| \int_{\partial T} f(z) dz \right| > 0$$

Since $z_0 \in S$ and S is open, we can find $\delta_1 > 0$ such that $B(z_0; \delta_1) \subset S$. Next, f is continuous at z_0 implies that there exists $\delta_2 > 0$ such that

$$|f(z) - f(z_0)| < 1 \quad \text{whenever} \quad |z - z_0| < \delta_2.$$

In other words,

$$|f(z)| \leq 1 + |f(z_0)| \quad \text{whenever} \quad |z - z_0| < \delta_2. \tag{4.4}$$

Next, let

$$\delta_3 = \frac{|I|}{8(|f(z_0)| + 1)}$$

and set $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Choose a and b on the triangle T such that the triangle T_1 formed by z_0, a and b lies inside $B(z_0; \delta)$. Note that

$$\int_{\partial T} f(z) dz = \int_{\partial T_1} f(z) dz.$$

Now, the length of T_1 is less than 4δ (the lengths of $[z_0, a]$ and $[z_0, b]$ are each less than δ and the length of $[a, b]$ is less than 2δ, the diameter of $C(z_0; \delta)$). By (4.4), $|f(z)|$ is bounded by $|f(z_0)| + 1$. Hence, by the ML-formula, we find that

$$\begin{aligned} 0 < |I| &= \left| \int_{\partial T_1} f(z) dz \right| \\ &< (|f(z_0)| + 1)\, 4\delta \\ &< \frac{|I|}{2}, \end{aligned}$$

where we have used the bound $\delta \leq \delta_3$ in the last inequality. Hence

$$\frac{|I|}{2} > |I|$$

and we have a contradiction. This implies that $|I| = 0$.

Case 3. If z_0 lies on the edge of the triangle, we just divide the triangle into two triangles having z_0 as a vertex and apply Case 2.

Case 4. If z_0 lies in the interior of T, we join z_0 to the three vertices to form three triangles with vertex z_0 and apply Case 2.

This completes the proof of the lemma. □

4.4 The Cauchy Integral Formula

We now apply Theorem 4.8 to obtain an important result.

THEOREM 4.10 Let f be analytic in a starshaped region S and let C be a simple closed contour in S traversed in the counterclockwise direction. If u is any point interior to C, then

$$f(u) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-u}\, dz.$$

This is called the Cauchy Integral Formula. It tells us that if a function f is analytic within and on a simple closed curve, then the values of f interior to C are completely determined by the values of f on C. In application, we choose C to be a circle centered at u with radius r such that $C(u;r) \in S$.

Sketch of the proof of Theorem 4.10
Define that function

$$g(z) = \begin{cases} \dfrac{f(z)-f(u)}{z-u} & \text{when } z \neq u \\ f'(u) & z = u. \end{cases} .$$

Note that $g(z)$ is continuous in S and analytic at $S - \{u\}$. If C is a simple closed curve in S, then by Theorem 4.8,

$$\int_C g(z)\, dz = 0.$$

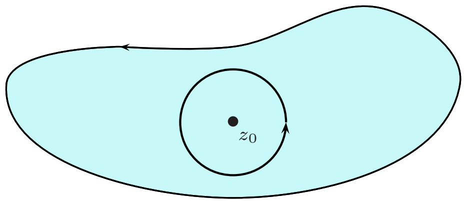
But

$$0 = \int_C g(z)\, dz = \int_C \frac{f(z)-f(u)}{z-u}\, dz = \int_C \frac{f(z)}{z-u}\, dz - \int_C \frac{f(u)}{z-u}\, dz. \tag{4.5}$$

But if C is a simple closed curve containing u then

$$\int_C \frac{1}{z-u}\, dz = \int_{C(u;r)} \frac{1}{z-u}\, dz, \tag{4.6}$$

where $C(u;r)$ is the circle centered at u with radius r traversed in the counterclockwise direction.



By parametrizing $C(z_0; r)$ using $z(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$, we find that

$$\int_{C(z_0;r)} \frac{1}{z - z_0}\, dz = 2\pi i.$$

Therefore,

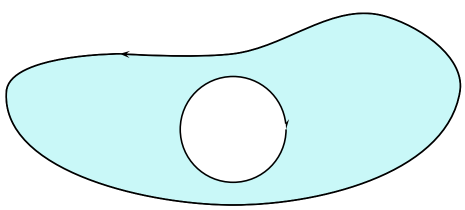
$$\int_C \frac{1}{z - z_0}\, dz = 2\pi i, \tag{4.7}$$

for any simple closed curve containing z_0. Hence, we may rewrite (4.5) as

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz,$$

which completes the proof of the theorem. □

REMARK 4.3 We have assumed (4.6) for arbitrary closed simple curve C. Note that the region bounded by the following curve



is not starshaped. One can cover this compact region by finitely many open balls with centers in the region. Subdivide the curves so that each subdivision is in an open ball. This would then imply that (4.6).

EXAMPLE 4.3

Let C be a positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, the Cauchy integral formula gives

$$\int_C \frac{z}{(9-z^2)(z+i)}dz = \int_C \frac{z/(9-z^2)}{z-(-i)}\,dz = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

4.5 Liouville's Theorem and the Fundamental Theorem of Algebra

THEOREM 4.11 (Liouville's Theorem) Let $f(z)$ be an entire function. If $f(z)$ is bounded, then f is a constant.

Proof

Suppose $|f| \leq M$ for some positive real number M. Let a and b be any two complex numbers and C be any positively oriented circle centered at 0 and with radius $R > \max(|a|, |b|.)$ Then, by Cauchy's Integral Formula,

$$\begin{aligned} |f(b) - f(a)| &= \left|\frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{z-b} - \frac{1}{z-a}\right\}dz\right| \\ &= \frac{1}{2\pi}\left|\int_C \frac{f(z)(b-a)}{(z-a)(z-b)}\,dz\right| \\ &\leq \frac{M|b-a|R}{(R-|a|)(R-|b|)}, \end{aligned}$$

since $|f| \leq M$, $|z-a| \geq |z| - |a| = R - |a|$, $|z-b| \geq R - |b|$ and the length of C is $2\pi R$. Since R can be taken as large as desired,

$$|f(b) - f(a)| \to 0 \quad \text{as} \quad R \to \infty.$$

Therefore $f(b) = f(a)$ for any two complex numbers a and b. Therefore $f(z)$ is a constant function. □

EXAMPLE 4.4

Let f be an entire function and suppose that $\operatorname{Re} f(z) < M$ for all $z \in \mathbf{C}$. Prove that f is a constant function.

Solutions. Let $g(z) = e^{f(z)}$. Then

$$\left|e^{f(z)}\right| = e^{\operatorname{Re} f(z)} \leq e^M.$$

This implies, using Liouville Theorem, that $e^{f(z)}$ is a constant function since $e^{f(z)}$ is entire. Let $f(z) = u + iv$. Then $e^{f(z)} = e^u \cdot e^{iv}$. The fact that $e^{f(z)} = C$ implies that

$$e^u = |C|.$$

Hence,

$$u = \ln |C|.$$

Since the real part of $f(z)$ is a constant, we conclude that $f(z)$ is a constant.

An application of this Theorem is that it yields a simple proof of the Fundamental Theorem of Algebra. This is surprising since we are now using results in analysis to prove results in Algebra. This further shows that topics in mathematics are inter-related.

Before we proceed with the proof of the next theorem, we quote a fact about continuous functions and subsets of **C** which are closed and bounded.

THEOREM 4.12 Let f be a continuous function on a region D and S be a closed and bounded set in D. Then $f(S)$ is also closed and bounded.

THEOREM 4.13 (Fundamental Theorem of Algebra) Every non-constant polynomial with complex coefficients has a zero in **C**.

Proof

Let $P(z)$ be any non-constant polynomial. Suppose $P(z) \neq 0$ for all $z \in \mathbf{C}$. This implies that $f(z) = \dfrac{1}{P(z)}$ is defined for all $z \in \mathbf{C}$ since $f(z)$ is bounded in **C**. The function $f(z)$ is entire because its derivative is given by

$$\frac{-P'(z)}{P(z)^2}.$$

Furthermore, if P is non-constant, $P \to \infty$ as $z \to \infty$ and so f is bounded. This is because since $f \to 0$ as $z \to \infty$, for $\epsilon = 1$ there exists M such that

$$|f| \leq 1 \quad \text{for all } |z| > M.$$

For $|z| \leq M$, we see that the set $S = \{z||z| \leq M\}$ is closed and bounded. By Theorem 4.12, we know that $f(S)$ is bounded, say,

$$|f(z)| \leq B$$

for all $|z| \leq M$. Hence for all $z \in \mathbf{C}$,

$$|f| \leq \max(B, 1)$$

and f is a bounded entire function.

By Liouville's Theorem, f must be a constant and so, P must be a constant. This contradicts our choice of P. □

By induction, we see that any polynomial with complex coefficients must factor into linear factors, in other words,

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 = a_n(z - \alpha_1) \cdots (z - \alpha_n).$$

In Chapter 1, we indicate that i may be defined as the root of $z^2+1=0$. A natural question to ask is if we consider all possible polynomials with coefficients in $\mathbf{C}$, say for example, $z^2+i=0$, will we discover new numbers analogous to that of i? Or do we have to define new numbers to solve these polynomials? The above observation says that we do not have to define more numbers. In fact, all polynomials factor into linear factors in $\mathbf{C}$. In other words, introducing an extra number i allows us to solve all polynomial equations, at least theoretically.

4.6 Appendix : Compact sets in C

DEFINITION 4.14 A set $S \subset \mathbf{C}$ is compact if it satisfies the Heine-Borel property, namely, or every open covering $\mathcal{C}$ of S, there exists a finite subcovering of S in $\mathcal{C}$.

This means that if $S \subset \bigcup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{C}$, then

$$S \subset \bigcup_{j=1}^{k} U_{\alpha_j}.$$

It can be shown that if S is a subset of $\mathcal{C}$ then S is a compact if and only if S is closed and bounded.

Let f be a function from a region X to Y. A function is continuous on a region X if for every set V that is "open in Y",[3] the set $f^{-1}(V)$ is "open in X". We can show that our old notion of continuous function satisfies the above property (see Appendix of Chapter 2).

We now sketch the proof of Theorem 4.12.

Sketch of the proof of Theorem 4.12

Let $\mathcal{C}$ be a covering of $f(S)$. Then

$$f(S) \subset \bigcup_{\alpha} \mathcal{O}_\alpha,$$

where $\mathcal{O}_\alpha$ is open in $\mathbf{C}$. Let $U_\alpha = \mathcal{O}_\alpha \cap Y$. Then $f(S) \subset \bigcup_\alpha U_\alpha$. This means that

$$S \subset f^{-1}\left(\bigcup_{\alpha} U_\alpha\right) = \bigcup_{\alpha} f^{-1}\left(U_\alpha\right).$$

Note that because f is continuous, $f^{-1}(U_\alpha)$ is "open in X". Let $f^{-1}(U_\alpha) = \mathcal{O}^*_\alpha \cap X$, with $\mathcal{O}^*_\alpha$ open set in $\mathcal{C}$. Since S is compact,

$$S \subset \bigcup_{j=1}^{k} \mathcal{O}^*_{\alpha_j}.$$

[3] We say that U is open in S if $U = O \cap S$ for some open set O in $\mathbf{C}$.

Hence,

$$S \cap X = S \subset \left(\bigcup_{j=1}^{k} \mathcal{O}^*_{\alpha_j} \right) \cap X = \bigcup_{j=1}^{k} f^{-1}\left(U_{\alpha_j}\right).$$

Hence

$$f(S) \subset \bigcup_{j=1}^{k} \left(\mathcal{O}_{\alpha_j}\right),$$

and therefore $f(S)$ is compact. □

Our aim is conclude that if S is compact and f is continuous then $|f(S)|$ is bounded since a set in **C** is compact if and only if it is closed and bounded.

We will next prove that a set S in **C** is compact if and only if S is closed and bounded.

We do this in several steps.

LEMMA 4.15 If $S \subset \mathbf{C}$ is compact, then S is bounded.

Proof
It is clear that

$$S \subset \bigcup_{s \in S} B(s;1).$$

Since S is compact, it is covered by finitely many such open balls and we have

$$S \subset \bigcup_{j=1}^{k} B(s_j;1).$$

Let $d = \max_{1 \le i < j \le k} |s_i - s_j|$. Given any $u, v \in S$, $u \in B(s_m;1)$ and $v \in B(s_n;1)$ for some integers m and n between 1 and k. Now, $|u-v| \le |u-s_m| + |s_m - s_n| + |v-s_n| < 2+d$. Hence the diameter of S is finite and S is bounded by $B(0;|u| + \text{diam} S)$. □

LEMMA 4.16 If $S \subset \mathbf{C}$ is compact, then S is closed.

Proof
To show that S is closed, we show that $S = \overline{S}$. Suppose $s \in \partial S$ and $s \notin S$. Then $B(s;\epsilon) \cap S \neq \phi$ and $B(s;\epsilon) \cap S^c \neq \phi$ for all $\epsilon > 0$. In particular, for $n \in \mathbf{Z}^+$,

$$B\left(s, \frac{1}{n}\right) \cap S \neq \phi.$$

This implies that

$$\overline{B\left(s, \frac{1}{n}\right)} \cap S \neq \phi.$$

Let $G_n = \left(\overline{B\left(s, \frac{1}{n}\right)}\right)^c$. Note that G_n is open. Now,

$$\bigcup \left(\overline{B\left(s, \frac{1}{n}\right)}\right)^c = \left(\bigcap \overline{B\left(s, \frac{1}{n}\right)}\right)^c.$$

But $\{\overline{B\left(s, \frac{1}{n}\right)} | n \in \mathbf{Z}^+\}$ is a collection of nested closed sets and by Cantor's Theorem (Lemma 4.4),

$$\bigcap \overline{B\left(s, \frac{1}{n}\right)} = \{s\}.$$

Therefore $\bigcup G_n = (\{s\})^c$ covers S ($s \notin S$). Since S is compact, S is covered by finitely many sets G_{n_j} or

$$S \subset \bigcup_{j=1}^{k} G_{n_j}.$$

This means that

$$S \cap \left(\bigcup_{j=1}^{k} G_{n_j}\right)^c = \phi,$$

with $n_1 < n_2 < \cdots < n_k$, or

$$S \cap \overline{B\left(s, \frac{1}{n_1}\right)} \cap \overline{B\left(s, \frac{1}{n_2}\right)} \cdots \cap \overline{B\left(s, \frac{1}{n_k}\right)} = S \cap \overline{B\left(s, \frac{1}{n_k}\right)} = \phi.$$

This contradicts the fact that $s \in \partial S$. Therefore S is closed. □

We are now left with the proof that if S is closed and bounded then S is compact [4]

DEFINITION 4.17 We say that a set S is totally bounded if for every $\epsilon > 0$, S can be covered by finitely many open balls of radius ϵ.

LEMMA 4.18 If S is bounded then it is totally bounded.

Proof
If the set is bounded and $\epsilon > 0$ is given, we can divide the set S using squares of side $\epsilon/\sqrt{2}$. Then the finite number of balls with vertices of the squares as centers would cover S. So, S is totally bounded. □

LEMMA 4.19 If S is closed subset of a complete set, S is complete.

[4] This is not true in general for complete metric space. It is true for $\mathbf{C}$ and $\mathbf{R}$.

Proof
Let $\{s_k\}$ be a sequence of numbers from S that is Cauchy. Since $\mathbf{C}$ is complete, $s_k \to a$ for some $a \in \mathbf{C}$. We claim that $a \in S$. If $a \notin S$, $a \in S^c$ and $B(a;\epsilon) \subset S^c$ for some $\epsilon > 0$ since S^c is open. This means that $B(a;\epsilon) \cap S = \phi$ and $|s_k - a| \geq \epsilon$. But this contradicts the fact that $s_k \to a$. Hence S is complete. □

LEMMA 4.20 If S is closed and bounded then S is compact.

Proof
From the previous two lemmas, we can assume that S is complete and totally bounded. Suppose $\mathcal{C} = \{\mathcal{O}_j\}$ is an open covering for S without any finite sub-covering.

Let $\epsilon_1 = \frac{1}{2}$. The set S is totally bounded implies that S can be covered by finitely many open balls of radius ϵ_1. Write

$$S = \bigcup_{\ell=1}^{m_1} (B(a_\ell;\epsilon_1) \cap S)$$

where we only include those open balls with non-empty intersection with S.

By assumption, S cannot be covered by finitely many sets $\mathcal{O}_j$'s in $\mathcal{C}$. Hence, there is an open ball $B(x_1;\epsilon_1)$ centered at x_1 with radius ϵ_1 such that then non-empty set $B(x_1;\epsilon_1) \cap S$ cannot be covered by finitely many open sets $\mathcal{O}_j$'s in $\mathcal{C}$. Now, $B(x_1;\epsilon_1) \cap S$ is totally bounded since it is a subset of a totally bounded set S. Therefore $B(x_1;\epsilon_1)$ can be covered by finitely many open balls of radius $\epsilon_2 = 2^{-2}$. Once again, among these open balls, there exists an open ball $B(x_2;\epsilon_2)$ such that the non-empty set $B(x_2;\epsilon_2) \cap S$ cannot be covered by finitely many $\mathcal{O}_j$'s in $\mathcal{C}$. Continuing with this construction, we obtain a sequence of sets $\{B(x_k;\epsilon_k) \cap S\}$ that cannot be covered by finitely many sets in $\mathcal{C}$. For each k, let $s_k \in (B(x_k;\epsilon_k) \cap S)$. Note that

$$\begin{aligned} |s_n - s_{n+1}| &\leq |s_n - x_n| + |x_n - x_{n+1}| + |x_{n+1} - s_{n+1}| \\ &< \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \leq \frac{3}{2^n}. \end{aligned}$$

Hence,

$$|s_m - s_{m+p}| \leq |s_m - s_{m+1}| + \cdots + |s_{m+p-1} - s_{m+p}| < \frac{6}{2^m}$$

Therefore, the sequence $\{s_k\}$ is a Cauchy sequence. Since S is complete, $s_k \to a$ with $a \in S$. Now S is covered by $\mathcal{O}_j$'s in $\mathcal{C}$ and therefore, $a \in \mathcal{O}$ for some open set $\mathcal{O}$ in $\mathcal{C}$. This implies that $B(a;\delta) \subset \mathcal{O}$ for some $\delta > 0$ since $\mathcal{O}$ is open. Now, since $s_k \to a$, there exists M such that

$$|s_k - a| < \frac{\delta}{3}$$

for all $k > M$. We choose $k > M$, say $k = K$, such that

$$\epsilon_K = 2^{-K} < \frac{\delta}{3}.$$

Now, consider the set $B(x_K;\epsilon_K)\cap S$. By assumption,

$$s_K \in B(x_K;\epsilon_K)\cap S.$$

Let $b\in B(x_K;\epsilon_K)\cap S$. Then

$$\begin{aligned}|b-a| &= |b-x_K+x_K-s_K+s_K-a|\\ &\leq |b-x_K|+|x_K-s_K|+|s_K-a|\\ &\leq \frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta.\end{aligned}$$

Therefore, $b\in B(a;\delta)\subset\mathcal{O}$. Hence, $(B(x_K;\epsilon_K)\cap S)\subset\mathcal{O}$ and is therefore covered by a finite number of sets in $\mathcal{C}$. This contradicts our choice of $B(x_K;\epsilon_K)$ and so S is covered by finitely many sets in $\mathcal{C}$. □

REMARK 4.4 In the proof of the above Lemma, one can start with open balls with centers in S. For suppose that $B(x;\epsilon/2)$ is one of the finitely many balls of radius $\epsilon/2$ covering S such that $B(x;\epsilon/2)\cap S\neq\phi$ and that $x\notin S$. Then let $s\in B(x;\epsilon/2)$. The ball $B(s;\epsilon)$ covers $B(x;\epsilon/2)\cap S$ and hence, we obtain an open covering of S by finitely by open balls $B(s;\epsilon)$ with $s\in S$. Using these balls with centers in S, we obtain another proof of the Lemma. For more details, see p. 61 of Ahlfors.

5 Cauchy's Integral formulas and their applications

5.1 Derivatives of Analytic Functions

We will prove in this section that if a function is analytic at a point, its derivatives of all orders exist at that point and are themselves analytic there. In other words, if $f(z)$ is analytic at $z = w$ then $f^{(n)}(z)$ is analytic at $z = w$ for $n \geq 1$.

THEOREM 5.1 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction. If w is any point interior to C, then

$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^2} d\zeta.$$

Proof

First, recall the Cauchy Integral formula

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - w}\, d\zeta.$$

Now, let z be sufficiently close to w so that both w and z are in the region bounded by C. Then

$$\begin{aligned}
&\frac{f(z) - f(w)}{z - w} - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^2} d\zeta \\
&\quad = \frac{1}{2\pi i} \left(\frac{1}{z - w} \int_C f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} - \frac{z - u}{(\zeta - w)^2} \right) \right) \\
&\quad = \frac{1}{2\pi i} \int_C f(\zeta) \frac{(z - w)}{(\zeta - z)(\zeta - w)^2} d\zeta.
\end{aligned}$$

Let $\delta > 0$ be such that $B(w;\delta)$ is contained in the region enclosed by C and $\overline{B(w;\delta)} \cap C = \phi$. Let d be the minimum distance from $\overline{B(w;\delta)}$ to C. Then

$$|(\zeta - z)(\zeta - w)^2| \geq d^3.$$

Since f is continuous, $|f(\zeta)| < M$ for some $M > 0$. Hence, the modulus of

$$g(\zeta) = \frac{f(\zeta)}{(\zeta - z)(\zeta - w)^2}$$

is bounded above by M/d^3 on C. Therefore, we may assume that $|g(\zeta)| < M/d^3$.

Denoting the length of C by L and using ML-formula, we deduce that

$$\left| \frac{f(z) - f(w)}{z - w} - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^2} d\zeta \right| < \frac{1}{2\pi} LM|z - w| < \epsilon$$

whenever $|z - w| < \min(\delta, 2\pi\epsilon/(LM))$. This concludes the proof of the theorem. □

THEOREM 5.2 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction. If w is any point interior to C, then

$$f''(w) = \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^3} d\zeta.$$

Proof
Let z be sufficiently close to w so that both w and z are in the region bounded by C. Then

$$\begin{aligned} &\frac{f'(z) - f'(w)}{z - w} - \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^3} d\zeta \\ &\quad = \frac{1}{2\pi i(z - w)} \int_C f(\zeta) \left(\frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - w)^2} - \frac{2(z - w)}{(\zeta - w)^3} \right) d\zeta \\ &\quad = \frac{1}{2\pi i} \int_C f(\zeta) \frac{(2\zeta - w - z)(\zeta - w) - 2(\zeta - z)^2}{(\zeta - z)^2(\zeta - w)^3} d\zeta \\ = &\quad = \frac{1}{2\pi i} \int_C f(\zeta) \frac{(z - w)(3\zeta - w - 2z)}{(\zeta - z)^2(\zeta - w)^3} d\zeta, \end{aligned}$$

where the last equality can be verified directly.

As in the proof of the previous Theorem, we let $\delta > 0$ be such that $B(w;\delta)$ is in the region enclosed by C $\overline{B(w;\delta)} \cap C = \phi$. Let d and d' be the minimum and maximum distance from $\overline{B(w;\delta)}$ to C respectively. This implies that$|(\zeta - z)^2(\zeta - w)^3| \geq d^5$ and $|3\zeta - w - 2z| \leq 3d'$. Since f is continuous, we conclude that on C, $|f(\zeta)| < M$ for some $M > 0$. Hence.

$$g(\zeta) = f(\zeta) \frac{3\zeta - w - 2z}{(\zeta - z)^2(\zeta - w)^3}$$

is bounded above by M' where $M' = 3Md'/d^5$.

Denoting the length of C by L and using ML-formula, we deduce that

$$\left| \frac{f'(z) - f'(w)}{z - w} - \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^3} d\zeta \right| < \frac{1}{2\pi} LM'|z - w| < \epsilon$$

whenever $|z - u| < \min(\delta, 2\pi\epsilon/(LM'))$. This concludes the proof of the theorem. □

We are now ready to prove an important result which we have mentioned in the previous chapter when we discussed harmonic functions and Cauchy-Goursat Theorem.

LEMMA 5.3 If f is analytic on a region D then f' is analytic on D.

Our next Theorem extends Theorems 5.1 and 5.2.

THEOREM 5.4 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction that encloses w. Then

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}}\,dz.$$

Proof

We proceed by using mathematical induction. Note that the case for $k = 1$ is Theorem 5.1. Suppose the result is true for $k = n-1$. By Lemma 5.3, we note that we may apply Cauchy's integral formula for $n-1$-th derivative to f' since f' is analytic in D whenever f is analytic in D. Hence,

$$(f')^{(n-1)}(w) = \frac{(n-1)!}{2\pi i}\int_C \frac{f'(\zeta)}{(\zeta-w)^n}\,dz. \tag{5.1}$$

On the other hand, we know that

$$\left(\frac{f(\zeta)}{(\zeta-w)^n}\right)' = \frac{f'(\zeta)}{(\zeta-w)^n} - n\frac{f(\zeta)}{(\zeta-w)^{n+1}}$$

and hence by the analogue of the Fundamental Theorem of Calculus, we find that

$$\frac{1}{2\pi i}\int_C \frac{f'(\zeta)}{(\zeta-w)^n}\,d\zeta = \frac{n}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-w)^{n+1}}\,d\zeta \tag{5.2}$$

since C is a simple closed curve. Combining (**??**) and (5.2), we complete the proof of the theorem. □

EXAMPLE 5.1

Evaluate the integral

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz$$

where C is the circle with center 0 and radius 2.

Solutions. By Cauchy's integral formula for $f''(z)$, we conclude that

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz = \pi i f''(i) = 10\pi i,$$

since $f''(i) = 10$.

EXAMPLE 5.2
When $f(z) = 1$,

$$\int_C \frac{1}{z - z_0}\, dz = 2\pi i,$$

and

$$\int_C \frac{1}{(z - z_0)^k}\, dz = 0$$

for $k \geq 2$.

5.2 Morera's Theorem and Extended Liouville Theorem

We see from Theorem 5.2 that if f is analytic on a region D then f' is analytic on the region D (since f'' exists). Hence, we have

We will now use Lemma 5.3 to prove an important result known as Morera's Theorem.

THEOREM 5.5 (Morera's Theorem) Let f be a continuous function on a region D. Let T be a closed trianglue in D and ∂T be the boundary of T traversed in the counterclockwise direction. If

$$\int_{\partial T} f(z) dz = 0$$

for all $T \subset D$, then f is analytic in D.

Proof
We first note that this is very similar to the statement when we wish to derive primitives for f in our proof of the Cauchy Goursat Theorem. However, we did not conclude that f is analytic.

Let $z_0 \in D$. Since D is a region, there exists $\epsilon > 0$ such that $B(z_0; \epsilon) \subset D$. Now, a open ball is a star shaped domain and the conditions guaranteed that we could construct a primitive for f, namely,

$$F(z) = \int_{z_0}^{z} f(\zeta) d\zeta.$$

Now, $F'(z) = f(z)$ and so, F is analytic on z_0. But by Corollary 5.3, we conclude that F' is analytic. This means that f is analytic. □

We now use Morera's Theorem to prove the following result.

THEOREM 5.6 Let f be an entire function and

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}.$$

Then g is entire.

Proof
From the definition of g we see that g is analytic at $\mathbf{C} - \{a\}$ and continuous at $z = a$. Let T be any triangle in $\mathbf{C}$. By the extended Cauchy Goursat Theorem (Theorem 4.8), we conclude that

$$\int_{\partial T} g(\zeta)d\zeta = 0.$$

Hence, by Morera's Theorem (Theorem 5.5), we conclude that g is analytic at a and hence g is entire. □

We are now in a position to prove the extended Liouville Theorem.

THEOREM 5.7 If f is entire and if, for some integer $k \geq 0$, there exist positive constants A and B such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k.

Proof
Note that the case $k = 0$ is the original Liouville Theorem (Theorem 4.11). We prove the theorem by induction. Consider

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}.$$

We have seen by Theorem 5.6 that g is entire. Now, if $z = 0$, $|g(0)| \leq |f'(0)|$. If $z \neq 0$ then

$$|g(z)| \leq \frac{A + |f(0)| + B|z|^k}{|z|}.$$

If $|z| > 1$, then using this bound on the first two terms on the right hand side of the inequality, we find that

$$|g(z)| \leq A + |f(0)| + B|z|^{k-1}.$$

If $|z| \leq 1$, then g is bounded by Theorem 4.12 and $|g(z)| \leq M$ for some $M > 0$. Hence we conclude that for all $z \in \mathbf{C}$,

$$|g(z)| \leq C + B|z|^{k-1},$$

where we take $C = \max(|f'(0)|, A + |f(0)|, M)$. By induction, $g(z)$ is a polynomial of degree at most $k - 1$ and hence $f(z)$ is a polynomial of degree at most k. □

EXAMPLE 5.3
Suppose $f(z)$ is entire and $|f'(z)| \leq |z|$ for all $z \in \mathbf{C}$. Show that

$$f(z) = a + bz^2$$

with $|b| \le 1/2$.

Solutions. The function $f'(z)$ is entire since f is entire. We have $|f'(z)| \le |z|$ implies that $f'(z) = Az + B$ by the extended Liouville theorem. But from the inequality $|f'(z)| \le |z|$ shows that $f'(0) = 0$ and hence $B = 0$. Now $|f'(z)| = |Az| \le |z|$ implies that $|A| \le 1$.

Next, $f' = Az$ implies that $f = Az^2/2 + C$ and $|A/2| \le \frac{1}{2}$.

5.3 Mean Value Theorem and the Maximum Modulus Theorem

We now examine some local behavior of analytic functions.

THEOREM 5.8 (Mean Value Theorem) If f is analytic in a region D and $\alpha \in D$, then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any ball centered at α and contained in D. That is

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta$$

when $B(\alpha; r) \subset D$.

Proof

From Cauchy Integral formula, we have

$$f(\alpha) = \frac{1}{2\pi i} \int_{C(\alpha;r)} \frac{f(\zeta)}{\zeta - \alpha} d\zeta.$$

Let $\zeta = re^{i\theta} + \alpha$. Then we find that

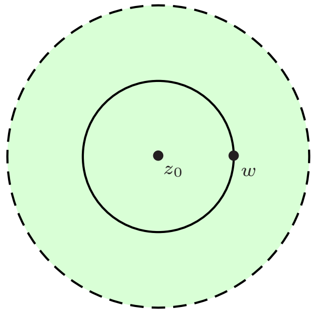
$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta.$$

□

THEOREM 5.9 (Maximum Modulus Theorem on an open ball) Suppose that $f(z)$ is analytic throughout a neighborhood $|z - z_0| < R$ of a point z_0. If $|f(z)| \le |f(z_0)|$ for each point z in that neighborhood, then $f(z)$ has the constant value $f(z_0)$ throughout the neighborhood.

Proof

Our aim is to show that if $|f(z)|$ is maximum for some $z = z_0$ in $B(z_0; R)$ then $f(z)$ is a constant on $B(z_0; R)$. Let w be an arbitrary point in $B(z_0; R)$.



Let $r = |u - z_0| < R$. From the Mean Value Theorem (Theorem 5.8), we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

It follows that

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|d\theta. \tag{5.3}$$

By assumption that $|f(z_0)|$ is maximum, we have $|f(z_0)| \geq |f(z)|$ for $z \in C(z_0; r)$. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|d\theta \leq |f(z_0)|.$$

Together with (5.3), we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|d\theta = |f(z_0)|,$$

or

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{i\theta})|d\theta = 0 \tag{5.4}$$

We claim that $|f(w)| = |f(z_0)|$ for all $w \in C(z_0; r)$. Suppose not. Then there exist T such that $|f(z_0 + re^{2iT})| < |f(z_0)|, 0 \leq T \leq 2\pi$. Let $F(t) = |f(z_0)| - |f(z_0 + re^{2it})|$. Then we have $|F(T)| > 0$. Let $|F(T)| = h > 0$. Since $F(t)$ is continuous, for $h/2 > 0$, there exists $\delta > 0$ such that if $|t - T| < \delta$ then

$$|F(t) - F(T)| < \frac{h}{2},$$

or

$$|F(T)| - |F(t)| < |F(T) - F(t)| < \frac{h}{2}.$$

Therefore,

$$|F(t)| > \frac{h}{2}.$$

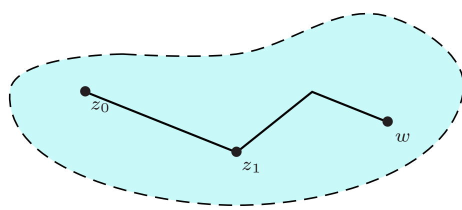
Hence,

$$\int_{T-\delta}^{T+\delta} |F(t)|dt > \frac{h}{2}2\delta > 0$$

and this contradicts (5.4). This implies $|f(z_0)| = |f(z_0 + re^{it})|, 0 \le t \le 2\pi$ and in particular, $|f(z_0)| = |f(w)|$. Since w is arbitrary, we conclude that $|f(z_0)| = |f(z)|$ for all $z \in B(z_0; R)$. Now, by Example 2.24, we conclude that f is constant on $B(z_0; R)$. □

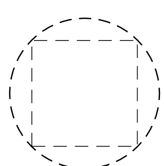
THEOREM 5.10 (Maximum modulus principle for a region) If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.

Proof
We wish to prove that for any $w \in D$, $f(w) = f(z_0)$. It suffices to show that $|f(w)| = |f(z_0)|$ for all $w \in D$. Since D is a region, there is a polygonal line from z_0 to w (see the diagram below):



To show that $|f(w)| = |f(z_0)|$, it suffices to show that if $[z_0, z_1]$ is a line, then $|f(z_0)| = |f(z_1)|$. By continuing along the polygonal line and using the result for line segment, we conclude that $|f(w)| = |f(z_0)|$.

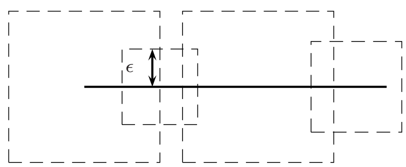
We now observe that if $z \in D$ and D is open then there exists a open "square" $S(z, s)$ with center z such that the $S(z, s) \subset D$. To see that, we see that there exists an open ball $B(z; \epsilon) \subset D$. Take $s = \sqrt{2}\epsilon)$ and we see that $S(z, s) \subset B(z; \epsilon) \subset D$ (see the diagram below).



Now, the line segment $[z_0, z_1]$ is in an open set, we find that for each $w \in [z_0, z_1]$ then there exists $s_w > 0$ such that the open square $S(w, s_w) \subset D$. Note that $\mathcal{C} = \{S(w, s_w) | w \in [z_0, z_1]\}$ is an open cover for $[z_0, z_1]$ and since $[z_0, z_1]$ is compact, we find that (there is a finite subcover)

$$[z_0, z_1] \subset \bigcup_{j=1}^{k} S(w_j, s_{w_j}).$$

Let $\epsilon = \min_{1 \le j \le k}(s_{w_j}/2)$ refer to the following diagram). Note that with this choice of ϵ, all open balls with centers on the line $[z_0, z_1]$ and radius ϵ will lie inside the region D.



Now, define

$$u_j = z_0 + j \frac{\epsilon}{2} \frac{z_1 - z_0}{|z_1 - z_0|}.$$

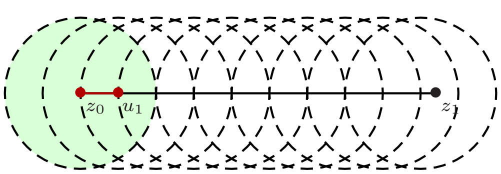
Note that (see the following diagram)

$$[z_0, z_1] \subset \bigcup_{j=0}^{m} B(u_j; \epsilon),$$

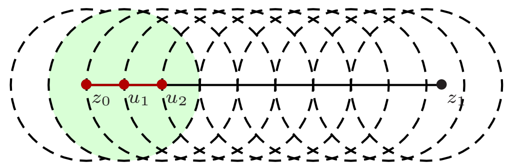
where

$$m = \left[\frac{|z_1 - z_0|}{(\epsilon/2)} \right]$$

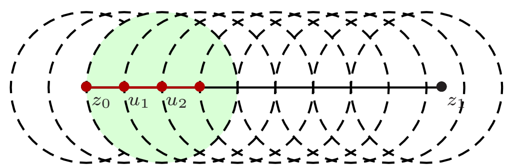
so that $z_1 \in B(u_m; \epsilon)$.



By the maximum modulus principle for open ball, we see that $f(z_0) = f(u_1)$. We then continue the process (see the following diagram where the red denotes those points t with $f(t) = f(z_0)$ and deduce that $f(u_1) = f(u_2)$.



To illustrate our point, we include one more diagram as follow to show that $f(z_0) = f(u_2)$.



This shows that $f(z_1) = f(z_0)$ if $[z_0, z_1] \subset D$ and by the remark in the beginning of the proof, we conclude that $f(z)$ is constant on D.

□

COROLLARY 5.11 Suppose a function f is continuous in a closed and bounded region D and it is analytic and not constant in the interior of D. Then the maximum value of $|f(z)|$ in D, which is always reached, occurs somewhere on the boundary of D and never in the interior.

Proof
The function f is a continuous function on a closed and bounded set and so, its image is closed and bounded by Theorem 4.12. Let $M = \sup_{z\in D} |f(z)|$. Then there exist $u \in D$ such that $|f(w)| = M$ since $f(D)$ is a closed set (see Remark 5.1). So the maximum modulus is attained. Now since f is not constant, this value cannot be attained in the interior of D, hence the maximum modulus is attained at ∂D.

□

EXAMPLE 5.4

(Bak-Newman, p. 84, Problem 6) Suppose f is a non-constant analytic function in the annulus: $1 \le |z| \le 2$, that $|f| \le 1$ for $|z| = 1$ and that $|f| \le 4$ for $|z| = 2$. Prove that $|f(z)| \le |z|^2$ throughout the annulus.

Solutions. Let $g(z) = f(z)/z^2$. Then from hypothesis,

$$|g(z)| = \frac{|f(z)|}{|z|^2} \le 1$$

when $|z| = 1$ and $|z| = 2$. Hence, $|g(z)| \le 1$ on the boundary of the annulus. Therefore, by maximum modulus principle, $|g(z)| \le 1$ on $1 \le |z| \le 2$. This shows that $|f(z)| \le |z|^2$ on $1 \le |z| \le 2$.

REMARK 5.1 Let $M = \sup_{z \in \overline{D}} |f(z)|$ where $\overline{D}$ is the closure of D. Then for $\epsilon = 1/n$, we know that $M - \epsilon$ cannot be a upper bound and hence, there exists z_n such that $M - 1/n < |f(z_n)| < M$. We thus create a sequence $\{|f(z_n)|\}$ that converges to M. Since $|f|$ is continuous, $|f(\overline{D})|$ is complete. Therefore $|f(z_n)|$ converges to $|f(z_0)|$ for some $z_0 \in \overline{D}$ and so, the maximum M is attained by some $z_0 \in \overline{D}$.

6 Series

6.1 Convergence of Sequences and Series

DEFINITION 6.1 An infinite sequence of complex numbers

$$z_1, z_2, \cdots, z_n, \cdots,$$

has a limit z if for each positive ϵ, there exists a positive integer N_ϵ such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n \geq N_\epsilon.$$

The following theorem has the similar flavor as Theorem 2.3. We have implicitly used this result to deduce that $\mathbf{C}$ is complete using the fact that $\mathbf{R}$ is complete.

THEOREM 6.2 Suppose that $z_n = x_n + iy_n, n = 1, 2, \cdots$, and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \quad \text{if and only if} \quad \lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Proof

We first show that (a) implies (b). Given $\epsilon > 0$ there exist N_ϵ such that

$$\operatorname{Re}(z_n - z) \leq |z_n - z| \leq \epsilon$$

and

$$\operatorname{Im}(z_n - z) \leq |z_n - z| \leq \epsilon$$

whenever $n \geq N_\epsilon$. Therefore

$$|x_n - x| < \epsilon \quad \text{and} \quad |y_n - y| < \epsilon$$

whenever $n \geq N_\epsilon$.

Conversely given $\epsilon > 0$, if

$$|x_n - x| < \frac{\epsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\epsilon}{2},$$

for $n \geq N_\epsilon$, then

$$|z_n - z| = |x_n - x + i(y_n - y)| \leq |x_n - x| + |y_n - y| < \epsilon,$$

for all $n \geq N_\epsilon$. □

Given a sequence $\{z_k\}_{k=1}^{\infty}$. We may construct the sequence $\{S_k\}_{k=1}^{\infty}$ where

$$S_k = \sum_{j=1}^{k} z_j.$$

An expression of the form

$$\sum_{k=1}^{\infty} z_k$$

is called an infinite series.

If the new sequence $\{S_n\}_{n=1}^{\infty}$ has a limit S, then we say that the infinite series

$$\sum_{k=1}^{\infty} z_k$$

converges and write

$$\sum_{k=1}^{\infty} z_k = S = \lim_{n\to\infty} S_n.$$

If the limit of the sequence $\{S_n\}_{n=1}^{\infty}$ does not exist then we say that the infinite series

$$\sum_{k=1}^{\infty} z_k$$

diverges.

THEOREM 6.3 Suppose that $z_k = x_k + iy_k$, $k = 1, 2, \cdots$ and $S = X + iY$. Then

$$\sum_{k=1}^{\infty} z_k = S \quad \text{if and only if} \quad \sum_{k=1}^{\infty} x_k = X \quad \text{and} \quad \sum_{k=1}^{\infty} y_k = Y,$$

where $S = X + iY$.

6.2 Taylor Series

We turn now to Taylor's Theorem.

THEOREM 6.4 Suppose that a function f is analytic throughout an open ball $B(z_0;R)$. Then at each $z \in B(z_0;R)$, $f(z)$ has the series representation

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \quad (|z-z_0|<R),$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!}, k = 0, 1, 2, \cdots .$$

The above theorem is the familiar Taylor series (when restricted to real variables) from Calculus.

Proof
First, set $z_0 = 0$ and suppose f is analytic in $|z| < R$. Let z be chosen and suppose that $|z| = r < R$. Choose R_1 such that $r < R_1 < R$ and let C_{R_1} be the circle $C(0;R_1)$ traversed in the anti-clockwise direction. By the Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s-z}\, ds.$$

Now,

$$\begin{aligned}
\frac{1}{s-z} &= \frac{1}{s}\left(\frac{1}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(1+\frac{1}{1-\frac{z}{s}}-1\right) \\
&= \frac{1}{s}\left(1+\frac{1-1+\frac{z}{s}}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(1+\frac{\frac{z}{s}}{1-\frac{z}{s}}\right) \\
&= \frac{1}{s}\left(1+\frac{z}{s}\left(\frac{1}{1-\frac{z}{s}}\right)\right) = \frac{1}{s}\left(1+\frac{z}{s}+\left(\frac{z}{s}\right)^2+\cdots+\frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right).
\end{aligned}$$

Therefore,

$$\frac{1}{s-z} = \frac{1}{s}+\frac{z}{s^2}+\frac{z^2}{s^3}+\cdots+\frac{z^{N-1}}{s^N}+\frac{z^N}{(s-z)s^N}.$$

Hence

$$f(z) = \frac{1}{2\pi i}\int_{C_{R_1}} f(s)\left\{\frac{1}{s}+\frac{z}{s^2}+\cdots+\frac{z^{N-1}}{s^N}+\frac{z^N}{(s-z)s^N}\right\} ds.$$

Now, the general Cauchy Integral formula says that

$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^{n+1}}\, ds.$$

Therefore,

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s}\,ds + \frac{z}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^2}\,ds + \cdots \\
&\quad + \frac{z^{N-1}}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^N} ds + \frac{1}{2\pi i}\int_{C_{R_1}} \frac{z^N}{(s-z)s^N}\,ds \\
&= f(0) + \frac{f'(0)}{1!}z + \frac{f^{(2)}(0)}{2!}z^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1} + \rho_N(z),
\end{aligned}
$$

where

$$\rho_N(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{z^N f(s)}{(s-z)s^N}\,ds.$$

To complete our proof, it suffices to show that $\rho_N(z) \to 0$ as $N \to \infty$. Recall that $r < R_1$. Hence,

$$|s-z| \geq |s| - |z| = R_1 - r.$$

This implies that

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \frac{M_1}{(R_1 - r)R_1^N} 2\pi R_1 = \frac{M_1 R_1}{R_1 - r}\left(\frac{r}{R_1}\right)^N,$$

where

$$M_1 = \max_{s\in C_{R_1}} |f(s)|.$$

But $\dfrac{r}{R_1} < 1$ and therefore,

$$\lim_{N\to\infty}\left(\frac{r}{R_1}\right)^N = 0.$$

Hence, $\rho_N(z) \to 0$ when $N \to 0$. Thus, for each point in $D(0;R)$,

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f^{(2)}(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots.$$

This special case of series is known as the Maclaurin series of $f(z)$. Setting $f^{(0)}(z) = f(z)$, we may rewrite the above series as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k.$$

We now prove the Taylor series expansion of $f(z)$. Suppose $f(z)$ is analytic in $|z - z_0| < R$. Then $g(z) := f(z + z_0)$ is analytic in $B(0;R)$. Therefore

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} z^k.$$

But

$$g^{(k)}(0) = f^{(k)}(0 + z_0) = f^{(k)}(z_0).$$

Therefore,

$$f(z+z_0)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}z^k, |z|<R.$$

Replacing z by $z-z_0$, we find that

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,$$

which is (6.4), with $|z-z_0|<R$. □

DEFINITION 6.5 The largest R for which a power series

$$S(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k$$

is convergent for all $|z-z_0|<R$ is called the radius of convergence of $S(z)$.

There are several ways of computing the radius of convergence of a given power series. We quote two of them.

THEOREM 6.6 Suppose $L=\lim_{k\to\infty}|C_k|^{1/k}$ exists.

(a) If $L=0$, then

$$\sum_{n=0}^{\infty}C_nz^n$$

converges for all z.

(b) If $L=\infty$, then the series

$$\sum_{n=0}^{\infty}C_nz^n$$

converges for $z=0$ only.

(c) If $0<L<\infty$ then

$$\sum_{n=0}^{\infty}C_nz^n$$

converges for $|z|<1/L$ and diverges for $|z|>\frac{1}{L}$.

The above result is true if we replace $\lim_{k\to\infty}|C_k|^{1/k}$ by $\lim_{n\to\infty}|C_{k+1}/C_k|$.

EXAMPLE 6.1

The radius of convergence of $\sum_{k=1}^{\infty}z^k$ is 1. This is because the series converges to $\dfrac{1}{1-z}$ for $|z|<1$.

EXAMPLE 6.2
The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is convergent everywhere. To show this, we compute

$$lim_{k\to\infty} |C_{k+1}/C_k| = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

By the above result, we conclude that the series is convergent for all z. This function turns out to be e^z.

EXAMPLE 6.3
Show that

$$\frac{1}{z^2} = \sum_{k=0}^{\infty} (k+1)(z+1)^k, \quad |z+1| < 1.$$

Solutions. We first write $1/z^2$ as $1/(1-(z+1))^2$ and then use the power series expansion for $1/(1-u)^2$.

6.3 Laurent Series

If a function f is not analytic at a point z_0, we cannot apply Taylor's Theorem at that point. It is, however, possible to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. A series representation of $f(z)$ that involves negative powers of $z - z_0$ is called a Laurent series of $f(z)$ about z_0.

EXAMPLE 6.4
Find the Laurent expansion of

$$f(z) = \frac{1+2z}{z^2 - z^3}$$

about $z = 0$.

Solutions. The expansion is

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(\frac{1+2z}{1-z}\right) \\ &= \frac{1}{z^2}(1+2z)(1+z+z^2+\cdots), \quad 0<|z|<1, \\ &= \frac{1}{z^2}\left(1+2z+z+2z^2+z^2+2z^3+\cdots\right) \\ &= \frac{1}{z^2}+\frac{3}{z}+3+3z+\cdots . \end{aligned}$$

This series expansion is convergent on $0<|z|<1$.

We will only discuss the case when $z_0 = 0$.

THEOREM 6.7 If f is analytic in the annulus

$$A = \{z \in \mathbf{C} | R_1 < |z| < R_2\},$$

then f has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

where

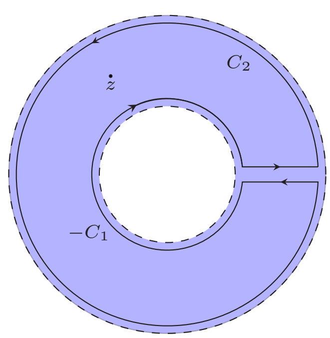
$$a_k = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{k+1}} d\zeta$$

and $C = C(0; R)$ with $R_1 < R < R_2$.

Sketch of proof

The proof of this theorem is similar to that of Theorem 6.4. Suppose $R_1 < r_1 < r_2 < R_2$. We consider the expansion of $1/(z-s)$, with $s \in \{z||z| = r_1\}$ or $s \in \{z||z| = r_2\}$ in two ways.

Let C be the contour $C(0; r_2)$ traversed in counterclockwise direction, and then meet $C(0; r_1)$, traversed in the clockwise direction and finally return to meet $C(0; r_2)$ (see the following diagram).



The result is a simple closed curve that encloses z. Suppose C_1 and C_2 are the respective paths along $C(0; r_1)$ and $C(0; r_2)$ traversed in the anticlockwise direction. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}d\zeta = I_1 + I_2. \end{aligned}$$

For the first integral, expand $1/(1 - z/\zeta)$ since or $|z/\zeta| < 1$. The result is the same as that for Taylor's series, i.e.

$$I_1 = \sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta.$$

For the second integral, we expand $1/(1 - \zeta/z)$ instead of $1/(1 - z/\zeta)$ since $|\zeta/z| < 1$. We have

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z(1 - \zeta/z)}d\zeta \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z}\left\{1 + \frac{\zeta}{z} + \cdots + \left(\frac{\zeta}{z}\right)^{N-1} + \left(\frac{\zeta}{z}\right)^{N} \frac{1}{(1 - \zeta/z)}\right\} d\zeta \\ &= \sum_{j=1}^{N-1} \frac{b_j}{z^j} + \sigma_N, \end{aligned}$$

where

$$b_j = \frac{1}{2\pi i}\int_{C_1} f(\zeta)\zeta^{j-1}d\zeta$$

and

$$\sigma_N = \frac{1}{2\pi i}\int_{C_1} \left(\frac{\zeta}{z}\right)^N \frac{f(\zeta)}{z - \zeta}d\zeta.$$

Now,

$$|\sigma_N| \leq \frac{1}{2\pi}2\pi r_1 \left|\frac{r_1}{z}\right|^N \frac{M}{|z| - r_1},$$

where $|f(z)| \leq M$ on C_1. Hence, $\sigma_N \to 0$ as $N \to 0$ since $|r_1/z| < 1$ and this completes the proof.

Note that C_1 and C_2 can now be replaced by a common circle $C = C(0; R)$ with $R_1 < R < R_2$ since the integrals are independent of z. □

COROLLARY 6.8 If f is analytic in the annulus $R_1 < |z - z_0| < R_2$, then f has a unique representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz$$

and $C = C(z_0; R)$ with $R_1 < R < R_2$.

We now give more examples of Laurent series expansions.

EXAMPLE 6.5
The function $f(z) = \dfrac{1}{(z-1)^2}$ is analytic in $0 < |z - 1| < \infty$. The Laurent expansion of $f(z)$ about $z = 1$ is just $\dfrac{1}{(z-1)^2}$.

EXAMPLE 6.6
The Laurent series expansion of $\dfrac{e^z}{z^2}$ about $z = 0$ is

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .$$

The region for which this is valid is $0 < |z| < \infty$.

EXAMPLE 6.7
$e^{1/z}$ has Laurent series expansion about $z = 0$ as

$$e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^n}, 0 < |z| < \infty.$$

Note that the b_n in this case are non-zero for infinitely many n.

EXAMPLE 6.8
Find the first 3 non-zero terms of the Laurent series expansion of

$$\frac{\sin z}{z^3(1 - z)}$$

about $z = 0$.

DEFINITION 6.9 We say that z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic on $B(z_0; r) - \{z_0\}$ for some $r > 0$.

We now classify singularities according to the Laurent series expansion of $f(z)$ about z_0. Suppose the Laurent series expansion of $f(z)$ about z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

If $a_n = 0$ for $n < 0$, then we say that z_0 is a removable singularity of $f(z)$.

If $a_n = 0$ for $n < -m$ for some negative integer $-m$, then we say that $f(z)$ has a pole of order m at z_0. The function $1/(z - z_0)^m$ is a function with pole of order m at z_0.

If $a_n \neq 0$ for infinitely many negative integers n, then we say that z_0 is an essential singularity of $f(z)$. The function $e^{1/z}$ is a function with essential singularity at $z = 0$.

6.4 Absolute Convergence, Uniform Convergence and continuity of power series

We say that a series

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

is convergent if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$ has a limit. If

$$S^*(z) = \sum_{n=0}^{\infty} |a_n z^n|$$

converges then we say that the series

$$\sum_{n=0}^{\infty} a_n z^n$$

is absolutely convergent.

THEOREM 6.10 If $S(z)$ converges when $z = z_1$, $(z_1 \neq 0)$ it is absolutely convergent for every value of z such that $|z| < |z_1|$.

Proof
Since $|z/z_1| < 1$, we set

$$\left|\frac{z}{z_1}\right| = r < 1.$$

Also, since

$$\sum_{n=0}^{\infty} a_n z_1^n$$

converges,

$$\lim_{n\to\infty} |a_n z_1^n| = 0.$$

This implies that for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$|a_n z_1^n| \le \epsilon$$

whenever $n \ge N_\epsilon$. Set $\epsilon = 1$. Then for $n \ge N_1$,

$$|a_n z_1^n| \le 1.$$

For $n \le N_1$,

$$|a_n z_1^n| \le \max_{0\le j\le N_1} |a_j z_1^j|.$$

Therefore there exists an $M > 0$ such that

$$|a_n z_1^n| \le M$$

for all integers $n \ge 0$.

Let

$$\rho_N := \sum_{n=0}^{\infty} |a_n z^n| - \sum_{n=0}^{N-1} |a_n z^n| = \lim_{m\to\infty} \sum_{n=N}^{m} |a_n z^n|.$$

Now,

$$\sum_{n=N}^{m} |a_n z^n| = \sum_{n=N}^{m} |a_n z_1^n| \left|\frac{z}{z_1}\right|^n \le \sum_{n=N}^{m} M r^n = M \sum_{n=N}^{m} r^n.$$

But the sequence of partial sums $\{G_\ell\}$ where $G_\ell := \sum_{n=0}^{\ell} r^n$ is a Cauchy sequence. Hence for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$|G_m - G_N| < \frac{\epsilon}{M} \quad \text{whenever } m > N \ge N_\epsilon.$$

Therefore,

$$\sum_{n=N}^{m} |a_n z^n| < \epsilon$$

whenever $m > N \ge N_\epsilon$ which implies that $S(z)$ is absolutely convergent. □

In general the rate at which $S_N(z) = \sum_{n=0}^{N} a_n z^n$ converges to $S(z)$ depends on z, i.e., for any $\epsilon > 0$,

$$|S_N(z) - S(z)| < \epsilon, \quad \text{whenever} \quad N \ge N_\epsilon(z).$$

However, if we choose z such that $|z| \le R_1 < R$, where R is the radius of convergence of $S(z)$, then there exist an N_ϵ which will work for all z in $|z| \le R_1 < R$. When N_ϵ is independent of z, we say that $S(z)$ is uniformly convergent. We also say that $S_N(z)$ converges uniformly to $S(z)$ if $|z| \le R_1 < R$.

THEOREM 6.11 The series

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

is uniformly convergent for $|z| \le R_1 < R$, where R is the radius of convergent.

Proof
Let z_0 be fixed and $|z_0| = R_1$. Let z be any point in $|z| \le R_1 = |z_0|$. Let

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n z^n = S(z) - \sum_{n=0}^{N-1} a_n z^n.$$

Now,

$$\left| \sum_{n=N}^{m} a_n z^n \right| \le \sum_{n=N}^{m} |a_n||z^n| \le \sum_{n=N}^{m} |a_n||z_0^n|.$$

Now let r be such that $R_1 < r < R$ and $|z_1| = r$. Note that

$$\sum_{n=0}^{\infty} a_n z_1^n$$

converges since $|z_0| < |z_1| = r < R$. Hence, by Theorem 6.10,

$$\sum_{n=0}^{\infty} a_n z_0^n$$

converges absolutely. Therefore

$$\sum_{n=0}^{\infty} |a_n z_0^n|$$

converges. This implies that

$$\sum_{n=N}^{m} |a_n z_0^n| < \epsilon \quad \text{whenever} \quad m \ge N_\epsilon.$$

Therefore,

$$\left| \sum_{n=N}^{m} a_n z^n \right| < \epsilon \quad \text{whenever} \quad m \ge N_\epsilon,$$

with N_ϵ independent of z. Hence, $S(z)$ converges uniformly. □

THEOREM 6.12 The series $S(z)$ is continuous in $|z| \le R_1 < R$, where R is the radius of convergence.

Proof
Let z_1 be fixed. Let $S(z) = S_N(z) + \rho_N(z)$, where

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N+1}^{m} a_n z^n.$$

We first observe that

$$\begin{aligned} |S(z) - S(z_1)| &= |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)| \\ &\le |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)|. \end{aligned} \tag{6.1}$$

Since $\rho_N(z)$ converges to 0 uniformly as $N \to \infty$, we conclude that

$$|\rho_N(z)| < \frac{\epsilon}{3} \quad \text{whenever} \quad N \ge N_\epsilon. \tag{6.2}$$

Now, let $N = N_\epsilon$ in (6.1) and note that since $S_N(z)$ is a polynomial in z, $S_N(z)$ is continuous at $z = z_1$. Therefore, given $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$|S_N(z) - S_N(z_1)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta_\epsilon. \tag{6.3}$$

Hence, we deduce from (6.1),(6.2) and (6.3) that

$$|S(z) - S(z_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

whenever $|z - z_1| < \delta_\epsilon$. This implies that $S(z)$ is continuous at z_1.

□

6.5 Power series and Analytic functions

We have shown that if f is analytic in $|z| \le R$ then f can be written in terms of power series. In this section, we will show the surprising result that if a power series

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

is convergent in $|z| \le R$, then $S(z)$ is an analytic function in $|z| \le R$.

THEOREM 6.13 Let C be any contour interior to the circle of convergence of $S(z)$, $g(z)$ continuous on C. Then

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)z^n\,dz.$$

In other words, we can integrate term by term.

Proof
The series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly in $|z| \leq R_1 < R$. Therefore

$$\left|\sum_{n=0}^{N} a_n z^n - S\right| < \epsilon \quad \text{whenever} \quad N \geq N_\epsilon$$

where N_ϵ is independent of z. Now,

$$\int_C g(z) \sum_{n=0}^{\infty} a_n z^n \, dz = \sum_{n=0}^{N} a_n \int_C g(z) z^n dz + \int_C g(z) \sum_{n=N+1}^{\infty} a_n z^n \, dz.$$

But

$$|g(z)| \left| \sum_{n=N+1}^{\infty} a_n z^n \right| < \epsilon M,$$

whenever $N \geq N_\epsilon$ and where $|g(z)| \leq M$ on C. Therefore, given $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$\left| \int_C g(z) \sum_{n=0}^{\infty} a_n z^n \, dz - \sum_{n=0}^{N} a_n \int_C g(z) z^n dz \right| \leq M\epsilon |C| \quad \text{whenever } N \geq N_\epsilon,$$

where $|C|$ is the length of the contour. This completes the proof of the theorem.

□

When $g(z) = 1$, we know that $\int_C z^n dz = 0$, and therefore, if $S(z)$ is a convergent power series, then

$$\int_C S(z) dz = 0$$

for any closed curve in $|z| \leq R_1 < R$. Now, by Theorem 6.12, we know that $S(z)$ is continuous. Hence, by Morera's Theorem, we conclude that the convergent power series $S(z)$ is an analytic function.

EXAMPLE 6.9
Let

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{when} \quad z \neq 0 \\ 1 \quad \text{when} \quad z = 0. \end{cases}$$

The power series for $\sin z / z$ is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots =: S(z)$$

and this series is convergent for all z except at $z = 0$. But $f(0) = 1$ and so, it agrees with the value of $S(0)$. Hence, $f(z)$ may be represented by $S(z)$, which is a power series. Since $S(z)$ converges for all z by the above Theorem, $S(z)$ is analytic everywhere, in other words, $S(z)$ is entire. Hence, $f(z)$ is entire.

We now give a companion of Theorem 6.13.

THEOREM 6.14 A convergent power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof

Let $B(z_0; R)$ be the region for which $S(z)$ is convergent. Let $z \in B(z_0; R)$ and let $C \subset B(z_0; R)$ be a simple closed curve enclosing z and suppose C is traversed in the counterclockwise direction. We let

$$g(\zeta) = \frac{1}{2\pi i} \frac{1}{(\zeta - z)^2}$$

at each point of $\zeta \in C$. Since $g(\zeta)$ is continuous on C, Theorem 6.13 implies that

$$\int_C g(\zeta) S(\zeta) d\zeta = \sum_{n=0}^{\infty} a_n \int_C g(\zeta)(\zeta - z_0)^n d\zeta.$$

Now, $S(\zeta)$ is analytic inside and on C and this enables us to write

$$\int_C g(\zeta) S(\zeta) d\zeta = \frac{1}{2\pi i} \int_C \frac{S(\zeta)}{(\zeta - z)^2} d\zeta = S'(z).$$

Furthermore,

$$\int_C g(\zeta)(\zeta - z_0)^n d\zeta = \frac{1}{2\pi i} \int_C \frac{(\zeta - z_0)^n}{(\zeta - z)^2} d\zeta = \frac{d}{dz}(z - z_0)^n, n = 1, 2, \cdots .$$

Thus, we find that

$$S'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1}.$$

□

6.6 Uniqueness of Power series

DEFINITION 6.15 A point u in D is said to be an accumulation point if for every $\delta > 0$,

$$B(u;\delta) \cap (D\backslash\{u\}) \neq \phi.$$

By choosing $\delta_n = 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$, we may pick $z_1, z_2, \cdots z_n$ from $B(u;\delta_n) \cap (D\backslash\{u\})$ such that $z_n \to u$ as $n \to \infty$.

We thus obtain the fact that for every accumulation point $u \in D$, there exist a sequence $\{z_n\}$ with limit u.

THEOREM 6.16 Let $z_n \to u$ where u is an accumulation point of D. Suppose

$$S(z) = \sum_{n=0}^{\infty} a_n (z-u)^n$$

is convergent in D such that $S(z_k) = 0$ for all k. Then $S(z) = 0$ in D.

Proof
Let

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

Since $S(z)$ is continuous,

$$a_0 = S(u) = \lim_{k\to\infty} S(z_k) = 0.$$

Next,

$$a_1 = \frac{S(z)}{z-u} - a_2(z-u) - a_3(z-u)^2 - \cdots .$$

Let $z = z_k$. We find that

$$a_1 = -a_2(z_k - u) - a_3(z_k - u)^2 - \cdots$$

since $S(z_k) = 0$. Hence,

$$a_1 = \lim_{k\to\infty} \left(-a_2(z_k - u) - a_3(z_k - u)^2 - \cdots\right) = 0$$

since the power series is continuous. Suppose $a_i = 0$, for $0 \leq i \leq n$. Then

$$a_{n+1} = \lim_{k\to\infty} \left(-a_{n+2}(z_k - u) - a_{n+3}(z_k - u)^2 - \cdots\right) = 0.$$

Therefore, $a_i = 0$ for all $i \in \mathbf{N}$. This implies that $S(z) = 0$. □

DEFINITION 6.17 Let f be continuous on a region D. We say that $z \in D$ is a limit of zeroes of f if there exists a sequence $\{z_k\}$ such that $z_k \to z$, with $f(z_k) = 0$.

Note that since f is continuous $f(z) = 0$.

THEOREM 6.18 Suppose f is analytic in a region D and $f(z_n) = 0$ where $z_n \to u \in D$. Then $f = 0$ in D.

Proof

Step 1: Let $A = \{z \in D : z \text{ is a limit of zeros of } f\}$. Note that $A \neq \phi$ since $u \in A$. Let $B = D \backslash A$. Note that $D = A \cup B$ and $A \cap B = \phi$.

Step 2: We now show that A is open. Let $z' \in A$. Then since f is analytic at z', $f(z)$ has a power series expansion about z', say $S(z)$, valid for $z \in B(z'; r)$ for some $r > 0$. Since z' is the limit of zeros of $f(z)$, it is the limit of zeros of $S(z)$, i.e., there exist a sequence $\{z_k\}$ such that $z_k \to z'$ and $S(z_k) = 0, k = 1, 2, 3 \cdots$. By Theorem 6.16, $S(z) = 0$ in $B(z'; r)$. This implies that each $z \in D(z', r)$ is a limit of zeros of $f(z)$ since they are all zeros of $f(z)$. Therefore $B(z'; r) \subset A$, which implies that A is open.

Step 3: Next we show that B is open. Suppose $z^* \in B$. Then since z^* is not a limit of zeros of f, there exist an open set $D(z^*, \delta)$ for which $f(z) \neq 0$ for all $z \in D(z^*, \delta)$ except possibly that $f(z^*) = 0$. This implies that none of the elements in the open set is a limit of zeros of f and so they are all in B. Hence, B is open. But D is connected and therefore cannot be a union of two open sets. Since we have noted that A is non-empty by Step 1, this implies that $B = \phi$. Hence $A = D$ and every element $z \in D$ is a limit of zeros of f and hence $f(z) = 0$ in D. □

COROLLARY 6.19 If $f(z)$ and $g(z)$ are analytic in a region D and agree at a set of points with an accumulation point in D, then $f(z) = g(z)$ in D.

This is known as the Uniqueness Theorem for analytic functions. This result gives the reason why series expansion for real functions such as e^x, $\sin x$ etc are the same as those for e^z, $\sin z$ etc..

6.7 Uniqueness theorem and Maximum Modulus Principle

In this section, we give another proof of Maximum Modulus Principle for a region D.

THEOREM 6.20 Let $f(z)$ be analytic in a region D. Suppose there is a point $a \in D$ such that $|f(a)| \geq |f(z)|$ for all $z \in D$, then f is a constant.

Proof
Let δ be such that $B(a;\delta) \subset D$. By Theorem 5.9, we know that $g(z) = f\big|_{B(a;\delta)}$, where $f\big|_{B(a;\delta)}$ denotes the function $f(z)$ restricted to $B(a;\delta)$, must be a constant. Hence, $f(z)$ and $g(z)$ agrees on $B(a;\delta)$ and by uniqueness theorem, $f(z)$ must also be a constant on D. This completes the proof of the theorem. □

6.8 Appendix : Polygonally connected and connected

In this appendix, we show that in $\mathbf{C}$, polygonally connected is equivalent to connected. Recall that a set D is disconnected if there exists nonempty disjoint open sets A and B such that

$$D = A \cup B.$$

Otherwise, D is said to be connected.

THEOREM 6.21 An open set D is connected if and only if it is polygonally connected

Proof
Suppose D is connected. Let $u \in D$ and let

$$A = \{s \in D | s \text{ is connected to } u \text{ by a polygonal line in } D\}.$$

Let $B = D \backslash A$, i.e., every point in B is not connected to u by a polygonal line in D. Note that $A \neq \phi$ because $u \in A$. Also, $D = A \cup B$ and $A \cap B = \phi$.

We now show that A is open. If $z \in A$, then $B(z;r) \subset D$ for some $r > 0$ since D is open. But in $B(z;r)$, any two points are polygonally connected. Now $z \in A$ implies that z is polygonally connected to u. This implies that all points in $B(z;r)$ are polygonally connected to u. Hence $B(z;r) \subset A$. This implies that A is open.

Suppose $B \neq \phi$. Let $z' \in B$. Then there exists $r' > 0$ such that $B(z';r') \subset D$. Note that none of the points in $B(z';r')$ is polygonally connected to u, for otherwise, z' would be polygonally connected to u and z' would not be in B. Hence, $B(z';r') \subset B$ and B is open. But now D is connected and $D = A \cup B$ where A and B are open sets. Since A is non-empty, the only way this can happen is $B = \phi$. Hence $D = A$ and every point in D is polygonally connected to u and hence, the open set D is polygonally connected.

Conversely, suppose D is not connected. Then let A and B be open disjoint sets such that

$$D = A \cup B.$$

Let $a \in A$ and $b \in B$. Suppose that there exists a polygonal line connecting a to b. We may assume this line to be the line segment $[a,b]$. For if not, there is a line contained in the polygonal path that joins a point a^* in A with a point b^* in B for the first time. We then replace a by a^* and b by b^*.

Now, let $\gamma : [0,1] \to D$ be $\gamma(t) = a(1-t) + bt$. Let

$$t^* = \sup\{t \in [0,1] | \gamma(t) \in A\}.$$

Since A is open, $t^* > 0$. Similarly since B is open, $t^* < 1$. Let $z^* = \gamma(t^*)$. Note that $z^* \notin A$. For if $z^* \in A$, then since A is open, there exists $\epsilon > 0$ such that $B(z^*;\epsilon) \subset A$. By continuity of γ, we conclude that there exists $\delta_\epsilon > 0$ such that if $|t - t^*| < \delta_\epsilon$ then $\gamma(t) \in B(z^*;\epsilon)$. But this means that $\gamma(t^* + \delta_\epsilon/2) \in A$ and t^* is not an upper bound for the set $\{t \in [0,1] | \gamma(t) \in A\}$.

Similarly, B does not contain z^*. For if B contains z^* then there exists $\epsilon' > 0$ such that if $|t - t^*| < \delta_{\epsilon'}$, then $\gamma(t) \in B(z^*;\epsilon') \subset B$. This implies that $\gamma(t) \in B$ for $t^* - \delta'_\epsilon < t$ and so, t^* is not the least upper bound for the set $\{t \in [0,1] | \gamma(t) \in A\}$.

We must therefore conclude that such a line segment does not exist. □

7 The Residue Theorem

7.1 Residues

When z_0 is an isolated singularity of f, there exist an R such that f is analytic on $0 < |z - z_0| < R$. Therefore $f(z)$ has a Laurent series expansion given by Theorem 6.7, namely,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, n = 0, 1, 2, \cdots,$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{(z - z_0)^{-n+1}}\, dz, n = 1, 2, \cdots.$$

When $n = 1$,

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz,$$

and this is called the residue of $f(z)$ at the isolated singular point z_0. It is denoted by $\text{Res}(f(z); z_0)$.

EXAMPLE 7.1
Consider

$$\int_C \frac{e^{-z}}{(z-1)^2} dz$$

where C is the circle $|z| = 2$, described in the positive sense.

The function $f(z) = \dfrac{e^{-z}}{(z-1)^2}$ is analytic in $|z| \leq 2$ except at the isolated singularity $z = 1$. By Cauchy Integral Formula,

$$\int_C \frac{e^{-z}}{(z-1)^2} dz = -\frac{2\pi i}{e}.$$

Thus, the residue of $f(z)$ at $z = 1$ is

$$\text{Res}(f, 1) = -\frac{1}{e}.$$

EXAMPLE 7.2
Show that if C is the contour $|z| = 2$, traversed in the positive sense, then

$$\int_C e^{1/z^2}\, dz = 0.$$

Solutions. We have

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \cdots .$$

The coefficient of $\frac{1}{z}$ is 0. Therefore

$$\int_C e^{1/z^2}\, dz = 0.$$

This example cannot be deduced from Cauchy's Integral Formula.

7.2 Residue Theorem

If a function f has only a finite number of singular points interior to a given simple closed contour C, they must be isolated. The following Theorem gives us a formula for evaluating

$$\int_C f(z)dz$$

if f has a finite number of singular points interior to C.

THEOREM 7.1 (The Residue Theorem) Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \cdots, z_n$ interior to C. If $B_1, B_2, \cdots, B_n$ denote the residues of f at these respective points, then

$$\int_C f(z)dz = 2\pi i(B_1 + B_2 + \cdots + B_n).$$

Proof

By Cauchy Goursat Theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

Since

$$\int_{C_i} f(z)dz = 2\pi i B_i,$$

we immediately obtain the result.

□

EXAMPLE 7.3

Suppose C is the circle $|z| = 2$ described anticlockwise. Evaluate

$$\int_C \frac{5z-2}{z(z-1)}dz.$$

Solutions. Note that $z = 0$ and $z = 1$ are the two singularities of the function

$$f(z) = \frac{5z-2}{z(z-1)}.$$

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + 3 + \cdots, \quad |z| < 1$$

and

$$\frac{5z-2}{z(z-1)} = \left(5 + \frac{3}{z-1}\right)(1-(z-1)+(z-1)^2+\cdots), \quad |z-1| < 1,$$

therefore, Residue of $f(z)$ at $z = 0$ is 2 and at $z = 1$ is 3. Hence,

$$\int_C f(z)dz = 2\pi i(2+3) = 10\pi i.$$

Alternatively one may use Cauchy Integral Formula.

THEOREM 7.2 If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2\pi i \text{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right).$$

Proof

Let $C(0; R)$ be a circle with R large enough so that the circle enclosed C. Then

$$\int_C f(z)dz = \int_{C(0;R)} f(z)dz.$$

Write

$$\int_{C(0;R)} f(z)dz = \int_0^{2\pi} f(Re^{it})iRe^{it}dt.$$

Replacing t by $-s$, we find that

$$-\int_0^{-2\pi} f(Re^{-is})iRe^{-is}ds.$$

We now convert this line integral back to contour integral by letting $z = e^{is}/R$ and deduce that

$$-i\int_0^{-2\pi} f(R/e^{is}) \cdot \frac{R^2}{e^{2is}} \cdot \frac{e^{is}}{R}ds = -\int_{C'(0;1/R)} \frac{f(1/z)}{z^2}dz$$

where $C'(0,1/R)$ is the circle centered at 0 with radius $1/R$ traversed in clockwise direction. Now, this gives

$$\int_{C(0;R)} f(z)dz = \int_{C(0;1/R)} \frac{f(1/z)}{z^2}dz$$

where $C(0;1/R)$ is traversed in anti-clockwise direction and the proof is complete.

□

EXAMPLE 7.4

Use Theorem 7.2 to solve Example 7.3.

Solutions. Let $f(z) = (5z-2)/(z(z-1))$. Then

$$\frac{1}{z^2}f(1/z) = \frac{5-2z}{z}\left(1+z+z^2+\cdots\right)$$

and so $\text{Res}(1/z^2 f(1/z), 0) = 5$ and hence the result.

7.3 Evaluations of improper integrals

An important application of the theory of residues is the evaluation of certain types of definite improper integral arising from real analysis.

In Calculus we encounter improper integral of continuous function $f(x)$ over semi infinite interval $x \geq 0$:

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx.$$

When the limit on the right exists, the improper integral is said to converge and its value is the value of the limit. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)dx.$$

When both integrals on the right hand side converges, we say that $\int_{-\infty}^{\infty} f(x)dx$ converges. It may happen that the integrals on the right side diverge but the limit

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

exists. In this case, we call

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

the Cauchy principal value of the integral of $\int_{-\infty}^{\infty} f(x)dx$ and write

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx.$$

REMARK 7.1 The Cauchy principal value of the integral $\int_{-\infty}^{\infty} f(x)dx$ may exist without $\int_{-\infty}^{\infty} f(x)dx$ being defined. For example, if $f(x) = 2x/(1+x^2)$, $\int_{-\infty}^{\infty} f(x)dx$ is divergent but its principal value is 0. However, when $f(x)$ is an even function, i.e., $f(x) = f(-x)$, both P.V.$\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ coincide.

In this section, we will use residue theorem to evaluate different type of integrals.

EXAMPLE 7.5
Evaluate

$$\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx.$$

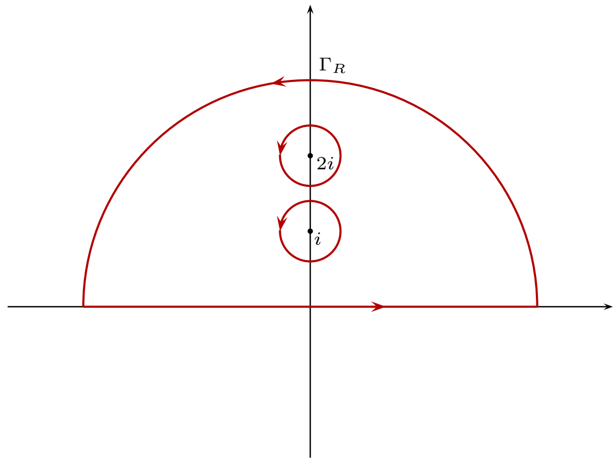
Since $f(x)$ is even it suffices to evaluate the Cauchy Principal value of the integral by Example 5.4.1. Consider the function

$$f(z) = \frac{2z^2-1}{z^4+5z^2+4}.$$

Note that

$$z^4+5z^2+4 = (z^2+4)(z^2+1).$$

Hence, $f(z)$ has poles at $\pm 2i$ and $\pm i$. Let $R > 2$.



From the Residue Theorem, we know that

$$\int_{C_R} f(z)dz = 2\pi i(\text{Res}(f(z); i) + \text{Res}(f(z); 2i)).$$

Now,

$$\text{Res}(f(z); i) = \frac{i}{2},$$

while

$$\text{Res}(f(z); 2i) = -\frac{3i}{4}.$$

Hence,

$$\int_{C_R} f(z)dz = \frac{\pi}{2}.$$

Let $C_R = [-R, R] \cup \Gamma_R$, where Γ_R is the arc from R to $-R$ and $[-R, R]$ is the line segment $[-R, R]$. Now, on Γ_R,

$$\left|\frac{2z^2 - 1}{z^4 + 5z^2 + 4}\right| \leq \frac{2R^2 + 1}{R^4 - 5R^2 - 4}.$$

Hence,

$$\left|\int_{\Gamma_R} f(z)dz\right| \leq \frac{\pi R(2R^2 + 1)}{R^4 - 5R^2 - 4},$$

which tends to 0 as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0.$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \int_{C_R} f(z)dz = \frac{\pi}{2}.$$

EXAMPLE 7.6
Show that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

7.4 Improper Integrals involving cos

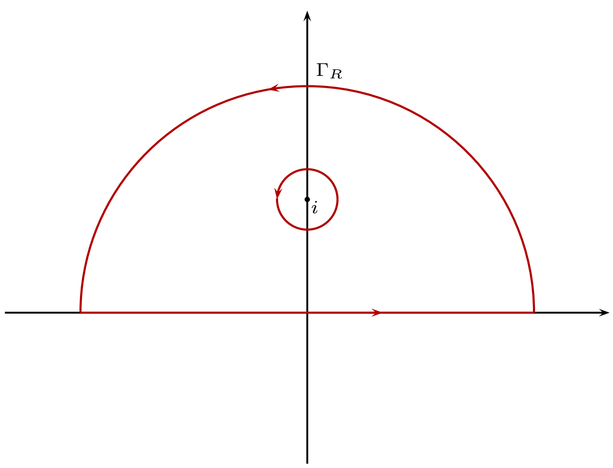
We now evaluate improper integrals of the type

$$\int_{-\infty}^\infty P(x)\cos x dx.$$

EXAMPLE 7.7
Show that

$$\int_{-\infty}^\infty \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

Consider $f(z) = \dfrac{e^{iz}}{(z^2+1)^2}$. Consider the following contour:



Note that

$$\int_{C_R} f(z)dz = 2\pi i \mathrm{Res}(f(z); i).$$

Let

$$\frac{\phi(z)}{(z-i)^2} = \frac{e^{iz}}{(z-i)^2(z+i)^2},$$

or

$$\phi(z) = \frac{e^{iz}}{(z+i)^2}.$$

By Cauchy Integral formula,

$$\int_{C_R} \frac{e^{iz}}{(z-i)^2(z+i)^2} dz = \int_{C_R} \frac{\phi(z)}{(z-i)^2} dz = 2\pi i \phi'(i).$$

Now,

$$\phi'(z) = \frac{-2e^{iz}}{(z+i)^3} + \frac{ie^{iz}}{(z+i)^2}.$$

Hence

$$\phi'(i) = \frac{-i}{2e}.$$

Therefore,

$$\int_{C_R} f(z) dz = \frac{\pi}{e}.$$

Now split $C_R = [-R, R] \cup \Gamma_R$. Then on Γ_R

$$\left| \frac{e^{iz}}{(z^2+1)^2} \right| \leq \frac{1}{(R^2-1)^2},$$

which implies immediately that

$$\int_{\Gamma_R} f(z) dz \to 0$$

as $R \to \infty$.

Hence, we conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

Splitting $e^{ix} = \cos x + i \sin x$, we deduce our result.

REMARK 7.2 In the computation of the residue $\text{Res}(f(z); i)$, we can also compute the Laurent series expansion of

$$\frac{1}{(z-i)^2} \frac{e^{iz}}{(z+i)^2} = \frac{1}{(z-i)^2} \frac{e^{-1}}{(2i)^2} (1 + (z-i)i + \cdots) \left(1 - 2\frac{(z-i)}{2i} + \cdots\right).$$

The coefficient of $(z-i)^{-1}$ in this expansion is $(2ie)^{-1}$.

7.5 An identity of Euler

In this section, we will show that

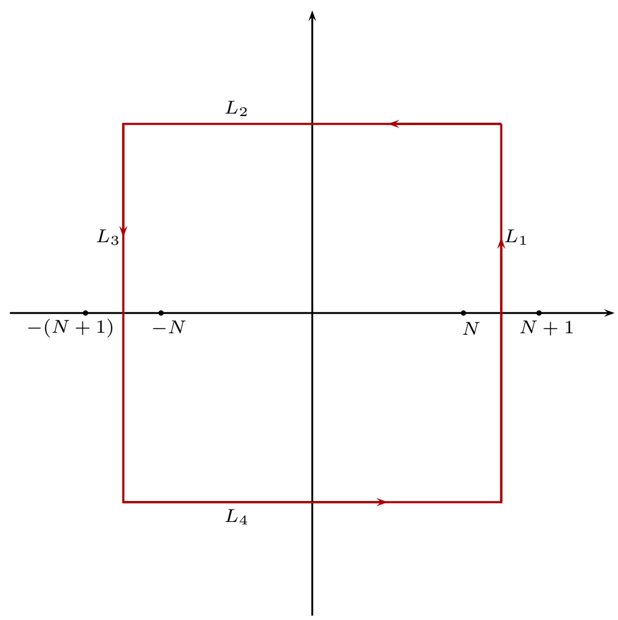
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We first study the property of

$$\varphi(z) = \cot \pi z.$$

First, note that the function $\varphi(z)$ has simple poles at $k \in \mathbf{Z}$. These are zeroes of $\sin z$.

Consider the following contour C_N:



We claim that on C_N,

$$|\varphi(z)| < 2.$$

On L_1, we set $z = (N + 1/2) + iy$, $-(N + 1/2) \le y \le N + 1/2$. We find that

$$\begin{aligned} |\varphi(z)| &\le \left| \frac{1 + e^{2\pi i N + \pi i - 2\pi y}}{1 - e^{2\pi i N + \pi i - 2\pi y}} \right| \\ &\le \frac{|1 - e^{-2\pi y}|}{1 + e^{-2\pi y}} \le \frac{1 + |e^{-2y}|}{1 + e^{-2y}} = 1. \end{aligned}$$

The bound for $|\varphi(z)|$ we obtained above is also valid on L_3.

On L_2, we set $z = x + i(N+1/2)$, $-(N+1/2) \leq x \leq N+1/2$. Then

$$|\varphi(z)| \leq \frac{1+e^{-\pi(2N+1)}}{1-e^{-\pi(2N+1)}}.$$

Since

$$\frac{1+t}{1-t} \leq 2$$

whenever $t \leq 1/3$ and that for sufficiently large N, $e^{-\pi(2N+1)} \leq 1/3$, we conclude that

$$|\varphi(z)| \leq 2.$$

The bound for $|\varphi(z)|$ we obtained above is also valid on L_2.

Now,

$$\left|\int_{C_N} \frac{1}{z^2}\pi \cot \pi z dz\right| \leq (8N+4)\frac{1}{(N+1/2)^2}2\pi.$$

This implies that the integral tends to 0 as $N \to \infty$. Next, we observe that

$$\int_{C_N} \frac{1}{z^2}\pi \cot \pi z dz = 2\pi i \left(\operatorname{Res}\left(\frac{1}{z^2}\pi \cot \pi z, 0\right) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \operatorname{Res}\left(\frac{1}{z^2}\pi \cot \pi z, k\right)\right)$$

For $k \neq 0$,

$$\operatorname{Res}\left(\frac{1}{z^2}\pi \cot \pi z, k\right) = \frac{1}{k^2}\lim_{z\to k} \pi \cot \pi z (z-k) = \frac{1}{k^2}\lim_{z\to k} \cos \pi z \frac{\pi(z-k)}{\sin \pi z} = \frac{1}{k^2}.$$

The term

$$\operatorname{Res}\left(\frac{1}{z^2}\pi \cot \pi z, 0\right) = \frac{1}{2\pi i}\left(\int_{\gamma} \frac{1}{z^2}\pi \cot \pi z dz\right),$$

where γ is a simple closed curve that enclosed 0 and no other poles of $\cot \pi z$. To compute this residue, we compute the Laurent series expansion of $\pi \cot \pi z/z^2$ and conclude that

$$\frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3}\left(1 - \frac{\pi^2}{2!}z^2 + \cdots\right)\left(1 + \frac{\pi^2}{6}z^2 + \cdots\right).$$

This implies that

$$\operatorname{Res}\left(\frac{1}{z^2}\pi \cot \pi z, 0\right) = -\frac{\pi^2}{3}.$$

Hence, we have

$$\int_{C_N} \frac{1}{z^2}\pi \cot \pi z dz = 2\pi i \left(-\frac{\pi^2}{3} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{1}{k^2}\right).$$

Letting $N \to \infty$ concludes our proof.

The above method can be used to obtain explicit values of

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$$

where m is a positive integer. We first define Bernoulli numbers as the numbers appearing in the following expansion:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

The first few values of B_m are

$$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$$
$$B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0, B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}.$$

Then, we see immediately that

$$\cot \pi z = i\frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} = i\left(1 + \frac{1}{i\pi z}\sum_{k=0}^{\infty} B_k \frac{(2\pi i z)^k}{k!}\right).$$

Using the same method as in the case of Euler's identity, we observe that

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^{2m}} = \operatorname{Res}\left(\frac{\pi}{z^{2m}} \cot \pi z, 0\right) = \frac{(-1)^{m+1}(2\pi)^{2m}}{(2m)!} B_{2m}.$$

This generalizes Euler's identity. Using $B_4 = -\frac{1}{30}$, we deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

EXAMPLE 7.8

Evaluate

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

Solutions. Observe that

$$\binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{5^n z^{n+1}} dz$$

where C is the circle $C(0;r)$ traversed in the counterclockwise direction and $r > 0$ to be chosen later. This implies that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \int_{C(0;r)} \frac{1}{z} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} dz = \frac{1}{2\pi i} \int_{C(0;r)} \frac{5}{3z - 1 - z^2} dz,$$

where r can be chosen to be 1 to ensure that on $|z| = 1$,

$$\left| \frac{(1+z)^2}{(5z)} \right| \leq \frac{4}{5}$$

and that the use of the expansion

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

is valid. The function $3z - 1 - z^2$ has zeroes $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ and only $\frac{3}{2} - \frac{\sqrt{5}}{2}$ is enclosed by $C(0;r)$. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} &= \frac{1}{2\pi i} \int_{C(0;r)} \frac{1}{z} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} dz \\ &= \frac{1}{2\pi i} \int_{C(0;r)} \frac{5}{3z - 1 - z^2} dz \\ &= \operatorname{Res}\left(-\frac{5}{z^2 + 1 - 3z}, \frac{3 - \sqrt{5}}{2} \right) = \sqrt{5}. \end{aligned}$$

EXAMPLE 7.9

Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

Solutions. We consider the contour similar to Example 7.5. Note that

$$\int_{C_R} \frac{e^{iz} - 1}{z} dz = 0.$$

Hence,

$$\int_{\Gamma_R} \frac{e^{iz} - 1}{z} dz + \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx = 0.$$

Now,

$$\int_{-R}^{R} \frac{e^{ix} - 1}{x} dx = -\int_{\Gamma_R} \frac{e^{iz} - 1}{z} dz = \pi i + \int_{\Gamma_R} \frac{e^{iz}}{z} dz.$$

We will show later that

$$\lim_{R \to \infty} \int_{\Gamma_R} \frac{e^{iz}}{z} dz = 0.$$

Assuming this for the moment, we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{R \to \infty} \operatorname{Im} \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx = \pi.$$

To show that

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z}dz = 0,$$

we observe that this integral can be bounded by

$$\int_0^\pi e^{-R\sin\theta}d\theta.$$

For $\delta > 0$, we find that $e^{-R\sin\theta} \le e^{-R\sin\delta}$ for $\delta \le \theta \le \pi - \delta$. Hence,

$$\begin{aligned}\int_0^\pi e^{-R\sin\theta}d\theta &\le \int_0^\delta e^{-R\sin\theta}d\theta + \int_{\pi-\delta}^\pi e^{-R\sin\theta}d\theta \\ &\quad + \int_\delta^{\pi-\delta} e^{-R\sin\theta}d\theta \\ &\le 2\delta + e^{-R\sin\delta}(\pi - 2\delta).\end{aligned}$$

Letting $R \to \infty$, we conclude that

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z}dz \le 2\delta.$$

Since $\delta > 0$ is arbitrary, we may let $\delta \to 0$ and conclude that

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z}dz = 0.$$

EXAMPLE 7.10

Evaluate

$$\int_0^{2\pi}\frac{1}{5+3\cos x}dx = \frac{\pi}{2}.$$

Solutions. Write the integral as

$$\int_{C(0;1)}\frac{1}{iz(5+3(z+z^{-1})/2)}dz.$$

This simplifies as

$$\int_{C(0;1)}\frac{2}{i(10z+3z^2+3)}dz = 2\pi i\text{Res}\left(\frac{2}{i(10z+3z^2+3)}, -\frac{1}{3}\right) = \frac{\pi}{2}.$$

7.6 Branch of logarithm and integral

Recall that in calculus, we define log to be

$$\ln x = \int_1^x \frac{1}{t} dt.$$

(For logarithm of real variable, we will use the symbol ln.)

For complex variable, we see that if $z = re^{i\theta}$ and assuming that "F" is the inverse of e^z, then since $e^{i\theta} = e^{i\theta+2k\pi i}, k \in \mathbf{Z}$, we see that F is not a function. It is a multi-valued function. For example

$$F(z) = \log|z| + i(\theta + 2k\pi).$$

To obtain a function from F we need to choose a branch by restricting for example, $-\pi < \theta < \pi$.

DEFINITION 7.3 We say that f is an analytic branch of $\log z$ in a region D if

(a) f is analytic in D,
(b) f is an inverse of the exponential function there; i.e.

$$\exp(f(z)) = z.$$

We have seen the function

$$F(z) = \int_1^z \frac{d\zeta}{\zeta}, -\pi < \arg z < \pi.$$

This function is analytic for $-\pi < \arg z < \pi$. We now show that $F(z)$ is a branch of log. The function $F(z)$ is denoted as Log$z$.

Consider the function

$$G(z) = z\exp(-F(z)).$$

Then

$$G'(z) = \exp(-F(z)) + z\exp(-F(z))(-F'(z)) = 0.$$

Hence $G(z)$ is a constant and the constant is 1 by setting $z = 1$. This shows that $F(z)$ is indeed a branch of logarithm.

Note that once we declare the range of arg z, say, $\delta < \text{arg} z < 2\pi + \delta$, we get a branch of logarithm by having

$$\log_{\{\delta\}} z = \ln|z| + i\text{arg}z.$$

Once we fix our branch, we may define, for example,

$$(z^{1/2})_{\{\delta\}} = \exp((\log_{\{\delta\}} z)/2).$$

Note that if $\delta = -\pi$, then $z^{1/2}$ is the positive square root of z for real number z. For if $z = r, r \in \mathbf{R}^+$ then the argument of r is 0 and $z^{1/2} = \sqrt{r}$.

When $\delta = \pi$, the argument of $z = r$ is 2π and hence

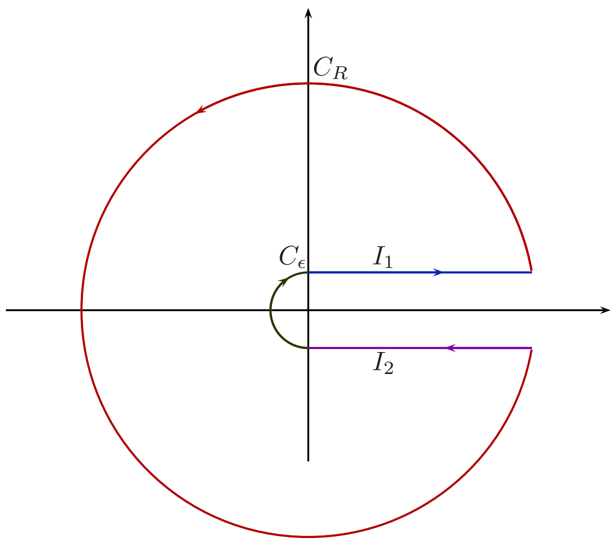
$$z^{1/2} = \sqrt{r}e^{\pi i} = -\sqrt{r}.$$

It is thus clear that the multiple value of $z^{1/2}$ corresponds to different branches of log.

EXAMPLE 7.11
Evaluate

$$\int_0^\infty \frac{1}{1+x^3}dx.$$

Solutions. Consider the following diagram:



Let $\gamma = C_R + C_\epsilon + I_1 + I_2$. Then

$$\int_\gamma \frac{\log z}{z^3+1}dz = 2\pi i\Bigg(\text{Res}\left(\frac{\log z}{z^3+1}, a\right) + \text{Res}\left(\frac{\log z}{z^3+1}, b\right) + \text{Res}\left(\frac{\log z}{z^3+1}, c\right)\Bigg)$$

where $a = e^{\pi i/3}, b = e^{\pi i}$ and $c = e^{5\pi i/3}$. Hence,

$$\int_\gamma \frac{\log z}{z^3+1}dz = 2\pi i\left(\frac{\pi i}{3}\frac{1}{(a-b)(a-c)} + \pi i\frac{1}{(b-a)(b-c)} + \frac{5\pi i}{3}\frac{1}{(c-a)(c-b)}\right).$$

Now, note that

$$(t-u)(t-v) = (t^3+1)/(t-w)$$

if u, v and w are roots of z^3+1. Therefore, by L'Hopital's Rule,

$$(w-u)(w-v) = 3w^2.$$

This gives

$$(a-b)(a-c) = 3e^{2\pi i/3}, (b-a)(b-c) = 3e^{2\pi i} \quad \text{and} \quad (c-a)(c-b) = 3e^{10\pi i/3}.$$

This implies that

$$\int_\gamma \frac{\log z}{z^3+1} dz = 2\pi i\Big(-\frac{\pi i}{9}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + \frac{\pi i}{3} - \frac{5\pi i}{9}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\Big)$$
$$= -2\pi i\left(\frac{2\sqrt{3}\pi}{9}\right).$$

We now estimate the individual integrals over C_R, C_ϵ, I_1 and I_2. On C_R, we have

$$\left|\int_{C_R} \frac{\log z}{z^3+1} dz\right| \le \frac{\pi R(\log R + 2\pi)}{R^3-1}$$

which tends to 0 as $R \to \infty$. On C_ϵ, we have

$$\left|\int_{C_\epsilon} \frac{\log z}{z^3+1} dz\right| \le \frac{\pi\epsilon(\log\epsilon + 2\pi)}{1-\epsilon^3}$$

which tends to 0 as $\epsilon \to 0$.

We have to deal with the integrals

$$\int_{i\epsilon}^{R+i\epsilon} \frac{\log z}{z^3+1} dz + \int_{R-i\epsilon}^{-i\epsilon} \frac{\log z}{z^3+1} dz.$$

We rewrite the above integrals as

$$\int_0^R \frac{\ln(\sqrt{t^2+\epsilon^2}) + i\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} dt + \int_R^0 \frac{\ln(\sqrt{t^2+\epsilon^2}) + i\theta(t-i\epsilon)}{(t-i\epsilon)^3+1} dt.$$

This yields

$$\int_0^R \ln(\sqrt{t^2+\epsilon^2})\left(\frac{1}{(t+i\epsilon)^3+1} - \frac{1}{(t-i\epsilon)^3+1}\right) dt$$
$$+ i\int_0^R \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} - \frac{\theta(t-i\epsilon)}{(t-i\epsilon)^3+1}\right) dt =: U + V.$$

The integral U is 0 if we let $\epsilon \to 0$. To simplify V, note that

$$\theta(t-i\epsilon) = 2\pi - \theta(t+i\epsilon).$$

Hence V is

$$i\int_0^R \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} + \frac{-2\pi + \theta(t+i\epsilon)}{(t-i\epsilon)^3+1}\right) dt.$$

It suffices to show that as $\epsilon \to 0$,

$$\int_0^R \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} + \frac{\theta(t+i\epsilon)}{(t-i\epsilon)^3+1} \right) dt = 0.$$

If this were true, then as $\epsilon \to 0$, we observe that

$$V = -2\pi i \int_0^R \frac{1}{t^3+1} dt.$$

Note that for any $\delta > 0$, we have

$$\int_0^R \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} + \frac{\theta(t+i\epsilon)}{(t-i\epsilon)^3+1} \right) dt = \int_0^\delta \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} + \frac{\theta(t+i\epsilon)}{(t-i\epsilon)^3+1} \right) dt$$
$$+ \int_\delta^R \left(\frac{\theta(t+i\epsilon)}{(t+i\epsilon)^3+1} + \frac{\theta(t+i\epsilon)}{(t-i\epsilon)^3+1} \right) dt =: U + V.$$

For S, we have $t \geq \delta > 0$ and

$$\theta(t+i\epsilon) \leq \tan^{-1}(\epsilon/t) < \tan^{-1}(\epsilon/\delta).$$

Hence,

$$\lim_{\epsilon \to 0} \theta(t+i\epsilon) = 0$$

and therefore, V is 0 as $\epsilon \to 0$.

For U, we observe that

$$\theta(t+i\epsilon) \leq \frac{\pi}{2}$$

in the interval $0 \leq t \leq \delta$. Write

$$\frac{1}{(t+i\epsilon)^3+1} + \frac{1}{(t-i\epsilon)^3+1} = \frac{2A}{|A^2+B^2|}$$

where

$$A + iB = (t+i\epsilon)^3 + 1 = (t^3 - 3\epsilon^2 t + 1) + i(3\epsilon t^2 - \epsilon^3).$$

Since $\operatorname{Re} z \leq |z|$, we conclude that

$$frac2A|A^2+B^2| \leq 2.$$

Hence, by ML-formula, U is bounded by

$$\frac{\pi}{2} cdot 2 \cdot \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that U is 0. Collecting all the information, we conclude that

$$\int_0^\infty \frac{1}{1+x^3} dx = \frac{2\pi\sqrt{3}}{9}.$$

8 Winding number

8.1 Winding number and Cauchy's Residue Theorem for closed curves

We have so far discussed only integral over simple closed curve. It is more natural to consider general closed curve, i.e., curve that intersects itself several times.

Consider the function $f(z) = z^m$, with $m \geq 2$ being a positive integer. Then the integral

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

can be written as

$$\int_{f(\gamma)} \frac{1}{\omega} d\omega,$$

where γ is $C(0;1)$ and $f(\gamma)$ is the image of the curve γ under f. Note that γ can be parametrized by $z(t) = e^{it}, 0 \leq t \leq 2\pi$. When t increases from 0 to $2\pi/m$, we see that z^m moves around the origin in one full circle. By the time t reaches 2π, z^m would have circled the origin m times. In other words, the curve $f(\gamma)$ "winds" around the origin m times and we observe that

$$\frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{\omega} d\omega = \frac{1}{2\pi i} \int_\gamma \frac{(z^m)'}{z^m} dz = m. \tag{8.1}$$

This motivates the following definition:

DEFINITION 8.1 Suppose γ is a closed curve (not necessarily simple) and that $a \notin \gamma$. Then

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$$

is called the winding number of γ around a.

THEOREM 8.2 For any closed curve γ and $a \notin \gamma$, $n(\gamma; a)$ is an integer.

Proof
Let γ be parametrized by $z(t)$, $0 \le t \le 1$. Set

$$F(s) = \int_0^s \frac{z'(t)}{z(t) - a} dt, 0 \le s \le 1.$$

Let

$$G(s) = \frac{e^{F(s)}}{z(s) - a}.$$

Now, since

$$F'(s) = \frac{z'(s)}{z(s) - a},$$

we conclude that

$$G'(s) = e^{F(s)} \frac{F'(s)}{z(s) - a} - \frac{e^{F(s)} z'(s)}{(z(s) - a)^2} = 0.$$

Hence,

$$G(s) = \frac{e^{F(s)}}{z(s) - a} = C,$$

where C is a constant. Let $s = 0$. Since $F(0) = 0$, we find that

$$C = G(0) = \frac{1}{z(0) - a}. \tag{8.2}$$

When $s = 1$, we find that

$$C = G(1) = \frac{e^{F(1)}}{z(1) - a}. \tag{8.3}$$

Now, γ is a closed curve and this implies that $z(0) = z(1)$ and we deduce from (8.2) and (8.3) that

$$e^{F(1)} = 1$$

or

$$F(1) = 2\pi i n(\gamma; a) = 2\pi i k, k \in \mathbf{Z}.$$

In other words, we have shown that $n(\gamma; a)$ is an integer. □

THEOREM 8.3 Suppose f is analytic in a star-shaped domain except for singularities at $z_1, z_2, \cdots, z_m$. Let γ be a closed curve not intersecting any of the singularities. Then

$$\int_\gamma f(\zeta) d\zeta = 2\pi i \sum_{k=1}^{m} n(\gamma; z_k) \mathrm{Res}(f, z_k).$$

Proof
We know that around each z_j, $f(z)$ can be written as

$$f(z) = \sum_{k=0}^{\infty} a_{j,k}(z - z_j)^k + \sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k}.$$

Let

$$g(z) = f(z) - \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k}.$$

Note that $g(z)$ is analytic in the region enclosed by γ and therefore, by Cauchy-Goursat's Theorem,

$$\int_\gamma g(\zeta) d\zeta = 0.$$

This implies that

$$\begin{aligned}\int_\gamma f(\zeta) d\zeta &= \sum_{j=1}^{m} \int_\gamma \sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k} d\zeta \\ &= \sum_{j=1}^{m} \int_\gamma \frac{b_{j,1}}{\zeta - z_j} d\zeta = 2\pi i \sum_{j=1}^{m} n(\gamma; z_j) \mathrm{Res}(f, z_j).\end{aligned}$$

□

REMARK 8.1 In the proof above, we use Laurent series expansions of $f(z)$ at various singularities and extract the residues from these expansions.

8.2 Counting zeroes and poles

THEOREM 8.4 Suppose γ is a simple closed curve. If f has finitely many zeroes and poles in the region enclosed by γ, then

$$Z_\gamma(f) - P_\gamma(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta,$$

where $Z_\gamma(f)$ is the number of zeroes of f inside γ (a zero of order k being counted k times) and $P_\gamma(f)$ is the number of poles of f inside γ (again with multiplicity).

Proof
Let $z_1, z_2, \cdots, z_m$ be distinct zeroes of f with multiplicity $\alpha_1, \alpha_2, \cdots, \alpha_m$ and $p_1, p_2, \cdots, p_\ell$ be distinct poles of f with multiplicity $\beta_1, \beta_2, \cdots, \beta_\ell$. Note that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{u=1}^{m} \mathrm{Res}\left(\frac{f'(z)}{f(z)}; z_u\right) + \sum_{v=1}^{\ell} \mathrm{Res}\left(\frac{f'(z)}{f(z)}; p_v\right).$$

If a were a zero of f with multiplicity s, then

$$f(z) = (z-a)^s g(z)$$

with $g(a) \neq 0$. This implies that

$$\frac{f'(z)}{f(z)} = \frac{s}{z-a} + \frac{g'(z)}{g(z)}$$

and the residue of f'/f at a is s.

Similarly if b were a pole of f with multiplicity t, then

$$f(z) = \frac{h(z)}{(z-b)^t}$$

with $h(z)$ analytic at b. This implies that

$$\frac{f'(z)}{f(z)} = \frac{-t}{z-a} + \frac{h'(z)}{h(z)}$$

and the residue of f'/f at b is $-t$. Hence, the integral

$$\frac{1}{2\pi i}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta = Z_\gamma(f) - P_\gamma(f).$$

□

If f has no poles in the region enclosed by γ, then we have

$$Z_\gamma(f) = \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta.$$

THEOREM 8.5 (Rouché's Theorem) Suppose f and g are analytic inside and on a simple closed curve γ and that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then

$$Z_\gamma(f+g) = Z_\gamma(f)$$

inside γ where $Z_\gamma(h)$ is the number of zeroes of $h(z)$ inside γ.

Proof

Note that by writing

$$f + g = f\left(1 + \frac{g}{f}\right)$$

and noting that

$$\frac{(f+g)'}{f+g} = \frac{f'}{f} + \frac{(1+g/f)'}{1+g/f},$$

we conclude that

$$Z_\gamma(f+g) = Z_\gamma(f) + \frac{1}{2\pi i}\int_\gamma \frac{1+(g(\zeta)/f(\zeta))'}{1+g(\zeta)/f(\zeta)}d\zeta.$$

But the last integral can be written as

$$\int_{F(\gamma)} \frac{1}{\omega} d\omega,$$

where

$$\omega = F(z) = 1 + \frac{g(z)}{f(z)}.$$

Now,

$$|f| > |g|$$

implies that

$$|g/f| < 1 \quad \text{or} \quad |g/f + 1 - 1| < 1.$$

Hence, $F(\gamma) \subset B(1;1)$ and the function $1/\omega$ is analytic in $B(1;1)$. Therefore, the last integral is 0 and we have

$$Z_\gamma(f) = Z_\gamma(f+g).$$

□

The above theorem is known as Rouché's Theorem. If we let $f(z) = a_n z^n$, with $a_n \neq 0$ and $g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$, we observe that for sufficiently large $R > 0$, we have

$$|f| > |g|$$

for all $|z| \geq R$. In particular, if $\gamma = C(0;R)$, $|f(z)| > |g(z)|$ for $z \in C(0;R)$. Hence,

$$Z_\gamma(f) = Z_\gamma(f+g).$$

This implies that the non-trivial polynomial $P(z) = a_n z^n + \cdots + a_0$ has exactly n-zeroes in $B(0;R)$ and conclude another proof of the Fundamental Theorem of Algebra.

8.3 Open mapping Theorem

In the beginning of this chapter, we define the winding number

$$n(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} d\zeta.$$

If we fix γ and treat $n(\gamma; z)$ as a function of z, then we observe that for $w \in \mathbf{C}$,

$$\begin{aligned} n(\gamma; z) - n(\gamma; w) &= \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} - \frac{1}{\zeta - w} d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{z - w}{(\zeta - z)(\zeta - w)} d\zeta. \end{aligned}$$

Hence
$$|n(\gamma;z)-n(\gamma;w)| \le \frac{1}{2\pi}L(\gamma)M|z-w|$$
where $L(\gamma)$ is the length of γ and M is the bound for
$$\frac{1}{|(\zeta-z)(\zeta-w)|}$$
on γ. The last expression is less than ϵ whenever $|z-w|<2\pi\epsilon/(L(\gamma)M)$. This shows that $n(\gamma;z)$ is a continuous function of z.

THEOREM 8.6 Suppose f is analytic at z_0 and $f(z)-w_0$ has a zero of order n at z_0. If $\epsilon>0$ is sufficiently small, then there exists a corresponding $\delta>0$ such that for all $a\in B(w_0;\delta)$ the equation $f(z)=a$ has exactly n roots in $B(z_0;\epsilon)$.

Proof
First, note that by hypothesis, $f(z_0)=w_0$. Choose $\epsilon>0$ so that $f(z)-w_0$ has only one zero, namely, $z=z_0$ in $\overline{B(z_0;\epsilon)}$. Let $\gamma=C(z_0;\epsilon)$ and let $\Gamma=f(\gamma)$. Note that
$$\frac{1}{2\pi i}\int_\gamma \frac{(f(\zeta)-w_0)'}{f(\zeta)-w_0}d\zeta = n.$$
But
$$\frac{1}{2\pi i}\int_\gamma \frac{(f(\zeta)-w_0)'}{(f(\zeta)-w_0)}d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-z_0} = n(\Gamma;w_0).$$
We have shown that $n(\Gamma;s)$ is a continuous function in s and hence, we know that for every $\epsilon'>0$, there exists a $\delta_{\epsilon'}>0$ such that
$$|n(\Gamma;s)-n(\Gamma;w_0)|<\epsilon'$$
for all $|s-w_0|<\delta_{\epsilon'}$. Choosing $\epsilon'=1$ and noting that $n(\Gamma;s)$ is an integer, we conclude that $n(\Gamma;s)=n(\Gamma;w_0)=n$ whenever $|s-w_0|<\delta$, with $\delta=\delta_1$. This translates to the statement that for all $s\in B(w_0;\delta)$, $f(z)-s$ has exactly n zeroes in $B(z_0,\epsilon)$.

□

As a corollary of the above, we have the open mapping theorem.

THEOREM 8.7 The image of an open set under a nonconstant analytic mapping is an open set.

Proof
Let U be an open set. We want to show that $f(U)$ is open. Let $w_0\in f(U)$. Then there exists a z_0 such that $f(z_0)=w_0$. If $f(z)-w_0$ has n zeroes in $B(z_0;\epsilon)$ for sufficiently small $\epsilon>0$. Then by the above theorem, there exists a $\delta>0$ such that for all $s\in B(w_0;\delta)$ $f(z)-s$ has n zeroes in $B(z_0;\epsilon)$. In other words,
$$B(w_0;\delta)\subset f(B(z_0;\epsilon))\subset f(U).$$
This implies that $f(U)$ is open. □

We can prove maximum modulus theorem for closed balls directly from the open mapping theorem. Let $f(z)$ be analytic on $B(z_0;r)$ and continuous on $\overline{B(z_0;r)}$. Suppose $|f(a)|$ is maximum for $a \in B(z_0;r)$. Now, $f(a) \in f(B(z_0;r)$, which is open by open opening theorem. This means that $B(f(a);s) \subset f(B(z_0;r))$ for some $s > 0$. This means that there exists a point ξ on the boundary of $B(f(a);s)$ such that $\xi = f(b)$ and $|f(b)| > |f(a)|$, which is a contradiction. (One can choose $\xi = f(a) + se^{i\theta}$ where $\theta = \arg f(a)$.)

It is possible to prove open mapping theorem using maximum modulus theorem. The proof is due to Carathéodory.