Curriculum Burst 48: Lengths in a Trapezoid

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In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$, $AB < CD$ and $AD = 7$. What is $AB \cdot CD$?

THE QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPIC

Geometry

COMMON CORE STATE STANDARDS

7.G.B6 Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

8. G.B Understand and apply the Pythagorean Theorem.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING**

SOURCE

This is question # 24 from the 2001 MAA AMC 10 Competition.

THE PROBLEM-SOLVING PROCESS:

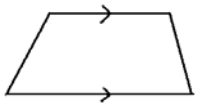
The appropriate start, as always, is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question feels hard to take in. It's about a trapezoid, and I know what that is - a four-sided figure with two sides parallel. But there are all these conditions about line segments. My suspicion is that this question will feel like a standard geometry problem from a textbook, once I can sort out what the question is!

I need to get started. It is clear I need to draw a picture.

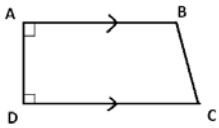
Now "$ABCD$ *is a trapezoid …*" means we have a trapezoid with vertices labeled A, B, C and D in a consistent clockwise or counterclockwise order. Here's a trapezoid:



but I am going to hold off labeling its vertices until I take in the information of the question.

"$\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$"

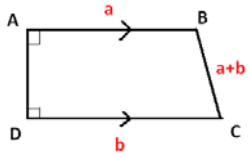
Given the order $ABCD$, the three line segments mentioned here really are sides of the trapezoid (not diagonals or something strange). So I need two sides to be perpendicular to another side. I guess the picture has to be something like:



"…with $AB + CD = BC$, $AB < CD$ and $AD = 7$."

This looks confusing! Let me go through these statements slowly.

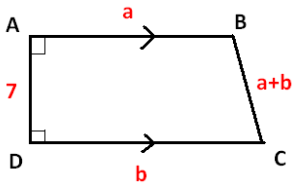
$AB + CD = BC$. Label the sides $\overline{AB}$ and $\overline{CD}$ as a and b. This is easier to read. Then we have:



(My picture is out of proportion. No matter.)

$AB < CD$: I lucked out. My picture has $a < b$.

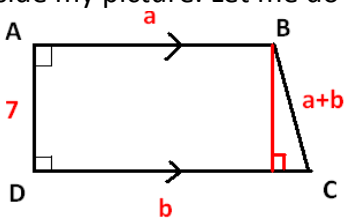
$AD = 7$: That's fine.



Okay … I've done something. What's next?

"What is $AB \cdot CD$?"

Hmm. In my picture this means we are being asked for the value of product ab. I have no idea how to get this value. Hmm again. I do feel compelled to draw a vertical line inside my picture. Let me do that.



Carrying on with my "do something" instinct I can't help but notice the right triangle. It has height 7, hypotenuse $a+b$, and base $b-a$. (Do see that?) The Pythagorean Theorem says:

$$49 + (b-a)^2 = (a+b)^2.$$

What next? It feels compelling to work on this equation!

$$49 + a^2 - 2ab + b^2 = a^2 + 2ab + b^2$$

$$49 = 4ab$$

Whao! $ab = 49/4$. This is the answer to the question!

Extension: Is it possible to determine a specific value for a (and hence b)? If not, find the range of possible values for a? (Could a be close to zero? Could a equal a billion?)

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