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Alternating Graphs*

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In this paper we generalize the concept of alternating knots to alternating graphs and show that every abstract graph has a spatial embedding that is alternating. We also prove that every spatial graph is a subgraph of an alternating graph. We define n-composition for spatial graphs and generalize the results of Menasco on alternating knots to show that an alternating graph is n-composite for $n = 0, 1, 2, 3$ if and only if it is "obviously n-composite" in any alternating projection. Moreover, no closed incompressible pairwise incompressible surface exists in the complement of an alternating graph. We then generalize results of Kauffman, Murasugi, and Thistlethwaite to prove that the crossing number of an even-valent rigid-vertex alternating spatial graph is realized in every reduced alternating projection with no uncrossed cycles and, if the graph is not 2-composite, the crossing number is not realized in any non-alternating projection. We give examples showing that this result does not hold for graphs with vertices of odd valence or graphs with uncrossed cycles. © 1999 Academic Press

Key Words: spatial graph; alternating graph; crossing number; incompressible surface.

1. INTRODUCTION

Over the years, knot theorists have devoted much attention to the classification of links and the study of the properties of those classifications. One particularly interesting category of links is the alternating links.

A projection of a link is alternating if as we travel along each component, the crossings alternate between over and under. A link is alternating if it has an alternating projection. It is a straightforward exercise to show that any projection of a link can be made alternating by changing crossings.

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In 1984 Menasco [5] proved that an alternating link is composite if and only if in the projection plane of its alternating projection there exists a disk with boundary punctured twice by the link separating two non-trivial segments of the link. In other words, an alternating link is composite if and only if it is "obviously" composite in its alternating projections. Menasco also showed that no closed incompressible pairwise incompressible surface exists in the complement of an alternating link.

In 1986, Murasugi [3], Kauffman [7], and Thistlethwaite [8] independently proved that the minimal crossing number of an alternating link is realized in every reduced alternating projection and, if the link is prime, nowhere else. These results show that a great deal of information about the properties of an alternating link is revealed in that link's alternating projection, making alternating links a very interesting set to study.

In this paper we extend the concept of alternating to spatial graphs. By a spatial graph, we mean a particular choice of an embedding of an abstract graph in the 3-sphere $S^3 = R^3 \cup \{\infty\}$, defined up to ambient isotopy. We say that a projection of a spatial graph is *alternating* if the following hold:

(1) As we travel along an edge, crossings alternate between over and under.

(2) Let W be the union of vertices and uncrossed edges of G. Let N be a regular neighborhood around W in the projection plane. Then as we travel around the boundary of N (denoted ∂N) in either direction, the first crossing between adjacent edges leaving the neighborhood will alternate between over and under.

A spatial graph G is *alternating* if it has some alternating projection $\pi(G)\colon S^3 \to S^2$, where $S^2 = R^2 \cup \{\infty\}$.

In Section 2 we discuss the relationship between alternating graphs and all spatial graphs. Specifically, we show that every abstract graph has a spatial embedding that is alternating. In addition, we show that every projection of any spatial graph is a subprojection of an alternating projection of a spatial graph. This shows that every spatial graph is a subgraph of an alternating graph. Although it is not true, as it is for knots, that any projection of any spatial graph can be made alternating by crossing changes, we do show that every spatial graph has some projection that can be made alternating by crossing changes. Moreover, every projection of every spatial graph with no odd-valent vertices can be made alternating by crossing changes. We also offer a regional definition of alternating for spatial graphs and show that this definition is equivalent to the one given above.

In Section 3 we generalize Menasco's work [5] on alternating knots to alternating graphs. We define n-composition for spatial graphs and show that an alternating graph is n-composite for $n = 0,1,2,3$ if and only if in any alternating projection there exists a disk with boundary punctured n times by the graph such that the boundary separates two non-trivial parts of the graph. Moreover, no closed incompressible pairwise incompressible surface exists in the complement of an alternating graph. In the case of an n-punctured surface with $n < 8$, we show that such an incompressible pairwise incompressible surface in the complement of an alternating graph has genus zero.

In Section 4 we generalize results of Kauffman [3], Murasugi [7], and Thistlethwaite [8] for alternating links to alternating graphs with no odd-valent vertices. We say that a projection of a spatial graph contains an uncrossed cycle if there is a disk in the projection plane bounded by uncrossed edges. Specifically, we show that the crossing number of an even-valent rigid-vertex alternating spatial graph is realized in every reduced alternating projection with no uncrossed cycles and, if the graph is not 2-composite, the crossing number is not realized in any non-alternating projection. A corollary to these results is that an alternating graph with no odd-valent vertices is nonplanar if it has a reduced alternating projection with no uncrossed cycles.

In Section 5 we discuss alternating projections containing uncrossed cycles. We offer a counterexample showing that the minimal crossing number result is false as stated for alternating graphs with odd-valent vertices. We leave as open the question of planarity in the case in which the graph contains odd-valent vertices. We show that the results from Section 4 hold for alternating projections that do not have certain kinds of uncrossed cycles, and offer counterexamples showing that the results are false as stated for alternating projections that do have certain kinds of uncrossed cycles. Other work has been done in [6] on crossing number of alternating spatial graphs. That paper offers a different definition of alternating graph than ours, and works in terms of the Yamada polynomial. We note that our definition in the even-valent case is more general and that Theorem 5.1 case (i) shows that in the even-valent case our results on crossing number hold for the definition of alternating presented in that paper.

2. CREATING ALTERNATING GRAPHS

In this section we show that a projection $\pi(G)$ of a spatial graph is alternating if and only if each region of $\pi(G)$ is alternating. We then show that there are several ways of adding edges to make a projection alternating if it is not so already. A direct corollary of this is that every connected link of n components is a sublink of an alternating link with $n+1$ components. We finish with a proof that every spatial graph has some projection that can be made alternating by changing a set of its crossings to crossings of the other type, and hence that every abstract graph has some embedding which is alternating.

Given a spatial graph G and a projection $\pi(G)$ of G onto S^2, define a *region* R of $\pi(G)$ to be a connected component of $S^2 - \pi(G)$ with $\partial R \subseteq \pi(G)$. Given a region R of $\pi(G)$ define a *segment* E of ∂R to be a connected subset of ∂R whose interior intersects no crossings of $\pi(G)$ and such that $\partial E = A$ where A is a non-empty set of crossings of $\pi(G)$. Note that a given region R need not have any segments on its boundary, as occurs when a region is bounded by a cycle of uncrossed edges.

An *alternating segment* is one with with an overcrossing at one end and an undercrossing at the other. A *nonalternating* segment is a segment which is not an alternating segment. There are two types of nonalternating segments, *under segments* which have undercrossings at both endpoints and *over segments* which have overcrossings at both endpoints.

An *alternating region* is a region with no nonalternating segments. A *nonalternating* region is a region which is not an alternating region.

Adjacent crossings C_1 and C_2 of a projection $\pi(G)$ are crossings which occur as the two endpoints of some segment of some region of $\pi(G)$. It is obvious that two crossings are adjacent if and only if they are separated only by a connected portion of an edge or are first crossings on adjacent edges coming out of the same neighborhood N of the union of vertices and uncrossed edges.

LEMMA 2.1. *A projection $\pi(G)$ of a spatial graph is alternating if and only if each region of $\pi(G)$ is alternating.*

Proof. Suppose that every region of $\pi(G)$ is alternating. Then adjacent crossings of $\pi(G)$ must alternate. Since this means that pairs of successive crossings along an edge must alternate and that the first crossings between adjacent edges as we travel around the boundary of the neighborhood of the vertices and uncrossed edges must also alternate, the two conditions for $\pi(G)$ to be alternating are fullfilled. Suppose that $\pi(G)$ is alternating. Then adjacent crossings must alternate and hence every segment is alternating. Therefore every region of $\pi(G)$ is alternating.

THEOREM 2.2. *Every projection $\pi(G)$ of a spatial graph can be made into an alternating projection $\pi'(G)$ of a spatial graph by*:

(a) *Adding vertices and uncrossed edges to $\pi(G)$*

or

(b) *Adding crossed edges to $\pi(G)$.*

We will prove this shortly but first we note the following corollary.

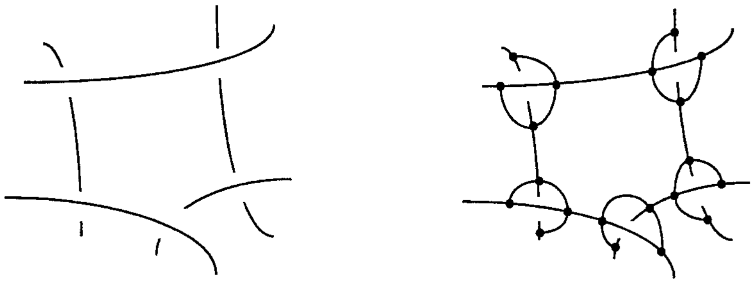

FIG. 1. Isolating a crossing in a nonalternating region.

COROLLARY 2.3. *Every spatial graph G is a subgraph of an alternating spatial graph G′.*

Proof. Consider any projection $\pi(G)$ of G. By Theorem 2.2, $\pi(G)$ is a subprojection of an alternating projection $\pi'(G)$ of a spatial graph G'. Thus G is a subgraph of the alternating spatial graph G'.

Proof of Theorem 2.2*a.* $\pi(G)$ will have a finite number of regions which are nonalternating. Create $\pi'(G)$ by isolating each crossing in each nonalternating region with two vertices and an edge as in Fig. 1. Once this has been done, each nonalternating region will have been divided into alternating subregions and by Theorem 1, $\pi'(G)$ will be alternating.

One can see that each region has an even number of nonalternating segments that alternate by crossing type as one traverses the boundary of the region (see Fig. 2). It will be useful to define a *problem segment* to be a nonalternating segment which contains a vertex V of valence greater than two. A *zone* Z of a projection $\pi(G)$ is a connected subset of the projection which is disconnected from the rest of the projection.

Furthermore if all the edges in $\pi(G)$ are crossed and there are no problem segments, each nonalternating segment is incident to exactly two nonalternating regions (see Fig. 3).

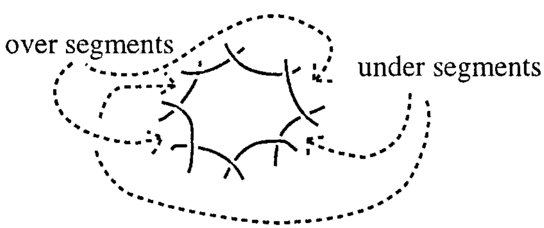

FIG. 2. A nonalternating region. Note that nonalternating segments alternate between under segments and over segments along its boundary.

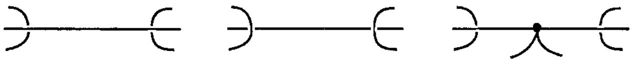

FIG. 3. The first two nonalternating segments are each incident to two nonalternating regions. Because of the vertex in the problem segment (i.e., the third diagram), we do not know how many nonalternating regions the segment is incident to.

LEMMA 2.4. *For any projection $\pi(G)$ with no problem segments there is a set of disjoint simple closed curves in the projection plane which together intersect each of the non-alternating segments exactly once, which are disjoint from the rest of $\pi(G)$ and such that as we travel along any one of them, the non-alternating segments that we intersect alternate between over and under segments.*

Proof. We will construct such a set of simple closed curves as follows: Create a path Q in the plane by beginning in any nonalternating region and exiting through some nonalternating segment. We will then travel through the projection plane crossing only nonalternating segments and never crossing any one nonalternating segment more than once. Each region that we enter will have at least one nonalternating segment on its boundary. Exit through the rightmost nonalternating segment on the boundary that has not previously been crossed. Note that if we enter a region through an over(under) segment, we exit through an under(over) segment. This process must eventually return us to the region from whence we came. Thus, Q can be made into a simple closed curve. We then create the other simple closed curves in the same way. Note that the curves will never intersect anything (including each other) other than nonalternating segments, each of which will be intersected exactly once. Thus we have constructed a set of simple closed curves which intersects each of the non-alternating segments exactly once and which is disjoint from the rest of $\pi(G)$.

Proof of Theorem 2.2b. This proof takes the form of an algorithm for creating $\pi'(G)$ by adding edges to any arbitrary $\pi(G)$ to make it alternating.

Step 1: Match odd valence vertices in pairs and then connect each pair up with a single edge E in the projection plane. The edge E can cross the original edges in $\pi(G)$ and the crossings on E can be chosen arbitrarily. As there must be an even number of vertices of odd valence this will eliminate all the vertices of odd valence.

Step 2: Cross all the uncrossed edges with a single edge E' that originates and terminates at a single vertex. As in Step 1, all the crossings that are added in this step can be chosen arbitrarily.

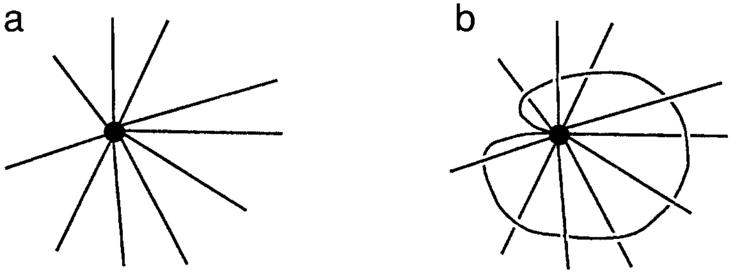

FIG. 4. The added edge (whose crossings alternate) will eliminate all problem segments incident to this vertex.

Step 3: Eliminate all the problem segments incident to each individual vertex V with problem segments by adding a single alternating edge E'' to V as shown in Fig. 4. As V is even valence, the crossings of E'' can always be chosen to be alternating.

Step 4: Divide sets of nonalternating regions into alternating subregions one zone Z at a time. This is accomplished by first picking some vertex V in Z and some region R incident to V. An edge C is then created which originates at V heading into R. Let $\{C_i\}$ be a disjoint set of simple closed curves which intersects each of the nonalternating segements in Z exactly once and which is disjoint from the rest of $\pi(G)$ (as in Lemma 2.4). C will then travel from V to one of the curves, C_1 in a path Q which alternates crossing type. It will then follow C_1 all the way around, adding crossings of the correct type to divide the nonalternating segments that C_1 passes through into alternating subsegments. C will then return to R along a path parallel to Q but traversed in the opposite direction, with the opposite crossings. This process is repeated for each curve in $\{C_i\}$ (as in Fig. 5). If this is repeated for each zone, $\pi'(G)$, the resultant graph, will

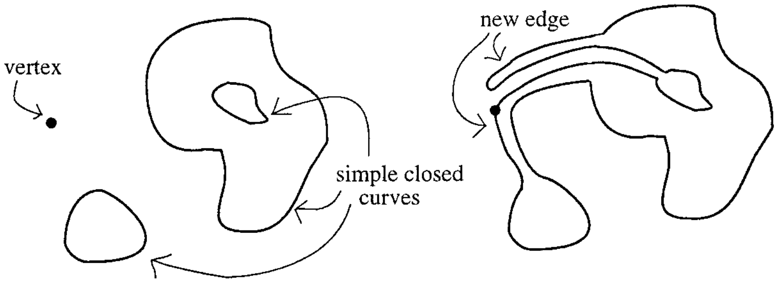

FIG. 5. An edge which traverses the plane in such a way as to divide each nonalternating segment into alternating subsegments.

FIG. 6. Splitting a vertex to make a $n+1$ component alternating link.

have only alternating regions and hence be an alternating projection of a spatial graph.

Once Step 4 is complete, each region of $\pi'(G)$ will be alternating and hence by Lemma 2.1, $\pi'(G)$ will be alternating.

COROLLARY 2.5. *Every connected projection of a link L of n components is a subprojection of an alternating projection of a link L′ of at most $n+1$ components.*

Proof. If L is already alternating, $L' = L$ and has n components. If L is not alternating, we apply the algorithm described in the proof to Theorem 2.2b to L. However, as all the vertices in L are valence two and all the edges are crossed we need only perform Step 4 of the algorithm. Furthermore as L is connected, it only has one zone, so only a single edge will be added when making the alternating projection. Once this edge has been added, there will be a single vertex of valence four in L' that can be split (as in Fig. 6) to make L' into an alternating link of $n+1$ components.

Given any projection of a knot or link, it is straightforward to show that one can always change the crossings to make the projection alternating. The same is not true for graphs, as Fig. 7 illustrates. However, the following is true.

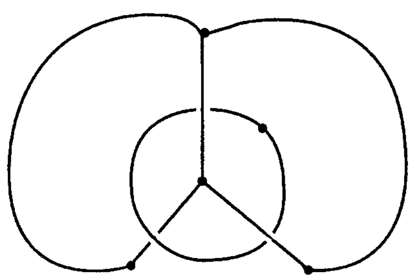

FIG. 7. This projection can never be made alternating by changing crossings.

THEOREM 2.6. *Every spatial graph has some projection $\pi(G)$ that can be made into an alternating projection $\pi'(G)$ by changing some set of its crossings to crossings of the other type.*

Proof. Choose a maximal tree T consisting of edges in the abstract graph. Because the tree is simply connected, the spatial realization of the abstract graph can be isotoped so that there exists a projection of the graph so that the entire tree projects to a set of edges with no crossings. Let $\pi(G)$ be such a projection. There are necessarily an even number of edge ends leading out of the maximal tree as each edge that is not in the maximal tree leaves it from two (not necessarily distinct) vertices. If we temporarily remove the tree and connect adjacent edges coming into a neighborhood of the tree in pairs so as to make a link, we can make that link alternating by simply changing some subset of its crossings. When we re-insert the maximal tree (removing the connections that we had put in), each region on our new graph will be alternating and hence by Lemma 2.1, our new graph $\pi'(G)$ is alternating.

If the graph has no even vertices the following theorem can be applied.

THEOREM 2.7. *Every projection $\pi(G)$ of a spatial graph with only even valence vertices can be made into an alternating projection $\pi'(G)$ by changing some set of its crossings to crossings of the opposite type.*

Proof. $\pi(G)$ can be arbitrarily changed into the projection of a four valent graph via edge expansion as in Fig. 8a. Once this has been done, if every vertex in $\pi(G)$ is replaced with an arbitrary crossing, we will have a link. As every link can be made alternating by changing some set of its crossings, there exists such a choice of crossings for this link. Neither changing a crossing to a vertex nor edge contraction (Fig. 8b) can make a

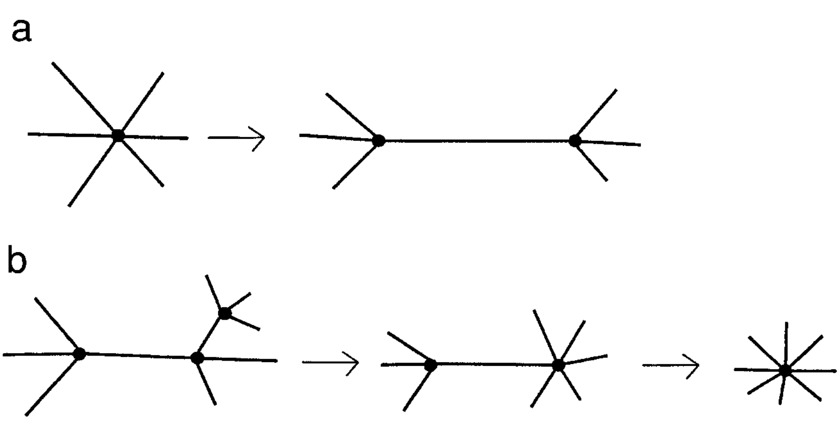

FIG. 8. (a) Edge expansion. (b) Edge contraction.

projection non-alternating. Thus when we replace all the crossings that used to be vertices with vertices and then contract edges until we have the original nodes and edges that we had in $\pi(G)$, we will have an alternating projection, $\pi'(G)$. Thus $\pi(G)$ will have been made into an alternating projection $\pi'(G)$ by changing some set of its crossings.

COROLLARY 2.8. *Every abstract graph has an alternating embedding.*

Proof. This follows directly from Theorem 2.6.

3. INCOMPRESSIBLE SURFACES IN GRAPH COMPLEMENTS

A link is defined to be composite if there is some twice-punctured sphere $F \subset S^3 - L$ separating L into two non-trivial components. Menasco [5] showed that an alternating link is composite if and only if it is obviously composite in its alternating projection, i.e, there is some disc D in the alternating projection $\pi(L)$ such that ∂D meets $\pi(L)$ transversely in two non-double points and $D \cap \pi(L)$ is not an embedded arc. Menasco also showed that L is splittable if and only if $\pi(L)$ is not connected. For spatial graphs we generalize the knot-theoretical definition of composite as follows: Given a spatial graph G and ball B in S^3, we say that $G \cap int(B)$ is non-trivial if it contains a vertex of valence three or greater or $G \cap int(B)$ cannot be isotoped to lie flat on ∂B while fixing its boundary points.

Let n be an integer greater than or equal to zero. A spatial graph G is *n-composite* if there is a sphere $F \subset S^3$ punctured n times transversely by edges of G such that for either ball B bounded by F, $G \cap int(B)$ is non-trivial, and there is no such m-punctured sphere for $m < n$.

THEOREM 3.1. *Let G be an n-composite alternating graph with reduced alternating projection $\pi(G)$, with $n < 4$. Then there is a disc $D \subset S^2$ such that ∂D meets edges of $\pi(G)$ transversely in n non-double points and $D \cap \pi(G)$ is non-trivial.*

Theorem 3.1 implies that an alternating graph is 0-, 1-, 2-, or 3-composite if and only if it is obviously so in its alternating projection. This is a direct generalization of Menasco's result for alternating links. (Menasco's results correspond to the case of two-valent graphs.)

In the case that there is a disc D as above, ∂D is an equator of the desired sphere. The following is a proof of the existence of such a disc in the case that G is known to be n-composite. The reader will find it helpful to reference [5].

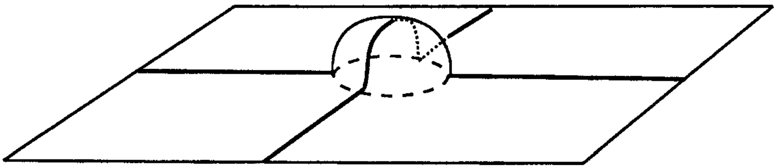

FIG. 9. A portion of S_+ with a crossing.

Following Menasco's [5] approach, we position G so that it lies on S^2 except near crossings of G, where G lies on a bubble.

Let $S_+(S_-)$ be S^2 with each disk inside a bubble replaced by the upper (lower) hemisphere of that bubble (Fig. 9). Let B_+ be the ball in S^3 bounded by S_+ and lying above S_+, and let B_- be the ball below S_- with $\partial B_- = S_-$. We will also use the notation $S_\pm$ to mean S_+ or S_- and similarly for other symbols with subscript $\pm$.

Let $F \subset S^3 - G$ be a surface without boundary that is punctured a finite number of times by edges of G. We may isotope F so that it meets the balls inside bubbles in saddle-shaped discs, and so that no punctures occur on bubbles (Fig. 10). Obviously each bubble in $S_\pm$ must be intersected an even number of times (twice for each saddle) by the components C_i of $F \cap S_\pm$. We may suppose F meets S_+ and S_- transversely. To each component C of $F \cap S_\pm$ we associate a cyclic word $w_\pm(C)$ in the letters P (puncture) and S (saddle), which records, in order, the intersections of C with G and with the bubbles, respectively. Strictly, $w_\pm(C)$ depends on an orientation for C. The number of punctures of F equals the total number of P's in all the $w_+(C)$'s.

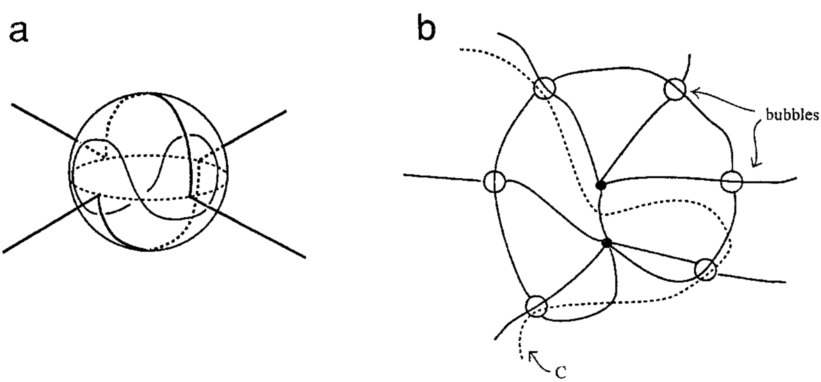

FIG. 10. (a) A saddle inside a bubble. (b) A portion of an alternating graph with an intersection curve, $w_+(C) = ...SPSPPS....$

We define a surface F to be in standard position if it satisfies the following three conditions:

(i) No word $w_+(C)$ associated to F is empty.

(ii) No curve intersects a bubble in more than one arc.

(iii) F intersects bubbles in saddles.

If F satisfies one of the following three conditions then F can be put into standard position by an argument similar to that in [5].

(1) $\pi(G)$ is connected and F is a 2-sphere not bounding a ball in $S^3 - G$;

(2) $S^3 - G$ is irreducible (contains no sphere as in (1)), and F is a sphere with at most three punctures separating G into two non-trivial components;

(3) $S^3 - G$ is irreducible, and F is incompressible and pairwise incompressible.

From this point on we assume that F is in standard position.

Note that we can restate Theorem 3.1 in terms of $w_\pm(C)$: Let G be an n-composite alternating graph, $n < 4$, and let F be some n-punctured sphere in $S^3 - G$ separating G into two non-trivial components, then $F \cap S_\pm$ is a simple curve C such that $w_\pm(C) = P^n$.

To prove Theorem 3.1 we will use one lemma, essentially identical in statement to Lemma 2 in Menasco's [5].

LEMMA 3.2. *There is no curve C in* $F \cap S_\pm$ *such that* $w_\pm(C) = P^i S^j$ *with* $j > 0$.

Lemma 3.2 follows from the proof of Lemma 2 in Menasco's [5] once it is noted that where Menasco uses the "alternating property," we can use the fact that saddles must alternate if there are no punctures between them. In the case of curves whose word is of the form $P^i S^j$, $j > 0$, these two properties are equivalent. Because of this property that extends from alternating links to alternating graphs, if there is a curve of the stated form in $w_+(C)$, then there must be a curve in $w_-(C)$ that intersects the same bubble twice, contradicting the fact that F is in standard position.

Proof of Theorem 3.1. We divide our proof into cases according to the type of composition of G.

Case 0. G is 0-composite (splittable). Assume $\pi(G)$ is connected (F can be put into standard position). For any closed curve C in $F \cap S_+$, $w_+(C) = S^i$. By Lemma 3.2, $i = 0$. So $F \cap S_+$ is empty. But then F bounds a ball in B_+ or B_-, contradicting the hypothesis that F separates G into two non-trivial components. It follows that $\pi(G)$ is not connected.

Case 1. G is 1-composite. Then $F \cap S_+ = C, w_+(C) = PS^i$. By Lemma 3.2, $i=0$. So $w_+(C) = P$.

Case 2. G is 2-composite. In $F \cap S_+{}^2$ there must be either two curves each with one puncture or one curve with two punctures. Suppose there are two curves. Then each is of the form PS^i. But then $i=0$ (Lemma 3.2) and G is 1-composite. So $F \cap S_+$ has one curve C with two punctures. Then any other curve, C' must intersect only bubbles, contradicting Lemma 3.2. So C is the only curve in $F \cap S_+$. Suppose that C intersects some bubble. Then there must be another curve which intersects the other side of the bubble. But no such curve exists. Hence, $w_+(C) = P^2$.

Case 3. G is 3-composite. If $F \cap S_+{}^2$ consists of multiple curves, then by an argument similar to that in Case 2, G would be 1-composite.

So $F \cap S_+$ has one curve C with three punctures. By the argument in Case 2, $w_+(C) = P^3$.

In the case of a four-composite graph the result does not extend, as there could be two curves in $F \cap S_+$, each with the word $PSPS$.

Menasco [5] also proves some results for incompressible, pairwise incompressible surfaces in the complements of alternating links. It is natural to consider such surfaces in the complements of alternating graphs, and we generalize some of Menasco's results in the following theorems.

THEOREM 3.3. *There is no closed incompressible, pairwise incompressible surface in the complement of an alternating graph.*

Proof. Let F be such a surface, and C a curve in $F \cap S_+$. Because F is closed, $w_+(C) = S^j$. By Lemma 3.2, $j=0$, contradicting the hypothesis that F is incompressible.

We use the following lemma in the proof of Theorem 3.5.

LEMMA 3.4. *Let F be a surface in the complement of an alternating graph G, and C an innermost curve on S_+ in $F \cap S_+$, then $w_+(C) \neq \ldots SS\ldots$, i.e., no innermost curve intersects two saddles without a puncture in between.*

Proof. Let C be such a curve. C bounds a disc D in $S_+ - F$. Because the two consecutive saddles must alternate left and right, the overstrand on one of the corresponding bubbles B must be inside D. Then some other curve C' must intersect the other side of B, contradicting the hypothesis that C is innermost.

THEOREM 3.5. *Given an alternating spatial graph G and an incompressible, pairwise incompressible surface $F \subset S - G$ with at most seven punctures, then $\bar{F}$ is a sphere.*

Proof. Because F is in standard position, each curve C in $S_\pm$ bounds a disc on F in $B_\pm$. So $\pi((F \cap S_+) \cup (F \cap S_-)$ yields a triangulation of F from which we can calculate the Euler characteristic of $\bar{F}$. It is $\chi(\bar{F}) = n_- + n_+ - n_s$, where $n_\pm$ is the number of components of $F \cap S_\pm$, and n_s is the number of saddles in F.

The cases in which F has less than four punctures are dealt with in the proof of Theorem 3.1, and the cases in which F has no saddles are straightforward. The remaining cases are proved as follows:

Case 4. F has exactly four punctures. By Lemmas 3.2 and 3.4 we have that $F \cap S_+ = C_1 \cup C_2$, $w_+(C_1) = w_+(C_2) = PSPS$. $\chi(\bar{F}) = 2 + 2 - 2 = 2$.

Case 5. F has exactly five punctures. We consider only the case in which $F \cap S_+$ contains two curves, one with three punctures and one with two. By Lemma 3.2 there can be no other curves in $F \cap S_+$, so both curves are innermost. Because of Lemma 3.2 and the fact that no innermost curve intersects more saddles then punctures, each word must contain exactly two S's. $\chi(\bar{F}) = 2 + 2 - 2 = 2$.

Case 6. F has exactly six punctures. We consider three subcases: (1) $F \cap S_+$ has two curves, one with four punctures and one with two. (2) $F \cap S_+$ has two curves, each with three punctures. (3) $F \cap S_+$ has three curves, each with two punctures. In all three subcases, by Lemma 3.2 $F \cap S_+$ has no other curves. In (1) both curves are innermost and both curves intersect exactly two saddles. So $\chi(\bar{F}) = 2 + 2 - 2 = 2$. In (2), either both curves intersect two saddles or both curves intersect three saddles. If both curves intersect two saddles then $\chi(\bar{F}) = 2 + 2 - 2 = 2$. Suppose both curves intersect three saddles and $F \cap S_-$ also contains exactly two curves. Then $\chi(\bar{F})$ would be odd. The same contradiction occurs if both $F \cap S_+$ and $F \cap S_-$ contain three curves each intersecting two saddles. Since these are the only two cases for six boundary components in which F contains three saddles, we conclude that if $F \cap S_+$ is of the form (2) then $F \cap S_-$ is of the form (3) and conversely, in which case $\chi(\bar{F}) = 3 + 2 - 3 = 2$.

Case 7. F has exactly seven punctures. We consider three subcases: (1) $F \cap S_+$ has two curves, one with five punctures and one with two. (2) $F \cap S_+$ has two curves, one with four punctures and one with three. (3) $F \cap S_+$ has exactly three curves, two with two punctures and one with three. In (1) $\chi(\bar{F}) = 2 + 2 - 2 = 2$. Subcases (2) and (3) are complementary in the same way as (2) and (3) above in which case $\chi(\bar{F}) = 3 + 2 - 3 = 2$.

For eight punctures we have a case with two curves, each with four punctures and each intersecting four saddles. In this case $\bar{F}$ is a torus. In general, for $n \geqslant 4$, the genus of $\bar{F}$ is at most $(n/4) - 1$.

People have taken the Menasco results generalized in Theorem 3.3 and extended them to two larger classes of links, namely almost alternating and toroidally alternating links. These extended results also hold for almost alternating and toroidally alternating graphs as defined below.

A connected graph, G, is *almost alternating* if there exists a projection $\pi(G)$ such that if one crossing is changed $\pi(G)$ is alternating.

Let T be a torus embedded in the 3-sphere S^3. Let G be a spatial graph that can be isotoped into a neighborhood $T \times I$ of T. Suppose that if $T \times I$ is retracted onto T, G projects to a connected graph on T such that the following hold:

(1) As we traverse an edge, crossings alternate between over and under when viewed from one side of T.

(2) Let W be the union of vertices and uncrossed edges of G. Let N be a regular neighborhood around W. Then as we traverse ∂N in either direction the first crossing between adjacent edges leaving the neighborhood will alternate between over and under.

In addition, assume that every nontrivial closed curve on T intersects the projection of G onto T. Then G is said to be *toroidally alternating with respect to T*.

THEOREM 3.6. *Let G be either an almost alternating graph or a toroidally alternating graph with respect to the standardly embedded torus in S^3. Then $S^3 - G$ contains no closed incompressible pairwise incompressible surfaces.*

Proof. The argument is a straightforward extension of the proofs found in [1, 2].

There is a coresponding result for toroidally alternating knots in more general 3-manifolds that appears in [1, Thm 3.1]. This result also extends to toroidally alternating graphs.

4. CROSSING NUMBER OF RIGID-VERTEX GRAPHS

A spatial graph G is called a *rigid-vertex graph* if for each vertex v of G there exists a neighborhood B_v of v and a small flat plane P_v such that $G \cap B_v \subset P_v$. For two rigid-vertex graphs G and G', G and G′ are *ambient isotopic as rigid-vertex graphs* if there exists an isotopy $h_t\colon R^3 \to R^3$,

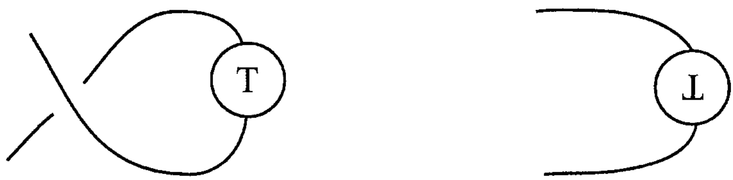

FIG. 11. A nugatory crossing.

$t \in [0, 1]$ such that $h_0 = \text{id}$, $h_1(G) = G'$, and $h_t(G)$ are rigid-vertex graphs for each $t \in [0, 1]$.

When we project a rigid-vertex graph, we will always assume that each of the planes P_v lies in the projection plane. We say that a projection of a spatial graph is *reduced* if it contains no nugatory crossings (see Fig. 11). From here on we assume that all projections are connected, since for a disconnected projection we can look at the connected components separately. For convenience, we also assume that there are no valence two vertices since such a vertex can be removed and the two incident edges can be replaced with a single edge without changing the topology.

In this section we show that the crossing number of an even-valent rigid-vertex alternating spatial graph is realized in every reduced alternating projection with no uncrossed cycles, and, if the graph is not 2-composite, the crossing number is not realized in any non-alternating projection. One consequence of this result is that an even-valent rigid-vertex alternating spatial graph is nonplanar if it has a reduced alternating projection with no uncrossed cycles.

Every region in the complement of an alternating projection of a graph except for those bounded entirely by uncrossed edges has an orientation given by travelling from the overcrossing to the undercrossing along any strand bordering the region. A graph is alternating if and only if all regions (including the outermost one) except those bounded entirely by uncrossed edges can be oriented in this way. This statement holds for links as well.

LEMMA 4.1. *Let G be an even-valent, rigid-vertex graph that is alternating with respect to the reduced projection $\pi(G)$. Assume that $\pi(G)$ contains no uncrossed cycles. Then $\pi(G)$ does not contain two adjacent regions with the same orientation separated by an uncrossed edge.*

Proof. Let e be an uncrossed edge in $\pi(G)$. Consider an ε-neighborhood N surrounding the maximal tree of uncrossed edges containing e. We wish to show that, starting on one side of ∂N adjacent to e and travelling around the ∂N in either direction to the other side of ∂N adjacent to e, there are an odd number of crossed edges coming out of ∂N. We argue inductively. Suppose that the maximal tree of uncrossed edges containing e consists of just e. Then, because G is even-valent, there must be an odd number of crossed edges coming out of the vertices at either end of e. As we travel in either direction around ∂N from one side of e to the other, we will pass through an odd number of crossed edges, each time entering a region with the reverse orientation from the previous one. Hence we will end in a region with the opposite orientation from the one in which we started.

Now suppose the result is true for maximal trees with n uncrossed edges. Then a maximal tree with $n+1$ uncrossed edges can be thought of as a maximal tree of n edges with one new edge added on. The new tree will have one fewer crossed edge at the vertex from which the new uncrossed edge originated, and an odd number of additional crossed edges at the vertex at which the new uncrossed edge terminates, increasing the number of crossed edges around the boundary of the ε-neighborhood N surrounding the maximal tree by an even number. Then as we travel in either direction around ∂N from one side of e to the other, we still pass through an odd number of crossed edges, and hence we still end in a region with the opposite orientation from the one in which we started.

A consequence of Lemma 4.1 is that any four-valent alternating graph can be made into an alternating link by replacing each vertex with a crossing so as to make the resulting projection alternating.

We can associate to an even-valent spatial graph G a four-valent spatial graph G' as follows: for each vertex $v \in V(G)$ of valence $2n$, $n > 2$, we expand v by creating two new vertices v_1 and v_2 connected by an uncrossed edge such that v_1 has valence 4, v_2 has valence $2n-2$, and the order of the edges coming into v is preserved around the neighborhood surrounding the uncrossed edge connecting v_1 and v_2. By our definition of alternating graph, G' is alternating if and only if G is alternating. This process of edge expansion clearly preserves planarity.

We define a geometric invariant of rigid vertex isotopy for an even-valent spatial graph G by the collection of four-valent spatial graphs, denoted $B(G)$, associated to G, obtained by repeated edge expansions used above to create a four-valent spatial graph from an even-valent spatial graph. Clearly $B(G)$ is an invariant of rigid vertex isotopy.

In [4] Kauffman defines an unoriented invariant of rigid vertex isotopy for a rigid-vertex, four-valent (denoted RV4) graph G by a collection of knots and links associated to G, denoted $C(G)$. An element of $C(G)$ is obtained by replacing each vertex locally by a configuration that connects the four edges in pairs. There are four ways to do this at any given vertex, as in Fig. 12.

Thus $C(B(G))$ is the Kauffman invariant outlined above applied to $B(G)$, the collection of four-valent graphs associated to G. Since $B(G)$ and

FIG. 12. The four ways of replacing each vertex to associate a collection of knots and links to an RV4 spatial graph under the Kauffman invariant.

$C(G)$ are both invariant, $C(B(G))$ is also an invariant of rigid vertex isotopy.

We treat as a special case the situation in which an alternating projection of G contains a disk intersecting the projection exactly once at a four-valent vertex, since the alternating link obtained from that projection will contain a nugatory crossing. If the valence of the vertex is greater than 4, there will always be a way to expand the vertex to avoid a nugatory crossing. Thus in the special case G must be two-composite.

Let v be the number of vertices of G and c be the number of crossings in some reduced projection $\alpha(G)$ of G. Let the valence of the ith vertex be $4+2(n_i)$, where i goes from 1 to v. Then every member of $B(G)$ will have $v+\sum_i n_i$ vertices, and each of the links in $C(B(G))$ will have at most $v+c+\sum_i n_i$ crossings. Moreover, if $\alpha(G)$ is alternating, then for each member of $B(G)$ there will be an alternating link $L_m \in C(B(G))$ with $v+c+\sum_i n_i$ crossings. Assuming we are not in the special case, if $\alpha(G)$ is reduced, L_m will also be reduced.

THEOREM 4.2. *Let G be an even-valent rigid-vertex spatial graph that is alternating with respect to the reduced projection $\pi(G)$. Assume that $\pi(G)$ contains no uncrossed cycles. Then the crossing number of G is realized in $\pi(G)$.*

Proof. First we assume that in $\pi(G)$ there does not exist a disk with its boundary intersecting the projection exactly once at a vertex and with parts of the graph both inside and outside the disk. Let v be the number of vertices of G and c be the number of crossings in $\pi(G)$. There exists in $C(B(G))$ an alternating link L with $v+c+\sum_i n_i$ crossings in its reduced alternating projection. Because we are not in the special case, L is reduced, and therefore there does not exist another projection of L with fewer than $v+c+\sum_i n_i$ crossings by [3, 7, 8].

Suppose there is some other projection $\gamma(G)$ of G with $k<c$ crossings. Then every four-valent graph $G' \in B(G)$ will have $v+\sum_i n_i$ vertices and k crossings, and thus every link $L' \in C(B(G))$ will have at most $v+k+\sum_i n_i$ crossings, contradicting the fact $C(B(G))$ is an invariant, and $L \in C(B(G))$. Thus there cannot exist another projection of G with fewer than c crossings.

Now suppose that there is one disk up to isotopy in the projection plane of $\pi(G)$ with boundary intersecting $\pi(G)$ once at a vertex and parts of the graph both inside and outside the disk. Let m be the number of crossings inside the disk and n be the number of crossings outside the disk. Then $\pi(G)$ has $n+m$ crossings, and the crossing number of G is at most $n+m$. We now consider the subgraph contained within the disk including the vertex and the subgraph outside the disk including the vertex. Because $\pi(G)$ is reduced and alternating, both of these subgraphs are also reduced and alternating, and therefore neither subgraph has a projection with fewer crossings. Therefore the crossing number of G is $n+m$. By induction this argument holds for any number of disks with boundary intersecting $\pi(G)$ once at a vertex.

We will need the following lemma in order to prove that non-alternating projections of prime alternating graphs do not have minimal crossing number.

LEMMA 4.3. *Let G be an even-valent rigid-vertex spatial graph that is not two-composite and is alternating with respect to the reduced projection $\pi(G)$. Assume that $\pi(G)$ contains no uncrossed cycles. Then either there exists a spatial graph $H \in B(G)$ such that H is not two-composite or G is planar.*

Proof. Assume that H does not exist. That is, assume that every method of edge expansion of G produces a two-composite graph. Starting in $\pi(G)$, we perform edge expansions of G one vertex at a time (because $B(G)$ is an invariant up to flat vertex isotopy, we can start in any projection of G). At each vertex, we avoid making our new graph two-composite whenever possible. But because every graph in $B(G)$ is two-composite, this process of edge expansion must eventually produce an alternating projection λ of some graph G_0, where G_0 is not two-composite and G_0 contains a vertex v_0 of valence n_0 such that every possible expansion of v_0 produces a two-composite graph. Because every possible edge expansion of v_0 creates a two-composite alternating graph, and every two-composite alternating graph is obviously so in any alternating projection, there must exist $n_0/2$ circles in $\lambda(G_0)$, each of which intersects one edge and v_0. Thus $\lambda(G_0)$ must look like the graph in Fig. 13a. But each of the tangles connected to v_0 by an edge must be a trivial arc, since a non-trivial tangle in an alternating projection is known to be non-trivially knotted, and if one of the tangles were non-trivially knotted the graph would be 2-composite, which is a contradiction. Thus G must have been a planar graph with a projection like the one in Fig. 13b. Therefore if every graph in $B(G)$ is two-composite then either G is two-composite or G is planar.

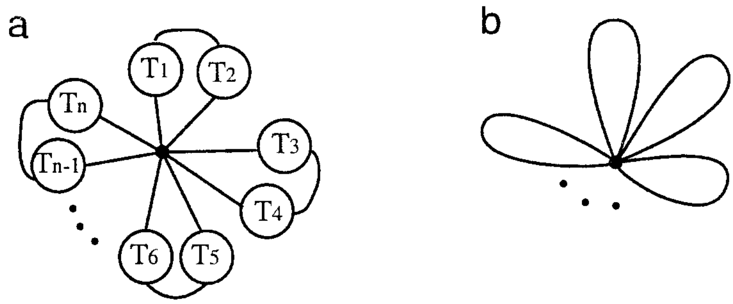

FIG. 13. (a) $\lambda(G_0)$. (b) A planar projection of G.

COROLLARY 4.4. *Let G be an even-valent rigid-vertex spatial graph that is not two-composite and is alternating with respect to the reduced projection $\pi(G)$. Assume that $\pi(G)$ contains no uncrossed cycles. Then every non-alternating projection of G has more crossings than $\pi(G)$.*

Proof. By Theorem 4.2 we know that no projection of G has fewer crossings than $\pi(G)$. So suppose $\alpha(G)$ is a projection of G with exactly c crossings. Assume $\alpha(G)$ is not alternating. Then if we construct $C(B(G))$ from $\alpha(G)$ we get a collection of links with $v+c+\sum_i n_i$ crossings and, because $\alpha(G)$ is not alternating, none of these links is alternating. But by Lemma 4.3, L is a reduced prime alternating link with $v+c+\sum_i n_i$ crossings, and there does not exist a non-alternating projection of L with $v+c+\sum_i n_i$ crossings. So L is not in $C(B(G))$. This is a contradiction. So $\alpha(G)$ must be alternating.

COROLLARY 4.5. *Let G be an even-valent spatial graph. Then if G has a reduced alternating projection with no uncrossed cycles, G is nonplanar.*

Proof. Because G is even-valent, any projection of G with no uncrossed cycles must have at least one crossing. Thus if G has a reduced alternating projection with no uncrossed cycles, then its crossing number must be at least 1.

In Section 5 we discuss alternating spatial graphs with some odd-valent vertices and uncrossed cycles.

5. COUNTEREXAMPLES AND OPEN QUESTIONS

In Section 4 we showed that the minimum crossing number of a rigid-vertex even-valent alternating graph is realized in its reduced alternating projections that contain no uncrossed cycles and, if the graph is not two-composite, the crossing number is not realized in any non-alternating projection. One corollary to this result is that a rigid-vertex even-valent alternating graph is nonplanar if it has a reduced alternating projection with no uncrossed cycles.

In this section we discuss rigid-vertex alternating spatial graphs with some odd-valent vertices and also rigid-vertex even-valent alternating spatial graphs whose reduced alternating projections contain uncrossed cycles. We offer a counterexample showing that Theorem 4.2 does not hold for alternating spatial graphs with odd-valent vertices. We also show that the results from Section 4 hold for alternating projections that do not contain certain kinds of uncrossed cycles, and offer a counterexample demonstrating that the result does not hold for alternating projections that do contain certain other kinds of uncrossed cycles.

Theorem 4.2 is false as stated for graphs with odd-valent vertices. Figure 14 shows a projection of a reduced alternating graph with a crossing that can be eliminated. However, we could classify the move that eliminates this crossing as a new reduction move. Thus this graph would no longer be a counterexample to Theorem 4.2 since it would not be reduced. Note that the graph in Figure 14 must have at least two odd-valent vertices. In fact, any graph that has a projection containing a disk with boundary intersecting the projection an odd number of times must have at least one odd-valent vertex inside the disk and at least one outside the disk.

Figure 15, however, shows a rigid-vertex alternating graph with no uncrossed cycles and odd-valent vertices with a crossing that can be eliminated by making the graph non-alternating. Thus, there is no possibility of repairing Theorem 4.2 for graphs with some odd-valent vertices by adding reduction moves that keep us in the category of alternating graphs.

Corollary 4.5 has not been disproved for graphs with odd-valent vertices. Figure 16 shows an alternating graph that can be made planar. However, the moves that make the graph planar can be done in an order that preserves the alternating property in the isotoped graph, and therefore can be classified as reduction moves.

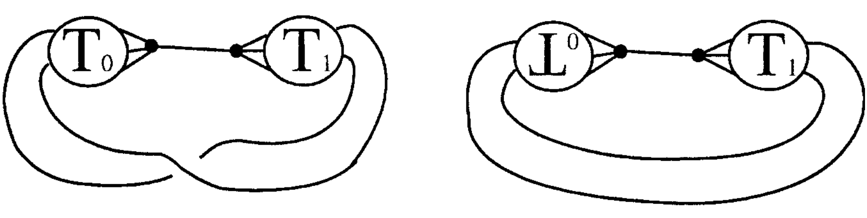

FIG. 14. An alternating projection of a spatial graph with odd-valent vertices that can be isotoped to eliminate a crossing.

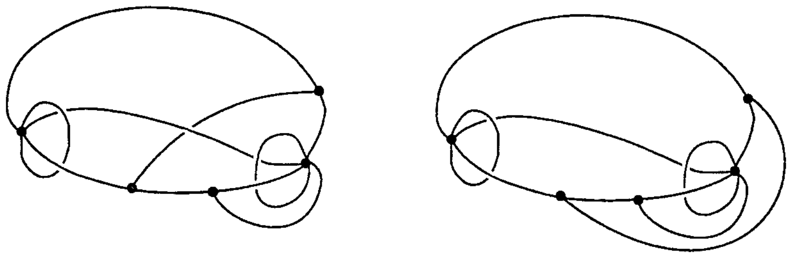

FIG. 15. A reduced alternating projection of a spatial graph with odd-valent vertices with five crossings, and a non-alternating projection of the same spatial graph with four crossings.

Conjecture. Let G be a rigid-vertex alternating spatial graph. If G has a "reduced" alternating projection and no uncrossed cycles, then G is nonplanar.

We leave as an open question whether Theorem 4.2 is true for graphs with some odd-valent vertices but no vertices of valence 3, noting that none of our counterexamples applies to this class of spatial graphs.

We now discuss alternating spatial graphs with uncrossed cycles.

Let G be an even-valent spatial graph with v vertices that is reduced and alternating with respect to the projection $\pi(G)$, and let $c(G)$ denote the crossing number of G. We say that an uncrossed edge e_1 in $\pi(G)$ is a *bad edge* if it separates two regions of the same orientation. An uncrossed edge e_2 in $\pi(G)$ is a *good edge* if it separates two regions of opposite orientation. By Lemma 4.1 if G contains a bad edge then that edge must be part of an uncrossed cycle of bad edges.

THEOREM 5.1. *Let G be an even-valent rigid-vertex spatial graph that is reduced and alternating with respect to the projection $\pi(G)$. Suppose that after some choice of orientation for the unoriented regions, either*:

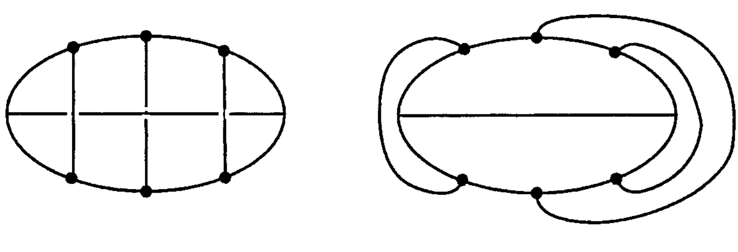

FIG. 16. An alternating projection of a spatial graph with odd-valent vertices that can be isotoped to eliminate a crossing.

(i) *there are no bad edges,*

(ii) *every vertex of a bad cycle has an edge* e_1 *going into the disk bounded by the cycle and an edge* e_2 *going out of the disk bounded by the cycle; moreover, in the projection we can travel from* e_1 *to some other vertex on the cycle and from* e_2 *to some other vertex outside the cycle, changing strands at crossings if necessary.*

Then the crossing number of G *is realized in* $\pi(G)$.

Proof. First let G be an RV4 spatial graph with v vertices that is reduced and alternating with respect to the projection $\pi(G)$. Let $\pi(G)$ have c crossings. If there are any unoriented regions in G (meaning they are bounded by an uncrossed cycle), we assign them an orientation so that the edges bounding the region are all good edges, if possible. Assume that all unoriented regions in G can be oriented in this way, and that G contains no bad edges. Then there will be at least one reduced alternating link $L \in K(G)$ with $v + c$ crossings (By Lemma 4.1 and our assumption that every uncrossed edge separates two regions of opposite orientation, L can be obtained from $\pi(G)$ by replacing each vertex with a crossing so as to make the resulting link alternating). Then the proof of Theorem 4.3 shows that the crossing number of G is c.

Now suppose that G is an even-valent rigid-vertex spatial graph with v vertices that is reduced and alternating with respect to the projection $\pi(G)$. Let the valence of the ith vertex be $4 + 2(n_i)$, where i goes from 1 to v. Suppose $\pi(G)$ has c crossings and contains uncrossed cycles but no bad edges after all unoriented regions of G have been assigned an orientation. Because edge expansion creates no new uncrossed cycles and does not change the orientation of regions, $B(G)$ will be a set of RV4 spatial graphs with no bad edges. By the above argument, there will be at least one reduced alternating link $L \in K(B(G))$ with $v + c + \sum_i n_i$ crossings. Then the proof of Theorem 4.3 shows that the crossing number of G is c.

Let G be an RV4 spatial graph with v vertices that is reduced and alternating with respect to the projection $\pi(G)$. Suppose that $\pi(G)$ has c crossings and contains uncrossed cycles. Also suppose that for every uncrossed cycle made up of bad edges, each vertex of the cycle has at least one edge e_1 going into the cycle and one edge e_2 leaving the cycle such that in the projection we can travel from e_1 to some other vertex of the cycle and from e_2 to some other vertex outside the cycle, changing strands at crossings if necessary.

Let G' be the subgraph obtained from G by removing the edges that are bad in $\pi(G)$. Because the edges we remove separate regions of the same orientation, $\pi(G')$ is alternating and all uncrossed edges in $\pi(G')$ are good edges. Because of our assumption about the edges coming out of the vertices of uncrossed cycles, $\pi(G')$ contains no nugatory crossings. By the above argument, $c(G') = c$. Because the crossing number of a subgraph is less than or equal to the crossing number of the original graph, $c \leqslant c(G)$. But $c(G)$ is less than or equal to the number of crossings in $\pi(G)$, so $c \leqslant c(G) \leqslant c$, so $c(G) = c$.

Now suppose that G is an even-valent rigid-vertex spatial graph with v vertices that is reduced and alternating with respect to the projection $\pi(G)$. Let the valence of the ith vertex be $4 + 2(n_i)$, where i goes from 1 to v. Suppose $\pi(G)$ has c crossings and contains uncrossed cycles such that for every uncrossed cycle made up of bad edges, each vertex of the cycle has at least one edge e_1 going into the cycle and one edge e_2 leaving the cycle such that we can travel in the projection from e_1 to some other vertex of the cycle and from e_2 to some other vertex outside the cycle, changing strands at crossings if necessary. Because edge expansion creates no new uncrossed cycles and does not change the orientation of regions, $B(G)$ will be a set of RV4 spatial graphs whose uncrossed cycles behave in the same manner as the uncrossed cycles in $\pi(G)$. By the above argument, $c(G) = c$.

Theorem 5.1 is false as stated for alternating projections with uncrossed cycles made of bad edges when there is at least one vertex on the cycle that does not have at least one edge e_1 going into the cycle and one edge e_2 leaving the cycle such that we can travel in the projection from e_1 to some other vertex of the cycle and from e_2 to some other vertex outside the cycle, changing strands at crossings if necessary. Figure 17 shows a four-valent spatial graph with such a cycle in an alternating projection. This projection has two crossings, but there is another alternating projection of this graph with one crossing. Furthermore, the isotopy from the first projection to the second cannot be classified as a reduction move, since such a move does not always preserve the alternating property (see, for example, Fig. 18).

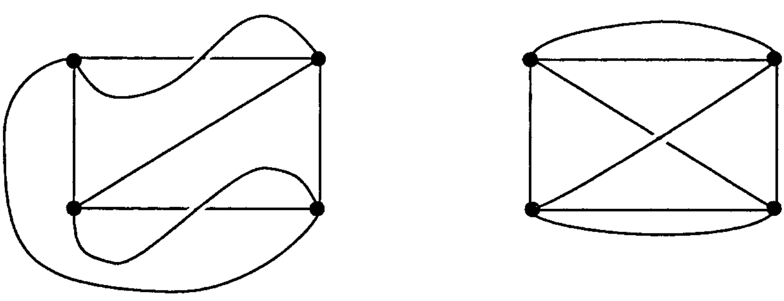

FIG. 17. A four-valent spatial graph with a reduced alternating projection with two crossings and a reduced alternating projection with one crossing.

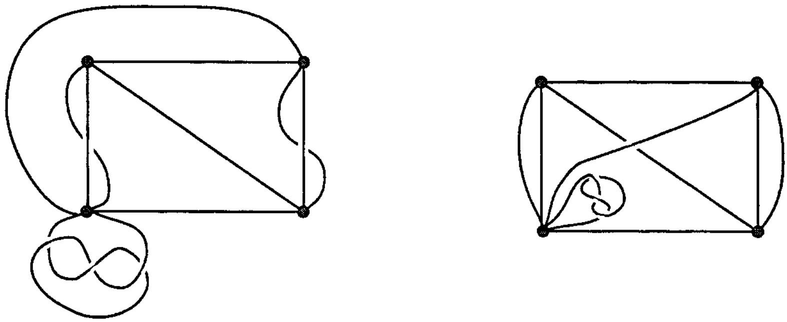

FIG. 18. The isotopy in Fig. 17 is not a reduction move since, as in this picture, it does not always preserve the alternating property.

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