ASSESSING ELEMENTARY STUDENTS' FUNCTIONAL THINKING SKILLS: THE CASE OF FUNCTION TABLES

Katherine L. McEldoon Bethany Rittle-Johnson
Vanderbilt University
katherine.l.mceldoon@vanderbilt.edu bethany.rittle-johnson@vanderbilt.edu

Functional thinking is an appropriate way to introduce algebraic concepts in elementary school. We have developed a framework for assessing and interpreting students' level of understanding of functional thinking using a construct modeling approach. An assessment was administered to 231 second- through sixth-grade students. We then developed a progression of functional thinking knowledge. This investigation elucidates the sequence of acquisition of functional thinking skills. This study also highlights the utility of a construct modeling approach, which was used to create criterion-referenced and ability-leveled assessment. This measure is particularly suited to measuring knowledge change and to evaluating instructional interventions.

Objectives

Research into the effectiveness of teaching and learning interventions has become increasingly more mainstream, particularly as the call for evidence-based research grows louder. Unfortunately, instruments that measure learning outcomes from focused interventions are often researcher-created and are not tested for reliability and validity. This makes it difficult to compare research results and generalize findings into instructional recommendations.

The goal of this project is twofold. The first is to develop an assessment of elementary-school student's functional thinking abilities, with a specific focus on students' ability to find a rule of correspondence within a function table. The second is to develop a model of knowledge progression of elementary-level functional thinking skills. We identified a set of skills important for elementary-level functional thinking, designed an assessment which tapped these skills, administered the assessment to students in Grades 2 to 6, and used a construct modeling approach (Wilson, 2005) to developed a construct map, or model of knowledge progression, based on the student performance data. The findings provide insight into the typical sequence in which learners acquire functional thinking skills. This proposed sequence of functional thinking skills, and the assessment, are useful tools for measuring student knowledge. Eventually, the assessment can be used to measure learning gains after instructional interventions.

Theoretical Perspective

The transition to algebra is a notoriously difficult one for many students. Our students must be better prepared in order to take on the challenges of an increasingly technical and complex world. Traditional mathematics instruction, which focuses on teaching arithmetic procedures, followed by a similarly procedure-based algebra instruction in middle and upper grades, has not been successful in supporting student learning (Blanton, 2005).

One way to overcome this difficulty is to inculcate appropriate forms of algebraic reasoning into early math instruction. Algebraic reasoning is defined as a type of reasoning in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways (Kaput, 1999).

Functional Thinking

Functional thinking is a particular kind of generalized thinking that lends directly to the development of algebraic thinking. It is a type of representational thinking that focuses on the relation between two varying quantities (Smith, 2008). Functional thinking is one of the core strands of Kaput's (2008) framework of algebraic reasoning and a core expectation for mathematics curriculum. For grades 3 through 5, students are expected to "describe, extend, and make generalizations about geometric and numeric patterns; represent and analyze patterns and functions, using words, tables, and graphs" (NCTM, 2008). At the heart of functional thinking is a relationship between two particular quantities; this can be referred to as a *rule of correspondence* (Blanton, 2005). This relationship can be used to find other sets of particular quantities that adhere to the same rule. The functional relationship binds together the set of numbers to which it applies.

Elementary-school students often focus on particular numbers as outcomes. In order to think in an algebraic way, and in a way which allows for generalization, one must understand that there are many possible outcome values. Finding a functional relationship between two sets of numbers is a way to jump from thinking of particulars to sets (Carrahar et al. 2008). This is an accessible way to get students thinking about numbers in a general way, and in a way which eases their transition to algebra. Thinking about functional relationships makes explicit the fact that many sets of values are possible for a given constraint (i.e., the rule) (Carrahar et al. 2008). This functional relationship can be described in terms of algebraic syntax, such a $y = 3x – 3$, and a prolonged consideration of functional relationships can scaffold a learner up to the logic behind such syntax (Carrahar et al. 2008).

Children's Functional Thinking Abilities

Exploring ways to support early functional thinking has been the focus of several recent teaching experiments, and commonalities across these studies indicate particular abilities that can be cultivated in elementary school.

Evidence from teaching experiments have shown that elementary-school students have the ability to understand the functional relation between X and Y values (Carraher, Schliemann, & Brizuela, 2003; Carraher, Martinez, & Schliemann, 2008), identify the rule or pattern (Carraher et al., 2003), use it to predict new values (Carraher et al., 2003, Carraher et al., 2008), and articulate the general rule verbally (Carraher et al., 2008; Cooper & Warren, 2008; Warren & Cooper, 2005; Warren & Cooper, 2006) and symbolically (Carraher et al., 2003; Carraher et al., 2008; Carraher & Earnest, 2003; Cooper & Warren, 2008; Warren & Cooper, 2006). In the teaching experiments, researchers typically use geometric or numeric patterns and introduce function tables as a representation for capturing functional relations. However, researchers have used their own assessments of student thinking and have not provided evidence for the validity of their assessments. In addition, little is known about the emergence of these skills in students in typical classrooms. In the current study, we focused on assessing students' knowledge of function tables, working with students in classrooms not receiving special interventions.

Assessment Development

We chose to focus on numeric patterns presented in function tables because they are a common and foundational component of functional thinking (Schliemann, Carraher & Brizuela, 2001). In an informal review of textbooks and national and state tests, we found function tables to be the most common problem format used at the elementary level. In particular, we focused on students' ability to identify rules of correspondence in function tables and use the rule to predict new instances.

A review of existing test items as well as a task analysis suggested at least 6 skills related to understanding function tables. (1) A precursor skill is to apply a given rule. When presented with a table with X and Y values, and a verbal or symbolic rule which describes how to compute the Y value from an X value, students should be able to use that rule to compute a Y value given a particular X value. (2) Students should also be able to recognize a function rule out of a selection of possible rules for a given table. In this case, students can test the rule against values in the table, rather than needing to generate the rule. (3) Students should be able to determine the next instance in a function table. However, students may be able to predict the next instance without thinking about the relation between the X and Y values (Schliemann et al., 2001). (4) Students should be able to extrapolate the function rule and articulate the function rule for a table in words. (5) Students should be able to predict a variety of instances in a function table, particularly instances that require identifying the function rule to make the prediction (e.g., predicting the 100th instance). (6) Students eventually learn to articulate the rule in the function table symbolically (Schliemann et al., 2001). Although there is a logical ordering of the relative difficulty of some of these skills (e.g., skill 1 is easier than skill 6), the relative difficulty of some skills could not be determined from previous research, and exploring the relative difficulty was one goal of the current research.

With these skills in mind, we created an assessment meant to tap each skill for working with functions presented in function tables. The functions were additive, multiplicative and two-operator functions. After data collection, we evaluated the progression of mastery of these skills, and used the data to inform the creation of a construct map (Wilson, 2005) of the development of functional thinking.

Method

The assessment was administered to a wide range of grade levels, as we expected differential performance and an increase in functional thinking skill through the grade levels. Participants were 231 2nd – 6th grade students attending two suburban schools. There were 52 second graders (24 girls), 50 third graders (30 girls), 25 fourth graders (15 girls), 60 fifth graders (28 girls), and 44 sixth graders (16 girls). Approximately 3% of the students were from minority groups. About 27% of students at the schools were eligible for free or reduced lunch.

The assessment contained 11 items, and was divided into three parts. Some of the items were broken into sub-items. The first section (one item) asked students to identify the number of eyes that a certain number of dogs would have, and to articulate the rule. It was designed to see to what extent students could engage in functional thinking given a supportive and grounded context without scaffolding from the problem formats that follow. The second section (six items) asked students to apply (two items) and recognize (four items) function rules. Of these four items, the function rules were presented as verbal statements (two items), and as algebraic equations (two items). The third section (four items) contained two problem types. Three of the items were function table problems, which required students to determine missing entries in the table, and then formulate both a verbal and symbolic rule. There was also a multiple-choice item asking students to identify the next value in a sequential numeric pattern.

The assessment was administered during a single class period. A member of the research team read the directions for each section, answered any questions, and enforced a time limit for each section. For 2nd and 3rd graders, each item was read aloud to reduce the reading demands. If a student had a question, the researcher would provide a helping prompt from a script.

All items were coded as correct or incorrect. A few items had open ended responses, and these were also coded as correct or incorrect according to a strict coding rubric. Each item or sub-item in the assessment only assessed one isolated skill.

Results

We used student accuracy on each item, in conjunction with other findings in the functional thinking literature, to place the skills into hierarchy and develop a construct map. A construct map is a representation of the continuum of knowledge that people are thought to progress through for the target construct (Wilson, 2005). Our preliminary construct map is presented in Table 1, with lower level skills at the bottom of the table. Student accuracy at each level of the construct map is presented in Table 2.

Table 1. *Elementary Function Skill Construct Map*

Level Description		Abilities	Examples
Level 4: Generate Symbolic Rule	• Generate an explicit symbolic rule	Can write rule of correspondence using algebraic symbols.	Y = X +4
Level 3: Generate & Use Verbal Rule	• Generate an explicit verbal rule • Complete a function table with missing values	Can write rule of correspondence verbally. Can fill in missing values of a function table, including a far instance.	-"You add 4 to the number in the X column to get the number in the Y column" -Completed function table
Level 2: Recognize Rule	Recognize Explicit Rule • Select a correct verbal rule out of several choices • Determine the next Y value in a function sequence	Can recognize a rule of correspondence for a table. Can also determine the next Y value in function tables and in patterns.	X Y 1 5 2 6 3 7 Is the rule: -multiply by 5 -multiply by 3 **-add 4**
Level 1: Apply Rule	Application of an Explicit Rule • Use a given rule to determine new Y values	When given a rule for a table, can use it to determine new values of the table.	Complete the Table: *X+4=Y* X Y 1 5 2 6 3 7 4 **8**

Level One: Apply Rule

The easiest items were the application of a given function rule items (85% correct). This ability level included items that asked students to apply a rule to determine missing values in a table. This application of an explicit rule is level one of our construct map. These items required that students understood enough about a relationship between X and Y values in a table that they could determine new Y values when the rule is provided. To complete items like these, computational skill is required, but not yet any deep understanding of a functional relationship.

Level Two: Recognize Rule

Level two of our construct map is Recognize Rule. Within this level are two of our original skills: recognizing a rule, and determining the next Y value. Performance on items at this level was at 61%. Recognizing a rule for a table from a number of other options is presumed to be a skill children acquire before they can generate a rule on their own, and our data bear this out. This level also includes determining the next sequential Y value in a function table. This is different than determining other missing values in a function table, as the next value can be determined by extending the pattern in the Y values, if the table is ordered sequentially.

Level Three: Generate & Use Verbal Rule

Level three in our construct map incorporates the main set of skills students must have in order to have a grasp of function table problems. The skills included within this level are determining further missing values in a table and generating a verbal rule. The ability to generate a correct rule for a table coincided with the ability to determine the missing values. This should not be so surprising, as one would have to determine the rule to find the missing values. Performance on items that utilized these skills was at 40%.

Level Four: Generate Symbolic Rule

The fourth and final level in our construct map is Generate a Symbolic Rule. These items entailed writing the rule using algebraic notation, such a Y = X + 4. This was more difficult for students, presumably not only because of the use of variables, but because of the generality that variables imply. Performance on Symbolic Rule items was at 28%.

Table 2. *Performance on Items by Level and Grade*

	Knowledge Level			
Grade	1	2	3	4
2 (n=52)	71%	35%	11%	1%
3 (n=50)	82%	45%	23%	5%
4 (n=25)	94%	74%	50%	32%
5 (n=60)	91%	71%	53%	38%
6 (n=44)	91%	87%	69%	68%
Average	85%	61%	40%	28%

Assessment and Construct Map Evaluation

To further evaluate the assessment and refine the construct map, classical test and item response methodologies were used. The alignment of the relative difficulty of the items and the leveled construct map was considered. An item-respondent map (i.e., a Wright map) generated by a Rasch model (a type of item response model) was used in this evaluation. The left column of a Wright map is the respondent, or participant, column. Respondents with the most estimated ability are placed near the top of the column, and those with the least estimated ability are on the lower end. In the right column, the most difficult items are listed at the top, and the easier items are at the bottom of the column. This Wright map allows for a visual evaluation of our construct map.

The purposed levels roughly clump together in the Wright map (Figure 1). There are some issues with Level 3 items overlapping with Level 2 and 4 items. The fact that there are different arithmetic operations in the underlying functional relationships in the items was the suspected cause for this overlap. Specifically, some items have an underlying functional relationship that involve addition (i.e., y=x+2), some multiplication (i.e., y=2x), and others, a combination of both (i.e., y=2x+2). Students in grades two through six have different proficiencies with these arithmetic skills, and so it is reasonable that this would affect item difficulty.

When the items were separated by operation of underlying function, the Wright maps have good grouping of items by level, and good separation between levels (Figures 2-4). There is some compression of levels 3 and 4 in the combination-only Wright map. This is likely due to the difficulty of the items; if a student is of high enough ability to do the level 3 combination items, they are also likely to be of high enough ability for the level 4 combination items. Viewing the Wright maps separately by underlying function type, or operation, seems to be quite useful for delineating and clarifying the underlying functional thinking ability progression. Now the levels have much more separation, and the item difficulties are no longer confounded by arithmetic difficulty. This multidimensional model, with operational difficulty considered as a separate factor, is a much clearer way of tracking the progression of early functional thinking knowledge. This model will be developed further in future work.

Several analyses were performed to evaluate the validity and reliability of the assessment. To evaluate validity from measures of internal structure, the expected rank order of difficulty from the construct map was compared to the empirical rank order difficulty from the Wright map analysis (r_s=0.916). An item-mean location analysis was performed to determine if getting an item correct implicated a greater ability level on the part of the respondent than getting it incorrect would. All items behaved appropriately according to this metric. Internal consistency of the assessment was high. The classical test index of internal consistency for binary data (the Kuder-Richardson 20) was quite high, indicating that 93% of the variance was accounted for by the model. The analogous measure from item response methodology, the separation reliability index, was at 0.99. These measures can be thought of as similar to Cronbach's alpha. Additionally, several other analyses from item response methodology were employed, and all metrics of item analyses and person and item fit indicated that the assessment functions well.

Discussion

Functional thinking has been argued to be a useful way to introduce young students to fundamental algebraic concepts. Several teaching experiments have suggested instructional techniques for bringing functional thinking to elementary classrooms. As this topic becomes a focus of more research studies, it becomes increasingly important to have a reliable and valid measure that can be used to capture knowledge change and to have a well defined knowledge construct, so that general claims can be made across studies. In this project, we have identified key skills that are important for elementary-level functional thinking, with a focus on function table problems. These skills were then incorporated into an assessment, which was given to 231 2nd through 6th grade students. Student performance data was used to develop a construct map, or proposed knowledge progression, of elementary-level functional thinking skills. The resulting construct map provided insight into the acquisition of functional thinking knowledge in elementary-school students, and can be used to guide future research.

Benefits of a Construct Modeling Approach

This approach to measurement development, based on Wilson's construct modeling approach (2005), was useful for several reasons. First, it elucidated the relative difficulty of functional thinking skills, and at times this was not in line with our predictions. Second, the resulting assessment is a criterion referenced measure which is particularly appropriate for assessing the affects of an intervention on individuals (Wilson, 2005).

In regard to relative skill difficulty, skills fell together in ways which we did not predict. For instance, Level 2 includes the ability to recognize a rule, as well as determine the next sequential Y value. Based on our literature review and intuitions, we predicted that determining the next Y value would be the easier skill. Also, Level 3 includes the skills of completing a function table and generating a verbal rule. We had predicted that there would be a gap in students' ability to master these two skills. This construct modeling approach allowed us to see that these skills were of the same difficulty, as they could be completed by students of the same ability level. Another aspect we did not predict to play such a large role was the arithmetic operation of the underlying function. We initially predicted that functional thinking skill would function somewhat independently of operation type, but the data indicated that operation profoundly effects students' ability to successfully complete a function table problem. As such, operation type must be given attention in future functional thinking research. We are working to more formally incorporate arithmetic operation of the underlying function into our model.

Future Directions and Conclusions

Our elementary-level functional thinking assessment and construct map are important first steps, but they could each be further refined. Based on the construct map, we can now edit the assessment to more evenly include items at the different skill levels. Additionally, since it is now clear that the operation in the underlying function is a highly important factor, we will include more items so that there are items of each difficulty using each operation type. This revised assessment will then be used in a new iteration of this project. We hope to develop a multidimensional item response model which incorporates both functional thinking and operational skill level. This will allow us to more accurately measure change in students' knowledge.

In conclusion, we have developed an assessment and proposed knowledge progression of elementary students' functional thinking skills, with a focus on function table problems. We believe that this and future iterations of this measure will be valuable tools for educational and psychological researchers who wish to measure knowledge change and evaluate instructional interventions.

References

Blanton, M., Kaput, J.J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*(5).

Carraher, D. W., Earnest, D. (2003). *Guess my rule revisited.* Paper presented at the Proceedings of the 27th International Conference for the Psychology of Mathematics Education, Honolulu, HI.

Carraher, D., Schliemann, A.D., & Brizuela, B. (2003). *Treating operations as functions.* Paper presented at the Psychology of Mathematics Education - NA XXII, Tucson, AZ.

Carraher, D., Martinez, M., Schliemann, A. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education*, *40*.

Cooper, T. J., Warren, E. (2008). The effect of different representations on Year 3 to 5 students' ability to generalize. *ZDM Mathematics Education, 40*.

Kaput, J., (1999). Teaching and learning a new algebra. In E. Fenne, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding*. Mahwah, NJ: Erlbaum.

Kaput, J., (2008). What is Algebra? What is Algebraic Reasoning? In J. J. Kaput, Carraher, D. W., Blanton, M. L. (Ed.), *Algebra in the Early Grades*. New York, NY: Lawrence Erlbaum Associates.

National Council for Teaching of Mathematics, (2008). NCTM Standards from http://standards.nctm.org/document/appendix/alg.htm

Schliemann, A. D., Carraher, D.W. & Brizuela, B.M. (2001). *When tables become function tables.* Paper presented at the Proceedings of the XXV Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands.

Smith, E. (2008). Representational Thinking as a Framework for Introducing Functions in the Elementary Curriculum. *Algebra in the Early Grades.* J. J. Kaput, Carraher, D. W., Blanton, M. L. New York, NY, Lawrence Erlbaum Associates.

Warren, E., Cooper, T. (2005). Introducing Functional Thinking in Year 2: a case study of early algebra teaching. *Contemporary Issues in Early Childhood, 6*(2).

Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom: Foundations of early algebraic reasoning. *The Journal of Mathematical Behavior, 25*(3).

Wilson, M. (2005). *Constructing Measures: An Item Response Modeling Approach*. Mahwah, NJ: Erlbaum.

Figure 1. Wright Map

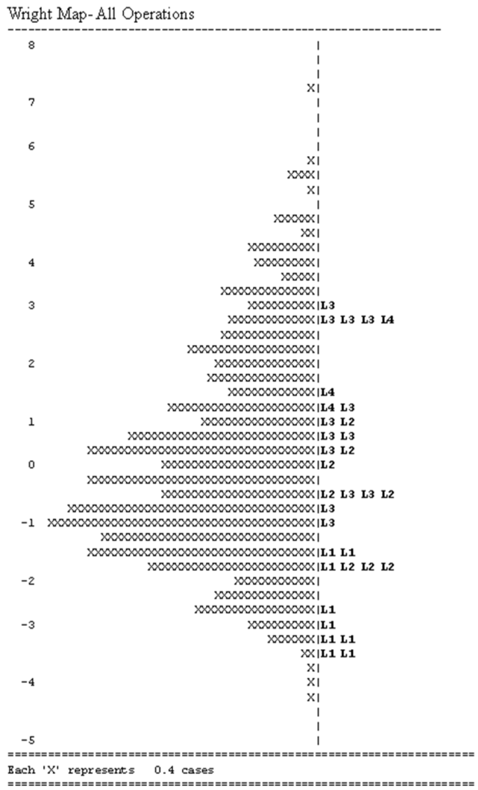

Figure 2. Wright Map, Addition

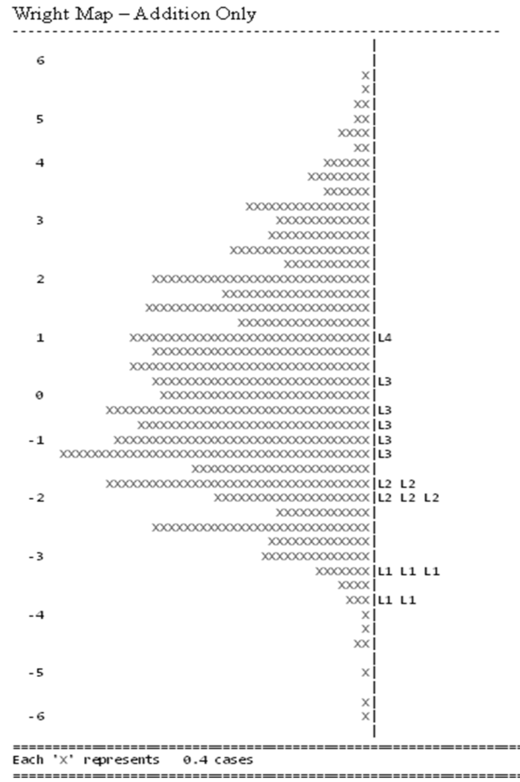

Figure 3. Wright Map, Multiplication

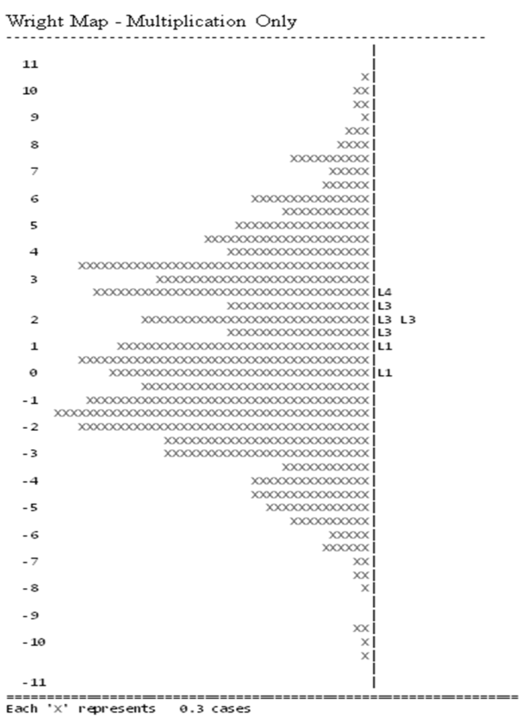

Figure 4. Wright Map, Combination